M303

Further pure mathematics

Metric spaces and continuity

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Contents

1 Introducing metric spaces

We now introduce the idea of a metric space, and show how this concept allows us to generalise the notion of continuity. We will then concentrate on looking at some examples of metric spaces and defer further discussion of continuous functions between metric spaces to the next chapter.

You should not expect to have a firm grasp of the idea of a metric space by the end of this section: this will come as you see more examples of metric spaces in Chapter 15 and later chapters.

Subsection 1.1 shows how properties of the Euclidean distance function on $\mathbb{R}^n$ can be generalised to define a metric on any set to create a metric space.

1.1 The definition of a metric space

Looking back at the definition of continuity for functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ we see that it depends on:

- the Euclidean distance in $\mathbb{R}^n$, $d^{(n)}$
- the Euclidean distance in $\mathbb{R}^m$, $d^{(m)}$
- the definition of convergence for sequences in both $\mathbb{R}^n$ and $\mathbb{R}^m$.

Recall that if $\mathbf{a}, \mathbf{b} \in \mathbb{R}^k$, then $d^{(k)}(\mathbf{a}, \mathbf{b})$ is given by $\sqrt{\sum_{i=1}^{k}(b_i - a_i)^2}$.

Moreover, the definition of convergence for sequences also depends on the notion of distance. Thus a notion of distance (in both the domain and the codomain) is a fundamental part of our definition of continuity. Hence to define a notion of continuity for functions between arbitrary sets X and Y, we must first have effective notions of distance appropriate to the sets X and Y. An obvious strategy to do this is to try to find some properties of the Euclidean distance that we would expect any sensible notion of distance to possess.

We showed that the Euclidean distance function has three properties.

The *Euclidean distance function* $d^{(n)} \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ has the following properties.

For each $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$:

(M1) $d^{(n)}(\mathbf{a}, \mathbf{b}) \geq 0$, with equality holding if, and only if, $\mathbf{a} = \mathbf{b}$
(M2) $d^{(n)}(\mathbf{a}, \mathbf{b}) = d^{(n)}(\mathbf{b}, \mathbf{a})$
(M3) $d^{(n)}(\mathbf{a}, \mathbf{c}) \leq d^{(n)}(\mathbf{a}, \mathbf{b}) + d^{(n)}(\mathbf{b}, \mathbf{c})$ (the *Triangle Inequality*).

Properties (M1)–(M3) do not use any special properties of the set $\mathbb{R}^n$, so it seems reasonable to say that for any set X, a function $d \colon X \times X \to \mathbb{R}$ is a *distance function* if it satisfies them. It turns out that these are indeed the appropriate properties of distance on which to base our definition. We call such a distance function a *metric*, and we call a set with such a function defined on it a *metric space*.

The word *metric* comes from the Greek word $\mu\varepsilon\tau\rho\acute{o}\nu$ (*metron*), meaning 'measure', an instrument for measuring.

Definition 1.1 *Metric*

Let X be a set. A **metric** on X is a function $d\colon X \times X \to \mathbb{R}$ that satisfies the following three conditions.

For each $a, b, c \in X$:

(M1) $d(a, b) \geq 0$, with equality holding if, and only if, $a = b$
(M2) $d(a, b) = d(b, a)$
(M3) $d(a, c) \leq d(a, b) + d(b, c)$ (the **Triangle Inequality**).

The set X, together with a metric d on X, is called a **metric space**, and is denoted by (X, d).

This definition was proposed by Maurice Fréchet, whose work is discussed in the *History Reader*.

Remarks

A 'point' may be nothing like a dot in the plane: X could be a set of functions and then a point in X would be a function.

1. Mathematicians usually refer to the members of the set X as **points**, to emphasise the analogy with the points of a line, a plane or three-dimensional space. Similarly, mathematicians usually refer to $d(a, b)$ as the **distance** between a and b. The theory of metric spaces is the study of those properties of sets of points that depend only on distance.
2. Condition (M1) says that distance is a non-negative quantity, and that the only point of a metric space that is at zero distance from a given point is that point itself.
3. Condition (M2) says that the distance from a point a to a point b is precisely the same as the distance from b to a. (The metric is *symmetric*. It doesn't matter whether you are going from a to b or from b to a – the distance is the same.)
4. Condition (M3) tells us that $d(a, c)$ gives the 'shortest' distance between a and c. For if we go directly from a to c, that gives a distance of $d(a, c)$. If we make a detour to b along the way, we must go a distance $d(a, b)$ to get to b, and then an additional distance $d(b, c)$ to get from b to c (Figure 1.1). Condition (M3) tells us that the total distance travelled must then be at least as great as the 'direct' distance $d(a, c)$.

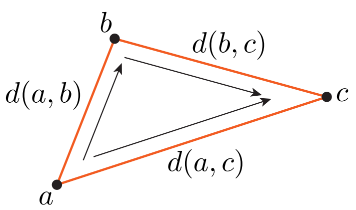


Figure 1.1 Shortest distance between a and c

5. Since our general definition is modelled on the corresponding properties of the Euclidean distance function $d^{(n)}$, it follows that $(\mathbb{R}^n, d^{(n)})$ is a metric space, for each $n \in \mathbb{N}$. It is known as **Euclidean n-space**. Furthermore, in the context of metric spaces, the Euclidean distance function $d^{(n)}$ is often referred to as the **Euclidean metric** for $\mathbb{R}^n$. These are our first examples of metric spaces.

If we look back at the proof of the Reverse Triangle Inequality for the Euclidean metric on $\mathbb{R}^n$ (Proposition **??**), we see that the proof makes use only of properties (M1)–(M3) of the Euclidean metric. So it is no surprise to learn that there is a Reverse Triangle Inequality for any metric space.

Proposition 1.2 *Reverse Triangle Inequality for metric spaces*

Let (X, d) be a metric space. For each $a, b, c \in X$,
(M3a) $\quad d(b, c) \geq |d(a, c) - d(a, b)|$.

The proof is essentially the same as that for Proposition **??**, and so we omit it.

1.2 Examples of metric spaces

We now give a few examples of metric spaces with the aim of getting you used to showing that a distance function is a metric. The examples we consider here are not very sophisticated and may not persuade you of the utility of this new concept. However, in the next chapter we investigate ways of defining distances between more interesting 'points', such as between pairs of functions or pairs of sequences – this is when we will start to see the real power of metrics.

The discrete metric

Our first example is simple but important, since it shows that it is possible to define a metric on any set X.

Definition 1.3 *Discrete metric*

Let X be a set. The **discrete metric** on X is the function $d_0 \colon X \times X \to \mathbb{R}$ defined by

$$d_0(a, b) = \begin{cases} 0, & \text{if } a = b, \\ 1, & \text{if } a \neq b. \end{cases}$$

We show that d_0 is a metric after the remarks.

Remarks

1. When $X = \emptyset$, the empty set, the definition still makes sense but is not very interesting.
2. The d_0-distance between any two *distinct* points of a set X is always equal to 1. In particular, if we consider the case when $X = \mathbb{R}$, the real line, we find that d_0 is a different metric from the usual Euclidean distance. (To see this, observe that $d_0(0, 2) = 1$ whereas $d^{(1)}(0, 2) = 2$.) Thus the discrete metric gives us a second notion of distance on $\mathbb{R}$. In fact, if (X, d) is any metric space whose metric d is not d_0, then (X, d_0) is a second metric space with underlying set X.

   The discrete metric is about the simplest possible notion of distance there can be and acts to 'isolate' points of a space: they are all at distance 1 from one another.

Proposition 1.4

Let X be a set. Then (X, d_0) is a metric space.

Proof Let X be a set. In order to show that d_0 is a metric on X, we must show that d_0 satisfies conditions (M1)–(M3).

If $X = \emptyset$, then X has no elements and so each of (M1)–(M3) vacuously holds: there is nothing to prove. So suppose instead that X is non-empty.

(M1) Since d_0 can take only the values 0 and 1, we have $d_0(a, b) \geq 0$ for each choice of $a, b \in X$.

The definition of d_0 implies immediately that $d_0(a, b) = 0$ if, and only if, $a = b$.

Thus d_0 satisfies (M1).

(M2) If $a = b$, then $d_0(a, b) = 0 = d_0(b, a)$.

If $a \neq b$, then $d_0(a, b) = 1 = d_0(b, a)$.

Thus, for each $a, b \in X$, (M2) holds.

(M3) Let $a, b, c \in X$. We examine the two possible cases: $d_0(a, c) = 0$ and $d_0(a, c) = 1$.

Suppose $d_0(a, c) = 0$ (so $a = c$). Since $d_0(a, b)$ and $d_0(b, c)$ are non-negative, it follows that

$$d_0(a, b) + d_0(b, c) \geq 0 = d_0(a, c).$$

Now suppose $d_0(a, c) = 1$; then $a \neq c$ and so b cannot equal both a and c. Hence, from (M1), at least one of $d_0(a, b)$ and $d_0(b, c)$ is non-zero and so must equal 1. Thus

$$d_0(a, b) + d_0(b, c) \geq 1 = d_0(a, c).$$

Hence, in both cases, (M3) holds.

Since d_0 satisfies (M1)–(M3), it is a metric on X and (X, d_0) is a metric space. ■

At first appearance, the simplicity of the definition of the discrete metric d_0 may make it seem unimportant, but this is not the case. Its utility lies in the fact that it gives an ‘extreme’ example of a metric space – no other definition of distance so completely ignores any structure that may be present in the underlying set X.

We use the discrete metric for a number of purposes, principally to test properties that we suspect may hold for all metric spaces. If such a property fails for (X, d_0), then we know that our suspicion was false.

Exercise 1.1

Let $X = \{x, y, z\}$ and define $d\colon X \times X \to \mathbb{R}$ by

$$\begin{aligned} d(x,x) &= d(y,y) = d(z,z) = 0,\\ d(x,y) &= d(y,x) = 1,\\ d(y,z) &= d(z,y) = 2,\\ d(x,z) &= d(z,x) = 4. \end{aligned}$$

Determine whether d is a metric on X.

The taxicab metric: an alternative metric for the plane

We have already observed that the usual definition of distance on the plane defines a metric and we have just seen that it is possible to define a very simple notion of distance on the plane, namely the discrete metric given previously. Here we introduce another natural way to measure distance between points in the plane, the taxicab metric.

Recall that the usual definition of distance on the plane is given by

$$d^{(2)}(\mathbf{a}, \mathbf{b}) = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2} \quad \text{for } \mathbf{a}, \mathbf{b} \in \mathbb{R}^2.$$

What happens if we change the formula we use for distance on the right-hand side of this expression to something that is perhaps easier to calculate? For example, consider the function $e_1\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$e_1(\mathbf{a}, \mathbf{b}) = |b_1 - a_1| + |b_2 - a_2| \quad \text{for } \mathbf{a}, \mathbf{b} \in \mathbb{R}^2.$$

The reason for the subscript 1 in the notation e_1 will become clear in Chapter 15.

As we will shortly see, this *does* define an alternative metric on the plane. First let us understand how this distance function behaves.

Exercise 1.2

Find:

(a) $e_1((0,0),(1,0))$; (b) $e_1((0,0),(0,1))$;

(c) $e_1((0,1),(1,0))$.

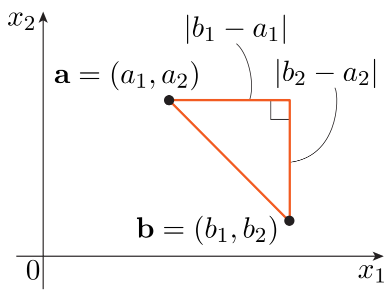


Figure 1.2 Visual representation of $e_1(\mathbf{a}, \mathbf{b})$

You may have realised from your work on Exercise 1.2 that one way to understand $e_1(\mathbf{a}, \mathbf{b})$ is to draw a right-angled triangle with $\mathbf{a}$ and $\mathbf{b}$ at its two non-right-angled vertices, as shown in Figure 1.2. Then $e_1(\mathbf{a}, \mathbf{b})$ is the sum of the distances along the sides parallel to the axes. Another way to interpret it is to consider $\mathbf{a}$ and $\mathbf{b}$ as representing intersections in a city where the roads form a rectangular grid, as Figure 1.3 illustrates. Then $e_1(\mathbf{a}, \mathbf{b})$ is the shortest distance that can be travelled by a vehicle, such as a taxi, to get from $\mathbf{a}$ to $\mathbf{b}$. This interpretation leads to the common name *taxicab metric* for e_1.

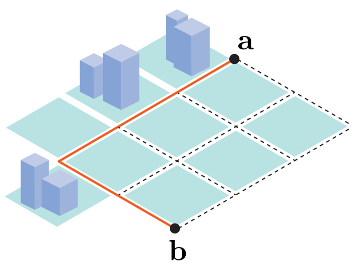


Figure 1.3 Alternative interpretation of $e_1(\mathbf{a}, \mathbf{b})$

Let us verify that e_1 does define a metric: we must check that it satisfies conditions (M1)–(M3).

(M1) For each $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$, $|b_1 - a_1|$ and $|b_2 - a_2|$ are non-negative, and hence so is their sum: thus $e_1(\mathbf{a}, \mathbf{b}) \geq 0$.

For each $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$,

$$e_1(\mathbf{a}, \mathbf{a}) = |a_1 - a_1| + |a_2 - a_2| = 0 + 0 = 0.$$

Conversely, suppose $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$ are such that $e_1(\mathbf{a}, \mathbf{b}) = 0$. Then

$$0 = |b_1 - a_1| + |b_2 - a_2|,$$

which implies that both $|b_1 - a_1| = 0$ and $|b_2 - a_2| = 0$. Thus $a_1 = b_1$ and $a_2 = b_2$; that is, $\mathbf{a} = \mathbf{b}$.

This proves that condition (M1) is satisfied.

(M2) Let $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$. Using that the modulus function has property (M2), we obtain

The modulus function is the Euclidean distance function on $\mathbb{R}$, and so is a metric.

$$\begin{aligned} e_1(\mathbf{a}, \mathbf{b}) &= |b_1 - a_1| + |b_2 - a_2| \\ &= |a_1 - b_1| + |a_2 - b_2| = e_1(\mathbf{b}, \mathbf{a}). \end{aligned}$$

This proves that condition (M2) is satisfied by e_1.

(M3) Let $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2), \mathbf{c} = (c_1, c_2) \in \mathbb{R}^2$. Then, using that the modulus function has property (M3),

$$\begin{aligned} e_1(\mathbf{a}, \mathbf{c}) &= |c_1 - a_1| + |c_2 - a_2| \\ &\leq (|c_1 - b_1| + |b_1 - a_1|) + (|c_2 - b_2| + |b_2 - a_2|) \\ &= (|b_1 - a_1| + |b_2 - a_2|) + (|c_1 - b_1| + |c_2 - b_2|) \\ &= e_1(\mathbf{a}, \mathbf{b}) + e_1(\mathbf{b}, \mathbf{c}). \end{aligned}$$

This proves that condition (M3) is satisfied by e_1.

We conclude that e_1 is a metric on the plane.

Exercise 1.3

Determine which of the following functions define a metric on $\mathbb{R}$:

(a) $d(a, b) = \sqrt{b^2 - a^2}$; (b) $d(a, b) = \sqrt{|b|^3 + |a|^3}$;

(c) $d(a, b) = \sqrt[3]{|b^3 - a^3|}$.

A 'mixed' metric in the plane

Our next example of a metric is a mixture of the discrete metric and the modulus function.

For points $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ in the plane, define the function $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d(\mathbf{a}, \mathbf{b}) = |b_1 - a_1| + d_0(a_2, b_2),$$

where d_0 denotes the discrete metric on $\mathbb{R}$.

Our goal is to show that this is indeed a metric on the plane. To do this, we make extensive use of the fact that the modulus function and the discrete metric both satisfy properties (M1)–(M3) on $\mathbb{R}$.

Exercise 1.4

Show that d has property (M1).

Exercise 1.5

Show that d has property (M2).

We verify the Triangle Inequality as a worked exercise.

Worked Exercise 1.5

Show that d has property (M3).

Solution

We must show that for any choice of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^2$, we have

$$d(\mathbf{a}, \mathbf{c}) \leq d(\mathbf{a}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c}).$$

As with the previous exercises, we use the fact that both $|\cdot|$ and d_0 have property (M3) when viewed as metrics on $\mathbb{R}$.

Let $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$, $\mathbf{c} = (c_1, c_2) \in \mathbb{R}^2$. Then

$$\begin{aligned}
d(\mathbf{a}, \mathbf{c}) &= |a_1 - c_1| + d_0(c_2, a_2), \text{ by the definition of } d\\
&\leq |a_1 - b_1| + |b_1 - c_1| + d_0(c_2, a_2), \text{ by (M3) for } |\cdot|\\
&\leq |a_1 - b_1| + |b_1 - c_1| + d_0(c_2, b_2) + d_0(b_2, a_2), \text{ by (M3) for } d_0\\
&= (|a_1 - b_1| + d_0(b_2, a_2)) + (|b_1 - c_1| + d_0(c_2, b_2))\\
&= d(\mathbf{a}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c}),
\end{aligned}$$

as required.

Since d has properties (M1)–(M3), we conclude that it is a metric for $\mathbb{R}^2$ but rather a strange one!

1.3 Understanding the geometry of metric spaces

Before we look at what it means for a sequence to be convergent with respect to a given metric, we spend a little time discussing one way of gaining some understanding about the geometric meaning of a given metric.

In the last subsection, we met three different metrics: the discrete metric, the taxicab metric on the plane and a mixed metric on the plane (which was formed from the usual distance in $\mathbb{R}$ together with the discrete metric).

An easy way to gain some insight into the behaviour of a metric is to look at the *balls* around a given point. For the usual Euclidean distance in $\mathbb{R}^n$, a ball of radius r around a point $\mathbf{a} \in \mathbb{R}^n$ consists of all those points whose distance from $\mathbf{a}$ is at most r, and this definition naturally extends to general metric spaces. However, in the following definition we take care to distinguish between balls that include points at exactly distance r from the centre $\mathbf{a}$ and those that do not.

Definition 1.6 *Open and closed balls*

Let (X, d) be a metric space, and let $a \in X$ and $r \geq 0$.

The **open ball** of **radius** r with **centre** a is the set

$$B_d(a, r) = \{x \in X : d(a, x) < r\}.$$

The **closed ball** of **radius** r with **centre** a is the set

$$B_d[a, r] = \{x \in X : d(a, x) \leq r\}.$$

The **sphere** of **radius** r with **centre** a is the set

$$S_d(a, r) = \{x \in X : d(a, x) = r\}.$$

When $r = 1$, these sets are called respectively the **unit open ball with centre** a, the **unit closed ball with centre** a and the **unit sphere with centre** a.

There is no universal agreement on the terminology and notation for balls. It is a wise precaution to check the conventions whenever reading a book on metric spaces.

Remarks

1. When we wish to emphasise the particular metric d on X, we refer to the *d-open ball*, the *d-closed ball* and the *d-sphere*.
2. When d is the usual Euclidean metric on $\mathbb{R}^2$, we recover the usual notions of open and closed balls in the plane.
3. Since $d(a, a) = 0$, $a \in B_d(a, r)$ whenever $r > 0$ and $a \in B_d[a, r]$ whenever $r \geq 0$.

Worked Exercise 1.7

Let (X, d) be a metric space, and let $a \in X$. Show that

$$B_d(a, 0) = \emptyset, \quad B_d[a, 0] = \{a\} \quad \text{and} \quad S_d(a, 0) = \{a\}.$$

Solution

It follows from (M1) that

$$B_d(a, 0) = \{x \in X : d(a, x) < 0\} = \emptyset,$$

$$\begin{aligned} B_d[a, 0] &= \{x \in X : d(a, x) \leq 0\} \\ &= \{x \in X : d(a, x) = 0\} = \{a\}, \end{aligned}$$

and

$$S_d(a, 0) = \{x \in X : d(a, x) = 0\} = \{a\}.$$

We now discover what open balls, closed balls and spheres look like for some of the metric spaces we have met already.

Let us start by determining the open and closed balls for the discrete metric, d_0.

Worked Exercise 1.8

Let X be a non-empty set and $a \in X$. Determine $B_{d_0}(a, r)$ for $r \geq 0$.

Solution

Let $a \in X$ and suppose that $r > 0$.

Since $B_{d_0}(a, r) = \{x \in X : d_0(a, x) < r\}$ and $d_0(a, x) = 1$ unless $a = x$ (when it is 0), we conclude that

$$B_{d_0}(a, r) = \begin{cases} \emptyset, & \text{if } r = 0, \\ \{a\}, & \text{if } 0 < r \leq 1, \\ X, & \text{if } r > 1. \end{cases}$$

Exercise 1.6

Let X be a non-empty set and $a \in X$. Determine $B_{d_0}[a, r]$ for $r \geq 0$.

Exercise 1.7

Let X be a non-empty set and $a \in X$. Determine $S_{d_0}(a, r)$ for $r \geq 0$.

Next, let us look at open balls defined using the taxicab metric.

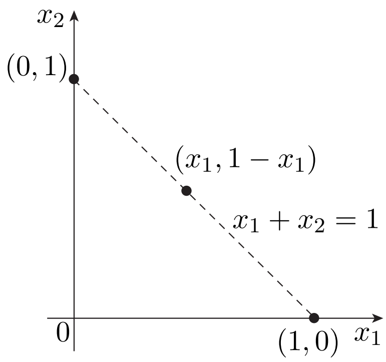


Figure 1.4 The line $x_1 + x_2 = 1$

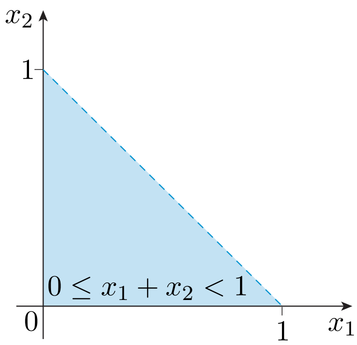


Figure 1.5 The points where $x_1 + x_2 < 1$

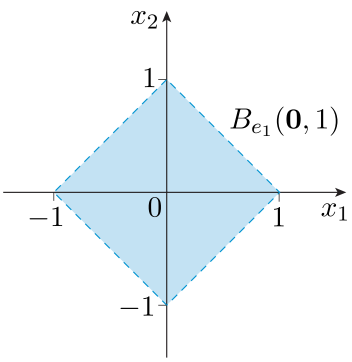


Figure 1.6 The open ball $B_{e_1}(\mathbf{0}, 1)$

Worked Exercise 1.9

Consider the metric space $(\mathbb{R}^2, e_1)$ – that is, the plane with the taxicab metric. Find the unit open ball $B_{e_1}(\mathbf{0}, 1)$.

Solution

The centre is $\mathbf{0} = (0,0)$, and we want to find all points $\mathbf{x} = (x_1, x_2)$ that satisfy

$$e_1(\mathbf{0}, \mathbf{x}) = |x_1| + |x_2| < 1.$$

We first consider points in the first quadrant, where $x_1, x_2 \ge 0$.

We want to find those points where $x_1 + x_2 < 1$. Consider the line $x_1 + x_2 = 1$, or equivalently $x_2 = 1 - x_1$. In the first quadrant, this line connects the points $(0,1)$ and $(1,0)$ and is shown dashed in Figure 1.4. The points on this line segment have coordinates $(x_1, 1-x_1)$. All points below the line segment have coordinates (x_1, x_2) with $x_2 < 1 - x_1$ and all points on or above it have coordinates (x_1, x_2) with $x_2 \ge 1 - x_1$. Hence the points where $x_1 + x_2 < 1$ are those strictly below the line segment, making up the shaded region in Figure 1.5.

By use of a similar argument for each of the other three quadrants, or by appealing to the symmetry of the situation, we obtain triangular regions in each quadrant. Combining these, we obtain the diamond-shaped region in Figure 1.6; the open ball $B_{e_1}(\mathbf{0}, 1)$ is the set of points strictly inside this diamond, shown shaded in the figure. The dashed boundary indicates that it is not included in the set.

Exercise 1.8

Sketch the open ball $B_{e_1}((2,3), 2)$, briefly justifying your answer.

Let us finish this discussion by looking at the balls for our mixed metric in the plane, d.

Worked Exercise 1.10

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$. Determine $B_d(\mathbf{a}, r)$ for $r > 0$, where d denotes the mixed metric in the plane given by

$$d(\mathbf{a}, \mathbf{b}) = |b_1 - a_1| + d_0(a_2, b_2) \text{ for } \mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbb{R}^2.$$

Solution

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ and suppose that $r > 0$. Then, using the definition of an open ball,

$$\begin{aligned} B_d(\mathbf{a}, r) &= \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : d(\mathbf{a}, \mathbf{x}) < r\} \\ &= \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_1 - a_1| + d_0(a_2, x_2) < r\}. \end{aligned}$$

We now consider the cases $x_2 = a_2$ and $x_2 \neq a_2$ separately. If $x_2 = a_2$, then $d_0(a_2, x_2) = 0$ and so

$$\begin{aligned} \{\mathbf{x} = (x_1, a_2) \in \mathbb{R}^2 : |x_1 - a_1| < r\} &= \{x_1 \in \mathbb{R} : |x_1 - a_1| < r\} \times \{a_2\} \\ &= (a_1 - r, a_1 + r) \times \{a_2\}. \end{aligned}$$

If $x_2 \neq a_2$, then $d_0(a_2, x_2) = 1$ and

$$\begin{aligned} &\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq a_2 \text{ and } |x_1 - a_1| + d_0(a_2, x_2) < r\} \\ &= \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq a_2 \text{ and } |x_1 - a_1| < r - 1\}. \end{aligned}$$

Let us unpack this set expression: x_2 can be any value other than a_2 but if $r \leq 1$, then there are no possible x_1 for which $|x_1 - a_1| < r - 1$, and so the expression is the empty set, $\emptyset$. Whereas if $r > 1$, then the set of $x_1 \in \mathbb{R}$ for which $|x_1 - a_1| < r - 1$ is the interval $(a_1 - (r-1), a_1 + (r-1))$. Hence for $r > 1$, the set expression is $(a_1 - (r-1), a_1 + (r-1)) \times (\mathbb{R} - \{a_2\})$. Summarising,

$$\begin{aligned} &\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq a_2 \text{ and } |x_1 - a_1| + d_0(a_2, x_2) < r\} \\ &= \begin{cases} \emptyset, & \text{if } r \leq 1, \\ (a_1 - (r-1), a_1 + (r-1)) \times (\mathbb{R} - \{a_2\}), & \text{if } r > 1. \end{cases} \end{aligned}$$

Hence, as shown in Figure 1.7,

$$B_d(\mathbf{a}, r) = \begin{cases} (a_1 - r, a_1 + r) \times \{a_2\}, & \text{if } r \leq 1, \\ ((a_1 - r, a_1 + r) \times \{a_2\}) \\ \qquad \cup ((a_1 - (r-1), a_1 + (r-1)) \times \mathbb{R}), & \text{if } r > 1. \end{cases}$$

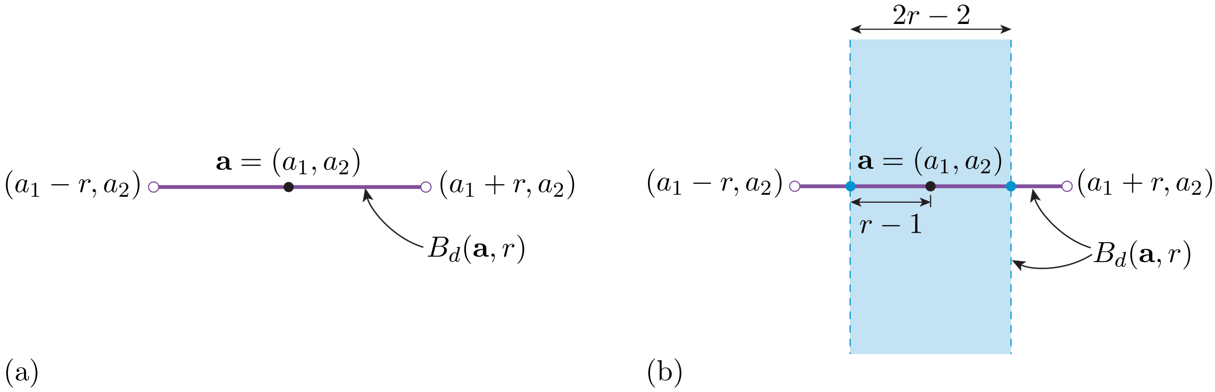


Figure 1.7 $B_d(\mathbf{a}, r)$: (a) for $r \leq 1$; (b) for $r > 1$

Exercise 1.9

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$. Determine $B_d[\mathbf{a}, r]$ for $r \geq 0$, where d denotes the mixed metric in the plane given by

$$d(\mathbf{a}, \mathbf{b}) = |b_1 - a_1| + d_0(a_2, b_2) \text{ for } \mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbb{R}^2.$$

We will return to our investigation of the geometry of metric spaces in Chapter 16.

2 Sequences in metric spaces

Now that we have several examples of metric spaces available to us, we return to the problem of defining continuous functions between metric spaces.

Since the definition of a general metric space is modelled on the properties of the Euclidean metric $d^{(n)}$ on $\mathbb{R}^n$, and we defined continuity of functions between Euclidean spaces in terms of convergent sequences, it is natural to attempt to extend our ideas about convergent sequences in $\mathbb{R}^n$ to general metric spaces. In fact, we did much of the hard work when we generalised from the notion of convergence for real-valued sequences to that of convergence of sequences in $\mathbb{R}^n$; it is now only a short step to develop these concepts for the metric space setting.

In Chapter 13, we observed that a real sequence can be thought of as a function $a\colon \mathbb{N} \to \mathbb{R}$, given by $n \mapsto a_n$. Note that the only role played by $\mathbb{R}$ here is as the codomain of the function $a\colon \mathbb{N} \to \mathbb{R}$; the structure of $\mathbb{R}$ becomes relevant only when convergence is considered. Since the codomain of a function is simply a set, the following definition is a natural generalisation.

Definition 2.1 *Sequence in a metric space*

Let X be a set. A **sequence** in X is an unending ordered list of elements of X:

$$a_1, a_2, a_3, \ldots.$$

The element a_k is the ***k*th term** of the sequence, and the whole sequence is denoted by (a_k), $(a_k)_{k=1}^{\infty}$ or $(a_k)_{k\in\mathbb{N}}$.

Note that this definition of a sequence does not require that we impose any additional structure (such as a metric) on the set X.

The definition of what it means for a sequence to converge in a metric space (X, d) is closely based on the definition of convergence in $\mathbb{R}^n$.

Definition 2.2 *Convergence in a metric space*

Let (X, d) be a metric space. A sequence (a_k) in X d-**converges** to $a \in X$ if the sequence of real numbers $(d(a_k, a))$ is a null sequence.

We write $a_k \xrightarrow{d} a$ as $k \to \infty$, or simply $a_k \to a$ if the context is clear.

We say that the sequence (a_k) is **convergent** in (X, d) with **limit** a.

A sequence that does not converge (with respect to the metric d) to any point in X is said to be d-**divergent**.

Remarks

1. Note that the limit a *must* be a point in X. For example, if X is the *open* interval $(0, 1)$ and we give it the usual notion of distance $d(a, b) = |a - b|$, then the sequence $(\frac{1}{n})$ is not d-convergent, since the only possible choice of limit is 0 – which is *not* a point of X.
2. In order to show that a sequence (a_k) converges to a for the metric d, we must show that $d(a_k, a) \to 0$ as $k \to \infty$ – that is, for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ for which $d(a_k, a) < \varepsilon$ for all $k > N$.

It is not hard to check that this defines a metric on $(0, 1)$ – we return to the general problem of defining metrics on subsets of metric spaces in the next chapter.

Exercise 2.1

Let $(\mathbb{R}^2, e_1)$ be the plane with the taxicab metric, and let $(\mathbf{a}_n)$ be the sequence given by $\mathbf{a}_k = (1 + \frac{1}{2k}, 2 - \frac{1}{2k})$. Show that $(\mathbf{a}_k)$ converges to $(1, 2)$ with respect to e_1.

Convergent sequences in $(\mathbb{R}^n, d^{(n)})$ have unique limits – that is, a sequence cannot simultaneously converge to two different limits. The next result establishes this as a fact in any metric space.

Theorem 2.3 *Uniqueness of limits in a metric space*

Let (X, d) be a metric space and let $a, b \in X$. If (a_k) is a sequence in X that d-converges to both a and b, then $a = b$.

Proof We use proof by contradiction.

Suppose that the sequence (a_k) d-converges to both a and b in X, with $a \neq b$. Then by property (M1) of Definition 1.1 for d, $d(a,b) > 0$ and so if we let $\varepsilon = \frac{1}{2}d(a,b)$, then $\varepsilon > 0$.

Since we are supposing that the sequence (a_k) converges to both a and b, the sequences of real numbers $(d(a_k, a))$ and $(d(a_k, b))$ are both null. Hence there is $N \in \mathbb{N}$ for which $d(a_k, a) < \varepsilon$ and $d(a_k, b) < \varepsilon$ whenever $k > N$.

The Triangle Inequality (property (M3) for d) tells us that, for each $k > N$,

$$\begin{aligned} d(a,b) \leq d(a, a_k) + d(a_k, b) < \varepsilon + \varepsilon &= 2\varepsilon \\ &= d(a,b), \end{aligned}$$

by the definition of ε. But this is impossible; hence our initial assumption that a sequence could converge to two distinct limits must be wrong. We conclude that any d-convergent sequence has a unique limit. ■

The next definition enables us to describe a particularly straightforward type of convergent sequence.

Definition 2.4 *Eventually constant*

Let (a_k) be a sequence in a set X. We say that (a_k) is **eventually constant** if there is $a \in X$ and $N \in \mathbb{N}$ such that $a_k = a$ whenever $k > N$.

Worked Exercise 2.5

Let (X, d) be a metric space and let (a_k) be a sequence in X that is eventually constant. Show that (a_k) converges for d and state its limit.

Solution

Since (a_k) is eventually constant, there is $a \in X$ and $N \in \mathbb{N}$ such that $a_k = a$ whenever $k > N$. Therefore

$$d(a_k, a) = d(a, a) = 0, \quad \text{for each } k > N.$$

Hence the sequence of real numbers $(d(a_n, a))$ is null, which means that $a_k \xrightarrow{d} a$ as $k \to \infty$. Thus the sequence (a_k) is d-convergent with limit a.

Convergence and the discrete metric

Let us consider the sequence (a_k) in $(\mathbb{R}, d_0)$ given by $a_k = \frac{1}{k}$. We see that

$$d_0(a_k, 0) = d_0\left(\tfrac{1}{k}, 0\right) = 1, \quad \text{for each } k \in \mathbb{N},$$

so (a_k) does not converge to 0 with respect to the discrete metric d_0. In fact, this sequence (a_k) is divergent for d_0, because given any non-zero $l \in \mathbb{R}$,

$$d_0(a_k, l) = d_0(\tfrac{1}{k}, l) = 1 \quad \text{for each } k > \tfrac{1}{|l|}.$$

$(\mathbb{R}, d_0)$ is the set of real numbers with the discrete metric d_0.

On the other hand, we know that the sequence $(\frac{1}{k})$ *does* converge to 0 with respect to the Euclidean metric $d^{(1)}$ on $\mathbb{R}$. The important conclusion to draw from this example is:

convergence depends on how we measure distance.

In other words, different metrics on the same set can give rise to different convergent sequences.

From the solution to Worked Exercise 2.5, we know that eventually constant sequences (a_k) in a metric space (X, d_0) are d_0-convergent. The next exercise asks you to show that the eventually constant sequences are the *only* convergent sequences for the d_0 metric on a set X.

Exercise 2.2

Let X be a set and d_0 the discrete metric for X. Suppose that (a_k) is a sequence in X that is d_0-convergent. Show that (a_k) is an eventually constant sequence.

3 The definition of continuity in metric spaces

Now that we know what it means for a sequence to converge in a metric space, we can formulate a definition of continuity for functions between metric spaces.

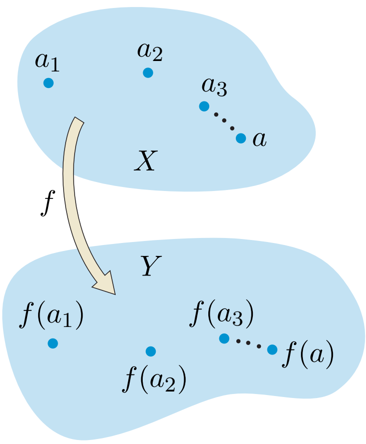


Figure 3.1 The function $f\colon X \to Y$

Definition 3.1 *Continuity for metric spaces*

Let (X, d) and (Y, e) be metric spaces and let $f\colon X \to Y$ be a function (Figure 3.1).

Then f is (d, e)**-continuous** at $a \in X$ if:

> whenever (a_k) is a sequence in X for which $a_k \xrightarrow{d} a$ as $k \to \infty$,
> then the sequence $f(a_k) \xrightarrow{e} f(a)$ as $k \to \infty$.

If f does not satisfy this condition at some $a \in X$ – that is, there is a sequence (x_k) in X for which $x_k \to a$ as $k \to \infty$ but $f(x_k)$ does not converge to $f(a)$ – then we say that f is (d, e)**-discontinuous** at a.

A function that is continuous at all points of X is said to be (d, e)**-continuous** on X (or simply **continuous**, if no ambiguity is possible).

Remarks

1. When $(X, d) = (\mathbb{R}^n, d^{(n)})$ and $(Y, e) = (\mathbb{R}^m, d^{(m)})$, we recover the definition of continuity given in Definition **??** of Subsection **??**.
2. We have not yet defined what it means to be continuous on a subset A of the metric space X; we return to this issue in the next chapter.

See Theorem **??** of Chapter 15.

Our next worked exercise shows that this definition can make some surprising functions continuous.

Worked Exercise 3.2

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$. Prove that f is always $(d_0, d^{(1)})$-continuous at a.

Solution

Let $a \in \mathbb{R}$ and suppose that (x_k) is a sequence in $\mathbb{R}$ that is d_0-convergent to a.

Then by Exercise 2.2 we deduce that there is $N \in \mathbb{N}$ so that for $k > N$, $x_k = a$. But then for $k > N$, $f(x_k) = f(a)$ and so for such k, $d^{(1)}(f(x_k), f(a)) = |f(a) - f(a)| = 0$. Hence $(d^{(1)}(f(x_k), f(a)))$ is a real null sequence and we conclude that f is $(d_0, d^{(1)})$-continuous at a.

This is a rather artificial example and it tells us that every function from $\mathbb{R}$ to $\mathbb{R}$ is $(d_0, d^{(1)})$-continuous on $\mathbb{R}$. However, it does illustrate that our intuitive notion of what continuity means breaks down when looking at metrics different from the Euclidean ones, and so highlights the importance of working from the definition.

In fact one can use a similar argument to show that any $f\colon X \to Y$ is (d_0, e)-continuous on X, no matter the choice of metric e for Y.

Worked Exercise 3.3

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x_1, x_2) = (x_1, 2x_2)$. Prove that f is $(d^{(2)}, e_1)$-continuous on $\mathbb{R}^2$.

Recall that e_1 denotes the taxicab metric on the plane and $d^{(2)}$ is the usual Euclidean distance on the plane.

Solution

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$. We must show that if $(\mathbf{x}_k)$ is a sequence in the plane that $d^{(2)}$-converges to $\mathbf{a}$, then $(e_1(f(\mathbf{x}_k), f(\mathbf{a})))$ is a real null sequence.

Suppose that $(\mathbf{x}_k = (x_{1,k}x_{2,k}))$ is a sequence in the plane that $d^{(2)}$-converges to $\mathbf{a}$, that is, a sequence for which $(d^{(2)}(\mathbf{x}_k, \mathbf{a}))$ is a real null sequence. Then

$$\begin{aligned} e_1(f(\mathbf{x}_k), f(\mathbf{a})) &= |x_{1,k} - a_1| + |2x_{2,k} - 2a_2|, \text{ by definition of } f \text{ and } e_1 \\ &= |x_{1,k} - a_1| + 2|x_{2,k} - a_2| \\ &\leq 2(|x_{1,k} - a_1| + |x_{2,k} - a_2|) \\ &\leq 2(d^{(2)}(\mathbf{x}_k, \mathbf{a}) + d^{(2)}(\mathbf{x}_k, \mathbf{a})) = 4d^{(2)}(\mathbf{x}_k, \mathbf{a}). \end{aligned}$$

But $e_1(f(\mathbf{x}_k), f(\mathbf{a})) \geq 0$ for every k and we are assuming that $(d^{(2)}(\mathbf{x}_k, \mathbf{a}))$ is a real null sequence. Hence by the Squeeze Rule, $e_1(f(\mathbf{x}_k), f(\mathbf{a})) \to 0$ as $k \to \infty$. That is, $(e_1(f(\mathbf{x}_k), f(\mathbf{a})))$ is also a real null sequence and so f is $(d^{(2)}, e_1)$-continuous at a.

Since $a \in \mathbb{R}^2$ was an arbitrary point in $\mathbb{R}^2$, we conclude that f is $(d^{(2)}, e_1)$-continuous on $\mathbb{R}^2$.

At the moment our stock of metric spaces is quite small: Euclidean spaces, the plane with the taxicab metric, the plane with a particular 'mixed' metric, and arbitrary sets with the discrete metric. In the next chapter we will look at more examples of metric spaces and examine further the notion of continuity. What we can do at this point, though, is prove a useful result that applies to all continuous functions and which is an extension of the Composition Rule for continuous functions between Euclidean spaces.

Proposition 3.4 *Composition Rule*

Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. Let $f\colon X \to Y$ be (d_X, d_Y)-continuous and let $g\colon Y \to Z$ be (d_Y, d_Z)-continuous. Then the composed function $g \circ f\colon X \to Z$ is (d_X, d_Z)-continuous.

Recall that $(g \circ f)(x) = g(f(x))$.

Proof We take an arbitrary point $a \in X$ and prove that $g \circ f$ is continuous at a.

We need to show that if (a_k) is a sequence in X with $a_k \overset{d_X}{\to} a$ as $k \to \infty$, then

$$(g \circ f)(a_k) \overset{d_Z}{\to} (g \circ f)(a). \tag{1}$$

Write $y_k = f(a_k)$ and $y = f(a)$. Since f is (d_X, d_Y)-continuous on X, we know that

$$y_k = f(a_k) \xrightarrow{d_Y} f(a) = y \text{ as } k \to \infty.$$

That is, the sequence $(d_Y(y_k, y))$ is a real null sequence.

But the function g is (d_Y, d_Z)-continuous on Y and so it follows by the definition of continuity for g that

$$g(y_k) \xrightarrow{d_Z} g(y) \text{ as } k \to \infty.$$

This means that equation (1) holds, because $g(y_k) = (g \circ f)(a_k)$ and $g(y) = (g \circ f)(a)$.

Thus $g \circ f$ is continuous at the arbitrary point $a \in X$, and so we conclude that $g \circ f$ is continuous on X. ■

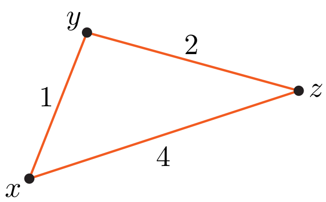
Exercise 3.1

Let (X, d) and (Y, e) be metric spaces, and let $b \in Y$ be fixed. Suppose that $f\colon X \to Y$ is (d, e)-continuous on X.

Use the Reverse Triangle Inequality to prove that $g\colon X \to \mathbb{R}$ given by $g(x) = e(f(x), b)$ is $(d, d^{(1)})$-continuous on X.

Solutions and comments on exercises

Solution to Exercise 1.1



The function d is not a metric. It does not satisfy (M3), the Triangle Inequality, since (as shown)

$$d(x,y)+d(y,z)=1+2=3<4=d(x,z).$$

Solution to Exercise 1.2

(a) $e_1((0,0),(1,0))=|1-0|+|0-0|=1$

(b) $e_1((0,0),(0,1))=|0-0|+|1-0|=1$

(c) $e_1((0,1),(1,0))=|1-0|+|0-1|=2$

Solution to Exercise 1.3

(a) This is not a metric on $\mathbb{R}$, since there are a and b in $\mathbb{R}$ for which $d(a,b)$ is not real-valued. For example, taking $a=1$ and $b=0$ gives $d(a,b)=\sqrt{-1}=i$, an imaginary number. Alternatively, $1\neq -1$, but $d(1,-1)=0$ and so (M1) does not hold.

(b) This is not a metric on $\mathbb{R}$, since (M1) does not hold. For example, if $a=b=1$, then

$$d(a,a)=\sqrt{1+1}=\sqrt{2}\neq 0.$$

(c) This is a metric on $\mathbb{R}$. We verify that conditions (M1), (M2) and (M3) hold.

(M1) Note first that $d(a,b)\geq 0$ for each $a,b\in\mathbb{R}$. Also,

$$d(a,a)=\sqrt[3]{|a^3-a^3|}=0,$$

and $d(a,b)=0$ implies that $|b^3-a^3|=0$ – that is, $a=b$.

(M2) Note that, for each $a,b\in\mathbb{R}$,

$$d(a,b)=\sqrt[3]{|b^3-a^3|}=\sqrt[3]{|a^3-b^3|}=d(b,a).$$

(M3) For each $a,b,c\in\mathbb{R}$, we must show that

$$d(a,c)\leq d(a,b)+d(b,c).$$

Since $d(a,c)$, $d(a,b)$ and $d(b,c)$ are all non-negative, it is enough to show that

$$d(a,c)^3\leq (d(a,b)+d(b,c))^3.$$

Now the Triangle Inequality for the modulus function gives

$$\begin{aligned}d(a,c)^3&=|c^3-a^3|=|c^3-b^3+b^3-a^3|\\&\leq |c^3-b^3|+|b^3-a^3|=d(b,c)^3+d(a,b)^3.\end{aligned}$$

Since $d(b,c)\geq 0$ and $d(a,b)\geq 0$, we have

$$d(b,c)^3+d(a,b)^3\leq (d(b,c)+d(a,b))^3,$$

and so

$$d(a,c)^3 \leq (d(b,c) + d(a,b))^3 = (d(a,b) + d(b,c))^3,$$

as required.

Hence (M1)–(M3) hold and we conclude that d is a metric for $\mathbb{R}$.

Solution to Exercise 1.4

It is certainly the case that $d(\mathbf{a},\mathbf{b})$ is always non-negative and for each $\mathbf{a} \in \mathbb{R}^2$, $d(\mathbf{a},\mathbf{a}) = 0$.

Now suppose that $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$ with $d(\mathbf{a},\mathbf{b}) = 0$. We must show that $\mathbf{a} = \mathbf{b}$.

If $d(\mathbf{a},\mathbf{b}) = 0$, then $|b_1 - a_1| + d_0(a_2, b_2) = 0$, which, since both $|\cdot|$ and d_0 are always non-negative, implies

$$|b_1 - a_1| = 0 \text{ and } d_0(a_2, b_2) = 0.$$

Hence, since $|\cdot|$ and d_0 are metrics on $\mathbb{R}$ (and so satisfy (M1)), both $a_1 = b_1$ and $a_2 = b_2$. Thus $\mathbf{a} = \mathbf{b}$ as required.

Solution to Exercise 1.5

We need to show that for any choice of $\mathbf{a} = (a_1, a_2), \mathbf{b} = (b_1, b_2) \in \mathbb{R}^2$, we have $d(\mathbf{a},\mathbf{b}) = d(\mathbf{b},\mathbf{a})$. This follows directly from the corresponding properties of $|\cdot|$ and d_0 when considered as metrics on $\mathbb{R}$:

$$\begin{aligned} d(\mathbf{a},\mathbf{b}) &= |b_1 - a_1| + d_0(a_2, b_2) \\ &= |a_1 - b_1| + d_0(b_2, a_2), \text{ by (M2) for } |\cdot| \text{ and } d_0 \\ &= d(\mathbf{b},\mathbf{a}), \end{aligned}$$

as required.

Solution to Exercise 1.6

Let $a \in X$ and suppose that $r \geq 0$.

Since $B_{d_0}[a,r] = \{x \in X : d_0(a,x) \leq r\}$ and $d_0(a,x) = 1$ unless $a = x$ (when it is 0), we conclude that

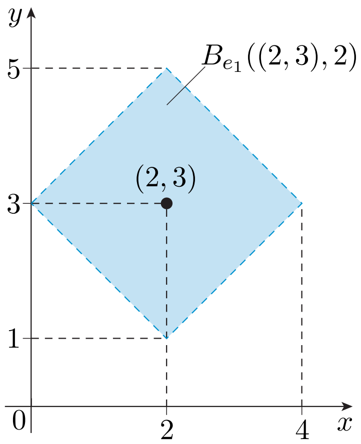
$$B_{d_0}[a,r] = \begin{cases} \{a\}, & \text{if } 0 \leq r < 1, \\ X, & \text{if } r \geq 1. \end{cases}$$

Solution to Exercise 1.7

Let $a \in X$ and suppose that $r > 0$.

Since $S_{d_0}(a,r) = \{x \in X : d_0(a,x) = r\}$ and $d_0(a,x) = 1$ unless $a = x$ (when it is 0), we conclude that

$$S_{d_0}(a,r) = \begin{cases} \{a\}, & \text{if } r = 0, \\ X - \{a\}, & \text{if } r = 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$



Solution to Exercise 1.8

The exercise is to find all $\mathbf{x}$ satisfying

$$e_1((2,3),\mathbf{x}) = |x_1 - 2| + |x_2 - 3| < 2.$$

The point $(2,3)$ is the centre of the ball, and its radius is 2. If we imagine the origin of coordinates at $(2,3)$ and put the vertices of the diamond two units from the centre along the axes, the required figure results (as shown).

Solution to Exercise 1.9

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$. If $r = 0$ then by Worked Exercise 1.7, $B_d[\mathbf{a}, r] = \{\mathbf{a}\}$.

Now suppose that $r > 0$. Then

$$\begin{aligned} B_d[\mathbf{a}, r] &= \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : d(\mathbf{a}, \mathbf{x}) \le r\} \\ &= \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_1 - a_1| + d_0(a_2, x_2) \le r\}. \end{aligned}$$

If $x_2 = a_2$, then $d_0(a_2, x_2) = 0$ and

$$\{\mathbf{x} = (x_1, a_2) \in \mathbb{R}^2 : |x_1 - a_1| \le r\} = [a_1 - r, a_1 + r] \times \{a_2\}.$$

If $x_2 \ne a_2$, then $d_0(a_2, x_2) = 1$ and

$$\begin{aligned} &\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 \ne a_2 \text{ and } |x_1 - a_1| + d_0(a_2, x_2) \le r\} \\ &= \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 \ne a_2 \text{ and } |x_1 - a_1| \le r - 1\} \\ &= \begin{cases} \emptyset, & \text{if } r < 1, \\ [a_1 - (r-1), a_1 + (r-1)] \times (\mathbb{R} - \{a_2\}), & \text{if } r \ge 1. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} &B_d[\mathbf{a}, r] \\ &= \begin{cases} [a_1 - r, a_1 + r] \times \{a_2\}, & \text{if } r < 1, \\ ([a_1 - r, a_1 + r] \times \{a_2\}) & \\ \qquad \cup ([a_1 - (r-1), a_1 + (r-1)] \times \mathbb{R}), & \text{if } r \ge 1. \end{cases} \end{aligned}$$

Solution to Exercise 2.1

To show that $(\mathbf{a}_k)$ converges to $(1,2)$ for e_1, we must show that the sequence of real numbers $(e_1(\mathbf{a}_k, (1,2)))$ is null.

We calculate

$$\begin{aligned} e_1(\mathbf{a}_k, (1,2)) &= e_1((1 + \tfrac{1}{2k}, 2 - \tfrac{1}{2k}), (1,2)) \\ &= \left|1 - (1 + \tfrac{1}{2k})\right| + \left|2 - (2 - \tfrac{1}{2k})\right| = \tfrac{1}{2k} + \tfrac{1}{2k} = \tfrac{1}{k}. \end{aligned}$$

Since $\left(\frac{1}{k}\right)$ is a basic null sequence, we conclude that $\mathbf{a}_k \to (1,2)$ with respect to e_1.

Solution to Exercise 2.2

Let (a_k) be a sequence in X and suppose that it converges (in d_0) to $a \in X$.

Then, by the definition of convergence, it must be the case that $(d_0(a_k, a))$ is a real null sequence and so, in particular, there is $N \in \mathbb{N}$ such that for $k > N$, $|d_0(a_k, a)| < 1$.

But $d_0(a_k, a)$ can only equal 1 or 0, so this means that there must be $N \in \mathbb{N}$ such that for $k > N$, $d_0(a_k, a) = 0$. Since d_0 is a metric on X, property (M1) tells us that for $k > N$, $a_k = a$. In other words, the sequence (a_k) is eventually constant, as required.

Solution to Exercise 3.1

We take a general point $a \in X$ and prove that g is $(d, d^{(1)})$-continuous at a. To do this, we take a sequence (x_k) in X with $x_k \overset{d}{\to} a$ as $k \to \infty$ and show that $g(x_k) \overset{d^{(1)}}{\to} g(a)$. This means we must show that

$$d^{(1)}(g(x_k), g(a)) \to 0 \quad \text{as } k \to \infty,$$

and so we find an upper bound for this quantity.

Using the definition of g, and the Reverse Triangle Inequality for e, we find that

$$\begin{aligned} d^{(1)}(g(x_k), g(a)) &= |g(x_k) - g(a)| \\ &= |e(f(x_k), b) - e(f(a), b)| \\ &\leq e(f(x_k), f(a)). \end{aligned} \tag{2}$$

Since f is (d, e)-continuous on X and $x_k \overset{d}{\to} a$ as $k \to \infty$, we have that

$$f(x_k) \overset{e}{\to} f(a).$$

So $(e(f(x_k), f(a)))$ is a null sequence. Hence, applying the Squeeze Rule to (2) shows that $\left(d^{(1)}(g(x_k), g(a))\right)$ is also a null sequence. That is, $(g(x_k))$ converges to $g(a)$ for $d^{(1)}$. Thus g is $(d, d^{(1)})$-continuous at a.

Since the choice of a is arbitrary, g is $(d, d^{(1)})$-continuous on X.

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