MST326 Mathematical methods and fluid mechanics

Unit 5

Kinematics of fluids

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Contents

Study guide

This unit, which starts Block 2 of the course, depends significantly on the vector calculus that was covered in *Unit 4*, and to some extent on the solution of first-order partial differential equations and a version of the Chain Rule from *Unit 3*.

The unit is relatively long, though this will be balanced by a shorter *Unit 6*. The first four sections here will probably take roughly equal times to complete, with Section 5 requiring a bit less. This last section departs from the general theme of the rest of the unit, but underpins *Unit 6* and much of the rest of the block.

There is a multimedia session associated with Subsection 3.2, for which you are referred to the *Media Guide*. There is no audio activity associated with this unit.

Introduction

Unit 1 introduced some of the basic properties of fluids, such as the physical ideas of viscosity and compressibility, and some of the measurable quantities like flow velocity, density and pressure. It also introduced the fundamental idea of the continuum model. Although in reality the fluid is composed of discrete molecules, the model is a continuous one in which, for instance, the density is defined at each point in a fluid. Some of the basic ideas were then used to solve problems involving fluids at rest.

Now we start to investigate fluids in motion. In this unit and in *Unit 6*, the fluid under investigation is assumed to be inviscid. All real fluids possess viscosity, and it is important to be aware of this fact in interpreting every solution obtained. However, the inviscid model does provide much insight into the actual behaviour of fluids in many cases. We noted in *Unit 1* that the flow of a real fluid past an object can be considered in two regions, one adjacent to the boundary, where the viscosity has a considerable effect (the boundary layer), and an outer region where the viscous effects are negligible. Thus *Units 5* and *6* relate principally to the outer region. *Units 7* and *8* consider a more complicated model which includes the effects of viscosity.

The strategy for solving fluid flow problems is often different from the familiar approach in solid mechanics. There, we start with a particle as a model of a body in motion subject to certain forces. These forces are modelled by vectors, and using the laws of vector algebra we obtain the total force on the particle. Newton's Second Law then provides a relationship between the acceleration of the particle and the total force acting on it, and this relationship is used to find the velocity and position of the particle at any instant of time, given the position and velocity at one instant. So the problem-solving strategy in particle mechanics may be summarised as in Figure 0.1.

There are some occasions when we go in the opposite direction — from the position or velocity to the total force.

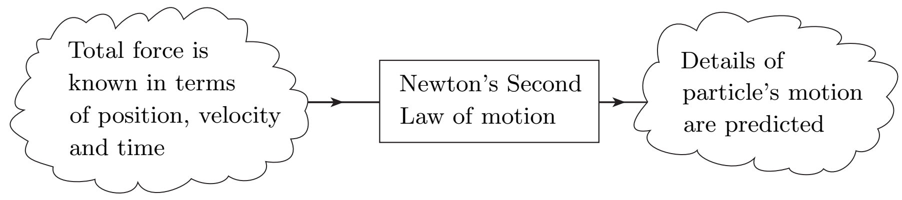


Figure 0.1

Many of the problems to be solved in fluid mechanics require a different approach, so it is important to identify at the outset the way in which to approach the subject. In a fluid flow problem, there are two main aims:

- to find the velocity field describing the velocity of the fluid at each of the points in the region occupied by the fluid, and
- to find the total force on any solid boundary in contact with the fluid.

In the *inviscid* case, there are two equations that enable us to achieve these aims. The *continuity equation* places a constraint on the velocity field; then the appropriate version of Newton's Second Law (called *Euler's equation* for the inviscid fluid model) provides both details of the velocity and also the pressure distribution in the fluid. The effect on any solid boundaries in the fluid can then be expressed in terms of the net surface force, which is obtained by integrating the pressure distribution over the boundary surface. This approach (for an inviscid fluid) is summarised in Figure 0.2.

The continuity equation was introduced in *Unit 4* Subsection 3.3. Euler's equation is derived in Section 5 of this unit.

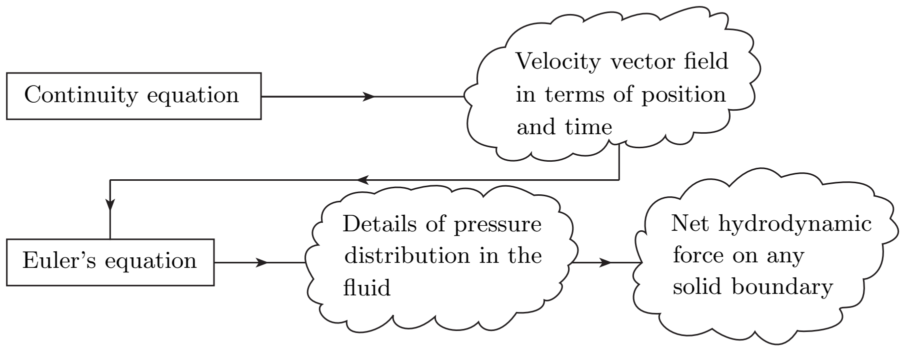


Figure 0.2

The initial and boundary conditions, not indicated in Figure 0.2, place additional constraints on the velocity field.

The last two steps in this problem-solving strategy were put to use in *Unit 1*, when evaluating the net force on a flat plate immersed in a static liquid. Since the liquid was at rest, the velocity field was zero, so the pressure distribution could be found without the use of Euler's equation (i.e. Newton's Second Law for an inviscid fluid).

You will see in Section 5 that the basic equation of fluid statics is just a special case of Euler's equation.

In general, the flow of a fluid is three-dimensional and unsteady. Mathematically, this means that the flow parameters, such as velocity and density, may depend on all three space coordinates and time; for example, in Cartesian coordinates, the velocity $\mathbf{u}$ may depend on x, y, z and t.

However, there are many physical situations in which, owing to the symmetry of the problem, the flow parameters depend on only one or two space coordinates. A flow is said to be *two-dimensional* if the flow parameters depend on only two space coordinates. Using Cartesian coordinates (x, y and z) or cylindrical polar coordinates (r, θ and z), the flow pattern for such a flow is the same in any plane $z =$ constant. The flow of water past a long circular cylinder is an example of a flow that can be modelled as two-dimensional. In practice, of course, all bodies and fluid regions are finite, so that the two-dimensional assumption applies only to that part of the cylinder where we can ignore 'end-effects'.

In Sections 1–3, all of the velocity fields to be considered are two-dimensional. To specify such a flow, we require:

- a plane $z = c$ (a constant) such that at no point on the plane is there a velocity component perpendicular to the plane, and
- the flow pattern to be identical in all planes parallel to the plane $z = c$.

Usually, we choose the plane $z = 0$ as a reference plane and the Cartesian coordinates x and y, or plane polar coordinates r and θ, as convenient space coordinates. This means that any solid boundaries in the flow must have a uniform cross-section which can be drawn as a curve in the plane $z = 0$. For example, the flow past a circular cylinder is represented diagrammatically as the flow past a circle. Also, all of the flow parameters will be functions of x, y and t or r, θ and t.

Sections 1–3 deal with the basic kinematics of two-dimensional fluid flows. Section 1 introduces the differential equations for *pathlines* and *streamlines*. Section 2 introduces a scalar field, called the *stream function*, which for an incompressible fluid provides an alternative method of modelling the flow and finding the streamlines. Sections 2 and 3 derive the stream functions for several simple two-dimensional flow types (the *uniform flow*, *source*, *doublet* and *vortex*), and suitable combinations of these are used to model more complicated flows.

Section 4 introduces the idea of *differentiation following the motion*, which is necessary for the development of *Euler's equation* in Section 5.

1 Pathlines and streamlines

Much of the fluid mechanics component of this course is concerned with finding the velocity vector fields for fluids in motion. With this in mind, we first investigate ways of visualising the velocity field. Clearly, when we predict a velocity field from mathematical theory, we shall require a means of validating the results by comparison with the actual flow under investigation. The visualisation of a real fluid flow is important for the modelling process, and many new modelling ideas owe their origin to some form of flow visualisation. Two methods of visualising fluid flows relate to the *pathlines* and *streamlines* of the flow. This section develops the mathematical equations for each of these.

You saw the use of pathlines and streamlines while studying *Unit 1*.

1.1 Visualisation of the flow field

In *Unit 1* Subsection 1.4, you saw some of the many methods that are used to give a pictorial representation of fluid flows. Such visualisations are often vital to the formulation of a mathematical model for the fluid flow. For example, the streamlines past an aerofoil at high incidence are quite different from those for the low-incidence case, and so we might expect that different mathematical models would be needed for the two cases.

See the corresponding *Media Guide* section.

Here 'incidence' refers to the angle at which the aerofoil faces the stream.

We now introduce the concepts of *pathline*, *streamline* and *streakline* on which flow visualisation methods are based.

Pathlines

Imagine that, at an instant of time, a particle of smoke is injected into a fluid (a gas) at a fixed point P, with the velocity of the fluid at P. It is reasonable to assume that this particle will move with the fluid, that is, it will have the velocity of the fluid at each point it occupies. Figure 1.1 shows the position of the smoke particle at several instants of time ($t = 0, 1, 2, 3$) for a two-dimensional flow.

For a liquid, a dye would replace smoke. Ideally the particle of smoke or dye has the same density as the fluid.

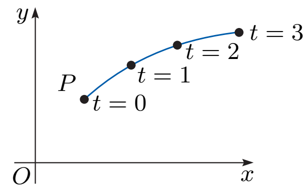


Figure 1.1

The set of all positions of the particle from $t = 0$ to $t = 3$ is a continuous path. A **pathline** is the path traced out by an individual fluid particle during a specified time interval.

If another particle is injected at P, at time $t = 1$ say, then, in general, the pathline for this particle will differ from that for the original particle. However, if the flow is *steady* (that is, at each point all conditions are independent of time), the pathline (for $t = 1$ to $t = 4$) of the second particle will be the same as that of the first particle (for $t = 0$ to $t = 3$); in fact, for steady flow, every particle passing through P has the same pathline.

Steady flow was defined in *Unit 1* Subsection 1.4.

By injecting smoke particles at several points at the same time, we can obtain a pathline visualisation of a flow.

Subsection 1.2 explains how to obtain the equations of pathlines of a flow for which the velocity field $\mathbf{u}$ is given.

Streamlines

Imagine that aluminium powder is scattered evenly over the region of interest in a particular flow, and that a photograph is taken with an exposure just long enough for the particles of powder to have made short trails. Each trail will be in the direction of the velocity of the particle (see Figure 1.2), and so the photograph represents the directions of the whole flow (the velocity direction field) at a given instant of time. We can sketch on the photograph the trajectories of this direction field by drawing smooth curves which have the trails as tangents. Since the trails are everywhere parallel to the velocity vectors, the trajectories of the direction field are the field lines of the velocity vector field.

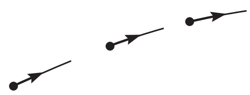


Figure 1.2 Trails and velocity vectors

Direction fields are discussed in MST209 *Unit 2*, and vector fields (including field lines) are discussed in MST209 *Unit 23*.

In fluid mechanics, the field lines of the velocity vector field are called *streamlines*. In more basic terms, a **streamline** at an instant of time is a curve such that, at each point along the curve, the tangent vector to the curve is parallel to the fluid velocity vector.

Streamlines are sometimes called *flowlines* or *lines of flow*.

In general, if we take another photograph of the particle trails at some later time, the streamline pattern revealed will be different from that obtained before. At each instant of time there is a corresponding streamline pattern. Figure 1.3 shows a set of streamlines at one instant of time, for flow past a plate that oscillates back and forth about its central axis (directed into the page). This pattern alters as the plate moves.

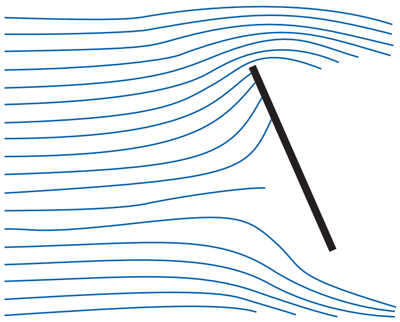


Figure 1.3 Streamlines at one instant for oscillating plate flow

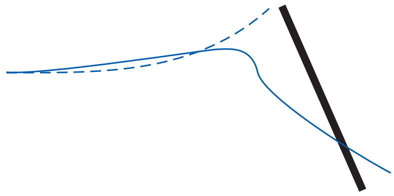


Figure 1.4 Comparison of (dashed) streamline and (solid) pathline

Figure 1.4 shows a streamline and a pathline through the same point for the oscillating plate flow; note that they are different curves, because this is an unsteady flow. If the flow is steady, then the streamlines do not change with time and, moreover, the streamline through a point P is the same *geometrically* as any pathline through P. In some cases, the pathlines and streamlines can coincide even if the flow is not steady.

The equations of streamlines are considered in Subsection 1.3, and the above remarks concerning steady flow are illustrated.

Streaklines

Unit 1 explained how *streaklines* were created by the continuous injection of dye into a fluid at a particular point. Streaklines differ in general both from streamlines (since the instantaneous fluid velocity vector may not be tangent to a streakline at a given time) and from pathlines (since a continuous stream of particles is involved, rather than the path of a single particle). However, for a steady flow the pathlines, streamlines and streaklines all coincide.

A visual comparison between pathlines, streamlines and streaklines, which refers to the oscillating plate of Figures 1.3 and 1.4, can be found via the *Media Guide*.

If the pathline of a particle, starting from a fixed point P at time τ, is given by $\mathbf{r}(t,\tau)$ for $t \geq \tau$, this position function can be regarded as varying with the starting time τ as well as with t. For each fixed value of τ, the function describes a pathline, with variable t. For each fixed value of t, the function describes a streakline, with variable τ.

We shall not pursue further the mathematical equations for streaklines, concentrating instead on pathlines and streamlines.

1.2 Pathlines

Motion of a single particle

To illustrate the idea of pathlines, consider the motion of a shot in athletics. The set of points through which the shot passes is the *path* of the shot.

Projectile motion is discussed in MST209 *Unit 14*.

To find this path, we start by deriving the equation of motion. Assuming that there is no air resistance, and that the only force acting on the shot is its weight, $\mathbf{W}$, the force diagram and choice of axes are as shown in Figure 1.5. The equation of motion is

$$m\mathbf{a} = m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{W} = -mg\,\mathbf{j}, \qquad \text{so that} \qquad \frac{d^2\mathbf{r}}{dt^2} = -g\,\mathbf{j}, \tag{1.1}$$

where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ is the position vector of the shot at any time, $\mathbf{a}$ is its acceleration, m is the mass of the shot and g is the magnitude of the acceleration due to gravity. Suppose that the shot is projected from height h above the horizontal ground with speed v_0 and at an angle θ_0 above the horizontal. Then the initial conditions are

$$\mathbf{r}(0) = h\,\mathbf{j}, \qquad \frac{d\mathbf{r}}{dt}(0) = v_0\cos\theta_0\,\mathbf{i} + v_0\sin\theta_0\,\mathbf{j}.$$

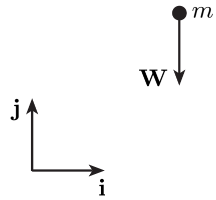


Figure 1.5 The forces acting on the shot

Integrating Equation (1.1), and substituting in the second initial condition, gives

$$\frac{d\mathbf{r}}{dt} = v_0\cos\theta_0\,\mathbf{i} + (-gt + v_0\sin\theta_0)\,\mathbf{j} = \mathbf{v},$$

where $\mathbf{v}$ is the velocity vector at time t. Integrating this equation, and substituting in the first initial condition, gives

$$\mathbf{r} = v_0\cos\theta_0\,t\,\mathbf{i} + (-\tfrac{1}{2}gt^2 + v_0\sin\theta_0\,t + h)\,\mathbf{j},$$

from which the x- and y-coordinates of the position of the shot at any time t are obtained as

$$x = v_0\cos\theta_0\,t \qquad \text{and} \qquad y = -\tfrac{1}{2}gt^2 + v_0\sin\theta_0\,t + h. \tag{1.2}$$

The time t provides a natural parameter for the path of the shot. The shape of the path is found by eliminating t between the two equations (1.2), to obtain

$$y = -\frac{g}{2v_0^2\cos^2\theta_0}\,x^2 + \tan\theta_0\,x + h,$$

which is the equation of a parabola (see Figure 1.6). This curve is the path along which the shot travels before hitting the ground; it is a pictorial record of the shot's motion.

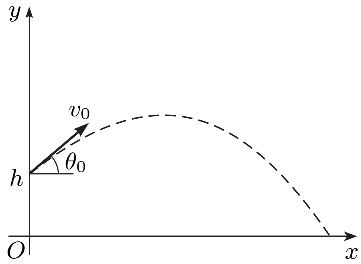


Figure 1.6 Path of a shot

Motion of many particles in a fluid

For a fluid in motion, there are infinitely many fluid particles that we could choose to follow, so there will be infinitely many pathlines. Suppose that we take one particular fluid particle that travels along a path through the point Q, shown in Figure 1.7. If the position vector of the fluid particle when at Q is $\mathbf{r}$, then its velocity $\mathbf{v}$ is given in terms of $\mathbf{r}$ by

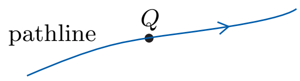


Figure 1.7

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}.$$

The velocity $\mathbf{v}$ of the fluid particle when at Q is the same as the velocity vector field $\mathbf{u}$ of the fluid at Q. Thus

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}. \tag{1.3}$$

In general, $\mathbf{u}$ depends on position and time, so that this vector equation represents coupled scalar differential equations linking x, y and t. The following example shows how the equations of the pathlines for a flow are found from this vector equation when $\mathbf{u}$ is known.

This vector equation also applies for three-dimensional flow problems, but here we deal with the two-dimensional case.

Example 1.1

Find the equations of the pathlines for a fluid flow with velocity field

$$\mathbf{u} = ay\,\mathbf{i} + bt\,\mathbf{j}, \qquad \text{where } a, b \text{ are positive constants.}$$

Sketch the pathlines of the fluid particles which pass through the points $(X, 0)$ at time $t = 0$, for $X = -1, 0, 1, 2, 3$.

Solution

The vector differential equation (1.3) for a pathline is

$$\frac{d\mathbf{r}}{dt} = \mathbf{u} = ay\,\mathbf{i} + bt\,\mathbf{j},$$

and since by definition

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j},$$

See MST209 *Unit 6* for the definition of the derivative of a vector function.

we have

$$\frac{dx}{dt} = ay, \qquad \frac{dy}{dt} = bt.$$

The second equation integrates to give

$$y = \tfrac{1}{2}bt^2 + C, \qquad \text{where } C \text{ is an arbitrary constant.}$$

Then from the first equation, we have

$$\frac{dx}{dt} = ay = \tfrac{1}{2}abt^2 + aC.$$

This integrates to give

$$x = \tfrac{1}{6}abt^3 + aCt + D, \qquad \text{where } D \text{ is an arbitrary constant.}$$

We now have $x = x(t)$ and $y = y(t)$, with t as a parameter. We can, in principle, eliminate t to obtain an equation relating y directly to x. This is done for the specific case below.

These equations for x and y in terms of t represent infinitely many curves in the region of fluid. For each pair of values for C and D we obtain a different pathline, depending on the fluid particle we choose to follow. The pathline of the particle which passes through the point $(X, 0)$ at time $t = 0$ must be such that $x = X$ and $y = 0$ at $t = 0$, that is,

$$y(0) = C = 0 \qquad \text{and} \qquad x(0) = D = X.$$

The pathline equations for this particle are then

$$x = \tfrac{1}{6}abt^3 + X, \qquad y = \tfrac{1}{2}bt^2.$$

These equations define the pathline for any specified time interval.

We can eliminate the parameter t from these two equations to obtain an explicit equation for the pathline:

$$(x - X)^2 = \frac{2a^2}{9b}y^3.$$

The pathlines corresponding to $X = -1, 0, 1, 2, 3$ are shown in Figure 1.8 (in which, for convenience, we have taken $b/a^2 = 16/9$).

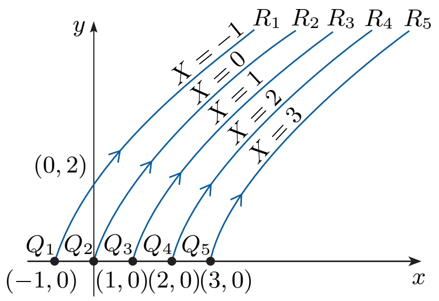


Figure 1.8

With $b/a^2 = 16/9$, the pathline Q_1R_1 crosses the y-axis at $y = 2$, as indicated. More generally, Q_1R_1 crosses the y-axis at $y = (9b/2a^2)^{1/3}$.

The direction of flow, shown by arrowheads, can be deduced from the sign of either component of $\mathbf{u}$. For example, $u_1 = ay$ is positive in the upper half-plane, where $y > 0$, so the flow is from left to right. Only the flow for $t \geq 0$ is shown here.

Note that the particle which travels along the pathline Q_1R_1, for example, was at the point Q_1 when $t = 0$. Note also that the same scales should be used for the x- and y-axes, since we wish the pathlines to be a visual representation of the real flow. ■

Exercise 1.1

Find the equations of pathlines for the two-dimensional flow with $\mathbf{u} = U\,\mathbf{i}$, where U is a positive constant. Sketch some of these pathlines.

The next example involves the use of plane polar coordinates, r and θ.

The ranges for r and θ are

$$r \geq 0, \quad -\pi < \theta \leq \pi.$$

The unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ for plane polar coordinates are the same as those for cylindrical polar coordinates in *Unit 4* Subsection 1.1. They were defined for the plane polar case in MST209 *Unit 20* and used in MST209 *Units 27* and *28*.

Example 1.2

Find the equations of pathlines for a fluid flow with velocity field

$$\mathbf{u} = \frac{m}{r}\,\mathbf{e}_r \qquad (r \neq 0),$$

where m is a positive constant. Find the pathline of the particle which passes through the point $r = 1$, $\theta = \frac{1}{4}\pi$ at time $t = 0$.

Does the particle speed up or slow down as time passes?

Solution

The polar form of the differential equation (1.3) for the pathlines, $d\mathbf{r}/dt = \mathbf{u}$, is

$$\frac{dr}{dt}\,\mathbf{e}_r + r\frac{d\theta}{dt}\,\mathbf{e}_\theta = \mathbf{u}.$$

The polar form of the velocity vector is given in MST209 *Unit 28* as

$$\dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta.$$

This is the derivative with respect to time t of the position vector, $\mathbf{r} = r\,\mathbf{e}_r$.

In this example $\mathbf{u} = (m/r)\,\mathbf{e}_r$, and hence

$$\frac{dr}{dt} = \frac{m}{r}, \qquad r\frac{d\theta}{dt} = 0 \qquad (r \neq 0).$$

It follows that

$$r\frac{dr}{dt} = m \qquad \text{and} \qquad \frac{d\theta}{dt} = 0,$$

which integrate to give

$$r^2 = 2mt + C, \qquad \theta = D, \qquad \text{where } C \text{ and } D \text{ are arbitrary constants.}$$

The pathline of the particle which passes through the point P_0 $(r = 1,\ \theta = \frac{1}{4}\pi)$ at time $t = 0$ has $C = 1$ and $D = \frac{1}{4}\pi$, so the pathline equation in this case is

$$r^2 = 2mt + 1, \qquad \theta = \tfrac{1}{4}\pi.$$

(Note that, since θ is constant on each pathline, the elimination of t between these two equations does not apply here.)

This is a ray $OP_0P_1P_2$, coming out from the origin O at the constant angle $\frac{1}{4}\pi$ (see Figure 1.9). Since r is an increasing function of t, or equivalently, since $dr/dt > 0$, the direction of flow along the pathline is outwards, as indicated by the arrowhead. (Note that O itself is excluded from the pathline, since $r \neq 0$, and that the motion occurs only for $t > -1/(2m)$.)

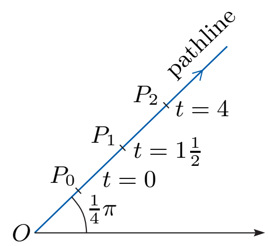


Figure 1.9

As time goes by, the particle slows down; after $1\frac{1}{2}$ seconds (with $m = 1$), it reaches P_1 $(r = 2)$, and after 4 seconds it reaches P_2 $(r = 3)$.

The full set of pathlines includes all equations of the form $\theta = D$ $(-\pi < D \leq \pi)$. Some of these rays starting from the origin O are shown in Figure 1.10, with O itself excluded in each case. ■

The velocity vector field in Example 1.2 is one of the basic models that can be used to describe many real fluid flow problems. Fluid particles emerge from the origin, O, and follow radial pathlines. We say that the flow field $\mathbf{u} = (m/r)\,\mathbf{e}_r$ represents a **source** of fluid at O. A source can be thought of as providing an injection of fluid along an axis perpendicular to the plane of the flow (and hence into the page in Figure 1.10). The fluid flows out equally in all directions perpendicular to this axis. We can interpret the constant m by considering the rate at which fluid crosses a circle of radius a (see Figure 1.11), representing a cylinder whose axis coincides with the line of the source. At any point on the circle, the speed of the fluid is m/a. The total volume of fluid flowing across the circumference of the circle in unit time and per unit depth into the paper is therefore

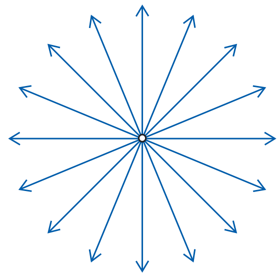


Figure 1.10

$$(2\pi a)\frac{m}{a} = 2\pi m.$$

Since the flow is steady, this volume of fluid must be provided in unit time by the inflow of fluid per unit length along the line of the source through O. So m is proportional to the rate at which fluid (by volume) is delivered into the flow region, per unit length of the source line, and $2\pi m$ (the volume flow rate across any circle containing O) is called the **strength** of the source.

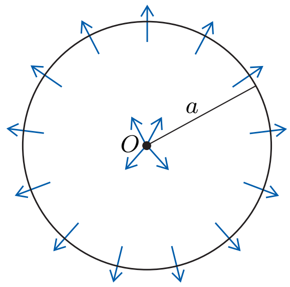


Figure 1.11

The velocity field

$$\mathbf{u} = -\frac{m}{r}\,\mathbf{e}_r \qquad (r \neq 0,\ m > 0)$$

is said to represent a **sink** of fluid at O. Its **strength** is also $2\pi m$. Both the source and the sink are basic flow patterns that will recur later in this unit.

The following procedure can be used for finding the equations of pathlines.

Procedure 1.1

To find pathlines for a two-dimensional flow, for which the velocity field $\mathbf{u}$ has been determined, proceed as follows.

(a) Write down the differential equation for the pathlines, either in vector form as

$$\frac{d\mathbf{r}}{dt} = \mathbf{u} \tag{1.3}$$

or in component form, with

$$\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} = \frac{dr}{dt}\,\mathbf{e}_r + r\frac{d\theta}{dt}\,\mathbf{e}_\theta. \tag{1.4}$$

(b) Solve these differential equations, if possible, to obtain the parametric equations

$$x = x(t), \quad y = y(t) \qquad \text{or} \qquad r = r(t), \quad \theta = \theta(t).$$

It may then be possible to eliminate t in order to obtain an equation relating x and y (or r and θ).

Integration of these equations can be achieved by any appropriate method.

The parametric equations in terms of t show a particle's progress along the pathline.

(c) Use a given initial point (if any) to determine the particular pathline required.

(d) If a sketch of pathlines is required, the direction of flow along each pathline (denoted by an arrowhead) can be found by considering the sign of a velocity component, dx/dt or dy/dt (or dr/dt or $d\theta/dt$), at any point on the pathline.

Use this procedure to solve the following exercise.

Exercise 1.2

Find the equations of pathlines for the following flows:

(a) a flow with velocity

$$\mathbf{u} = \frac{k}{r}\,\mathbf{e}_\theta \qquad (r \neq 0), \qquad \text{where } k \text{ is a positive constant;}$$

(b) a flow with velocity

$$\mathbf{u} = x(3t+1)\,\mathbf{i} + 2y\,\mathbf{j} \qquad (x > 0,\ y > 0).$$

Sketch some pathlines for part (a) only, and comment on how the fluid speed varies as r increases.

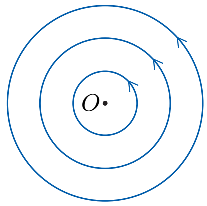


Figure 1.12

The talcum powder particles in the flow round the plughole, in the bath experiment of *Unit 1* Section 2, travel along almost circular paths. A vortex could be used to model the flow near the plughole.

The velocity vector field $\mathbf{u} = (k/r)\,\mathbf{e}_\theta$ in Exercise 1.2(a) is another of the basic flow patterns that will be used to model more complicated fluid flow problems. In contrast to the source, whose pathlines radiate from one point, the pathlines are circles centred on the origin, and fluid particles move along these circles (see Figure 1.12). The further the circle is from the origin, the lower is the speed. This type of flow is called a **vortex** of **strength** $2\pi k$, and is studied further in *Unit 7*. The flow along pathlines is anticlockwise if $k > 0$ (as in Exercise 1.2(a)) and clockwise if $k < 0$.

As with the source, a vortex should be thought of as extending along a line through O, perpendicular to the plane of the flow. The reason for the factor 2π in the definition of vortex strength will be seen in *Unit 7*.

1.3 Streamlines

Important properties of streamlines

Subsection 1.1 explained that the field lines of the velocity vector field at a given instant of time are called streamlines, and that the flow visualisation method associated with streamlines is to photograph particle trails over a short time interval. By 'joining up' the trails with smooth curves, the resulting curves give a good approximation to the streamlines. (Also, the relative lengths of the trails indicate the variation in magnitude of the velocity field.)

If the velocity field depends on time, then the streamline pattern will change each instant (because the direction of the velocity vector at each point changes). However, for steady flows (those which do not depend on time), the streamline pattern remains the same. In steady flow, the fluid particles travel along the streamlines so that the pathlines and streamlines then coincide.

This is illustrated in Example 1.4 below.

It is often useful to imagine streamlines being drawn in the real fluid, and we often speak of them as if they were. Further, for any streamline at an instant of time, at each point of it we can picture the fluid velocity vector as tangential to the curve. An important property of streamlines can be deduced from this characterisation. At any point, the flow is tangential to the local streamlines. Hence the following is also true.

At any instant of time, there is no fluid crossing any streamline.

This follows because, at any point on a streamline, the velocity vector is perpendicular to $\mathbf{n}$, a vector normal to the streamline. Hence $\mathbf{u} \cdot \mathbf{n} = 0$, and $\mathbf{u} \cdot \mathbf{n}$ provides a measure of the flow rate across a curve or surface in the fluid.

In *Unit 4* we derived the volume flow rate across a surface S as $\int_S \mathbf{u} \cdot \mathbf{n}\, dA$.

A second important property of streamlines is the following.

> At any instant of time, distinct streamlines cannot cross.

We establish this fact by using a contradiction argument. Suppose that two streamlines do cross at a point, A say. Then, since at each point of a streamline the velocity vector is parallel to the tangent to the streamline, the velocity vector at A has two different directions (see Figure 1.13), and so the velocity field is not uniquely defined. This contradiction establishes the result. (Nor can $\mathbf{u}$ physically have two directions at a point.)

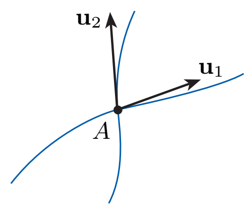


Figure 1.13

A streamline may *appear* to cross itself, or to divide into more than one branch, at a point where the velocity is zero; see Figure 2.15 on page 28. Such cases are not covered above.

Finding the streamline equations

In order to derive the streamline equations, we use the condition that at any instant of time the velocity vector $\mathbf{u}$ is parallel to the instantaneous streamline.

If P and Q are neighbouring points on the same streamline (see Figure 1.14), then the velocity vector $\mathbf{u}$ at P is approximately parallel to the chord PQ. As Q approaches P then, in the limit, the slope of the tangent to the streamline at P is equal to the slope of the velocity vector at P. If the streamline has equation $y = f(x)$ and $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$, then

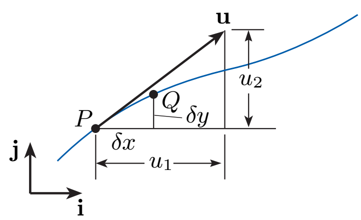


Figure 1.14

$$\frac{dy}{dx} = \frac{u_2}{u_1} \qquad (u_1 \neq 0). \tag{1.5}$$

Both sides of this equation represent the instantaneous slope of the streamline at P at time t.

Similar equations hold in plane polar coordinates (see Figure 1.15), for which $\mathbf{u} = u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta$; the 'slope' of the tangent to the streamline at P $(= r\,d\theta/dr)$ is equal to the 'slope' of the velocity vector at P $(= u_\theta/u_r)$, and so

In this case, 'slope' is referred to the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ at the point P.

$$r\frac{d\theta}{dr} = \frac{u_\theta}{u_r} \qquad (u_r \neq 0).$$

More conventionally, the polar coordinate form is written as

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{u_r}{u_\theta} \qquad (u_\theta \neq 0). \tag{1.6}$$

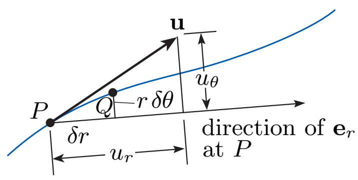


Figure 1.15

The streamline example below uses the flow field of Example 1.1.

Example 1.3

Find the streamlines through the point (x_0, y_0) for the following two-dimensional flow field:

$$\mathbf{u} = ay\,\mathbf{i} + bt\,\mathbf{j}, \qquad \text{where } a, b \text{ are positive constants.}$$

Solution

In this unsteady flow, Equation (1.5) defining the streamlines is

$$\frac{dy}{dx} = \frac{u_2}{u_1} = \frac{bt}{ay} \quad (y \neq 0), \qquad \text{or} \qquad ay\frac{dy}{dx} = bt.$$

In this differential equation, t is held constant, because a set of streamlines is defined at each instant of time. Integrating with respect to x, we obtain

$$\tfrac{1}{2}ay^2 = btx + C, \qquad \text{where } C \text{ is a constant.}$$

The point (x_0, y_0) lies on this streamline if

$$\tfrac{1}{2}ay_0^2 = btx_0 + C,$$

and so, by subtraction of the last equation from the one before,

$$\tfrac{1}{2}a(y^2 - y_0^2) = bt(x - x_0).$$

These streamlines are parabolas; for example, if $x_0 = y_0 = 0$ and $b/a = 2$, then the streamlines are given by $y^2 = 4tx$. The streamlines through $(x_0, 0)$ at time $t = 1$ are $y^2 = 4(x - x_0)$, some of which are shown in Figure 1.16.

These equations are quite different from those for the pathlines derived in Example 1.1 and shown in Figure 1.8. When the flow is unsteady, the pathlines and streamlines will normally be different, as in this case. ■

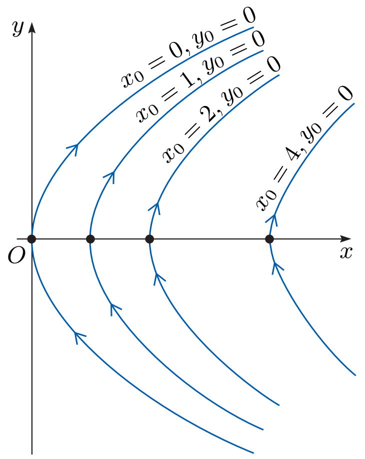


Figure 1.16 Streamlines at time $t = 1$ (with $b/a = 2$)

Note that the direction of flow has been shown on the streamlines in Figure 1.16. This may be found by considering a point (x, y) on a streamline. At this point, $\mathbf{u} = ay\,\mathbf{i} + bt\,\mathbf{j}$. Now $u_1 = ay$ has the same sign as y, and u_2 is positive. This information is shown in Figure 1.17, and justifies the direction of arrowheads shown in Figure 1.16. Note also that the same scale should be used on each axis in order that the streamlines represent the real flow.

For many simple flows, the direction of flow is obvious from other considerations.

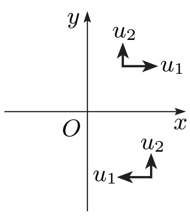


Figure 1.17

Three-dimensional direction fields can model fluid flows, and these also lead to streamlines, but we shall deal only with two-dimensional flows.

Exercise 1.3

Find the streamlines through the point (x_0, y_0) for the two-dimensional vector field

$$\mathbf{u} = U\cos\alpha\,\mathbf{i} + U\sin\alpha\,\mathbf{j}, \qquad \text{where } U, \alpha \text{ are constants.}$$

(This vector field represents a uniform flow at angle α to the x-axis.)

The next example requires use of the polar coordinate form of the streamline equations.

Example 1.4

Find the streamlines for the fluid flow with velocity field

$$\mathbf{u} = \frac{m}{r}\,\mathbf{e}_r \qquad (r \neq 0),$$

where m is a positive constant.

The pathlines for this flow were found in Example 1.2.

Solution

In this case of the flow due to a source, $\mathbf{u} = (m/r)\,\mathbf{e}_r$ implies that

$$u_r = \frac{m}{r} \qquad \text{and} \qquad u_\theta = 0.$$

We use the first form of the streamline equation,

$$r\frac{d\theta}{dr} = \frac{u_\theta}{u_r} = \frac{0}{m/r} = 0,$$

The second form of this equation, which is Equation (1.6), is not defined when $u_\theta = 0$.

which integrates to give $\theta =$ constant. Thus, the streamlines are rays from the origin, and are identical to the pathlines obtained in Example 1.2 (see Figure 1.10). This illustrates the fact that for steady flow, streamlines and pathlines are identical. ■

These examples suggest the following procedure for determining streamlines. A major point, which makes this an easier process than that for pathlines, is that for unsteady flows, t is taken as constant during the integration.

Procedure 1.2

To find streamlines for two-dimensional flows, for which the velocity field $\mathbf{u}$ has been determined, either as $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ or as $\mathbf{u} = u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta$, proceed as follows.

(a) Write down the differential equation for the streamlines either as

$$\frac{dy}{dx} = \frac{u_2}{u_1} \qquad (u_1 \neq 0) \tag{1.5}$$

or as

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{u_r}{u_\theta} \qquad (u_\theta \neq 0). \tag{1.6}$$

(b) Solve this differential equation, regarding t as a constant.

(c) Substitute one point, (x_0, y_0) say, which is on the streamline of interest, to obtain its equation.

(d) Sketch the graphs of several typical streamlines at a selected time t; this gives an instantaneous picture of the flow at time t. The direction of flow along each streamline (denoted by an arrowhead) can be found by considering the sign of a velocity component, u_1 or u_2 (or u_r or u_θ), at any point on the streamline for the given time.

The cases for which $u_1 = 0$ or $u_\theta = 0$ at all points are discussed below.

Each of Equations (1.5) and (1.6) can be used also in the 'upside down' form, that is,

$$\frac{dx}{dy} = \frac{u_1}{u_2} \qquad (u_2 \neq 0),$$

$$r\frac{d\theta}{dr} = \frac{u_\theta}{u_r} \qquad (u_r \neq 0).$$

If a velocity component is zero everywhere, then the equation of streamlines is easy to write down. For example, if $u_1 = 0$, then the velocity is parallel to the y-axis, and so the streamlines are given by $x = \text{constant}$. Similarly,

if $u_2 = 0$, the streamlines are given by $y = \text{constant}$;
if $u_r = 0$, the streamlines are given by $r = \text{constant}$;
if $u_\theta = 0$, the streamlines are given by $\theta = \text{constant}$.

This case arose in Example 1.4.

Exercise 1.4

Find the equations of the streamlines for the following flows:

(a) $\mathbf{u} = \dfrac{k}{r}\,\mathbf{e}_\theta \qquad (r \neq 0)$;

(b) $\mathbf{u} = x(3t+1)\,\mathbf{i} + 2y\,\mathbf{j} \qquad (x > 0,\ y > 0)$.

For part (b), sketch the streamlines which pass through the point $(1,1)$ for times $t = 0$, $t = \frac{1}{3}$ and $t = 1$.

At this stage, you might ask which of pathlines and streamlines are the more relevant. The answer is that, while pathlines have a role to play, streamlines are usually more relevant since they depict the flowlines of a region of fluid at a particular instant of time. This is what can most conveniently be photographed, and most pictures of fluid flows show the positions of coloured particles over a small interval of time.

For more on pathlines, streamlines and streaklines, see the *Media Guide*.

The flow in Exercise 1.4(a) is the vortex flow discussed after Exercise 1.2. You have seen that for a source and for a vortex (each of which is a steady flow), the streamlines and the pathlines coincide. These flow patterns, together with that for a uniform flow, are very important in attempts at modelling real fluid flows in Section 3 of this unit and in *Unit 7*. The results for these flows are summarised as follows.

Summary of results

(a) The velocity field of *a source of strength* $2\pi m$ at the origin is

$$\mathbf{u} = \frac{m}{r}\,\mathbf{e}_r \qquad (r \neq 0,\ m > 0),$$

and the streamline pattern is the set of all radial lines

$$\theta = \text{constant} \qquad \text{(see Figure 1.18)}.$$

The strength, $2\pi m$, is the volume rate of inflow of fluid into the region.

(b) The velocity field of *a sink of strength* $2\pi m$ at the origin is

$$\mathbf{u} = -\frac{m}{r}\,\mathbf{e}_r \qquad (r \neq 0,\ m > 0),$$

and the streamline pattern is the set of all radial lines

$$\theta = \text{constant}.$$

The strength, $2\pi m$, is the volume rate of outflow of fluid from the region.

(c) The velocity field of *a vortex of strength* $2\pi k$ at the origin is

$$\mathbf{u} = \frac{k}{r}\,\mathbf{e}_\theta \qquad (r \neq 0),$$

and the streamline pattern is the set of all circles

$$r = \text{constant} \qquad \text{(see Figure 1.19)}.$$

(d) The velocity field of a *uniform flow* of speed U at an angle α to the x-axis is

$$\mathbf{u} = U\cos\alpha\,\mathbf{i} + U\sin\alpha\,\mathbf{j},$$

and the streamline pattern is the set of all (parallel) lines making an angle α with the x-axis (see Figure 1.20).

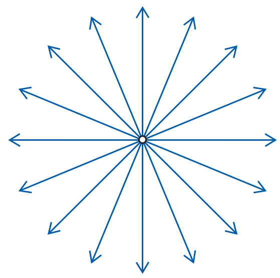


Figure 1.18 Source at origin

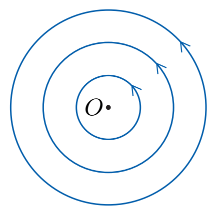


Figure 1.19 Vortex at origin

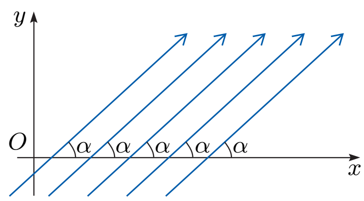


Figure 1.20 Uniform flow

End-of-section exercises

Exercise 1.5

Determine the pathline and the streamline which pass through the origin at $t = 0$ for the velocity field

$$\mathbf{u} = a\cos(\omega t)\,\mathbf{i} + a\sin(\omega t)\,\mathbf{j}, \qquad \text{where } a, \omega \text{ are positive constants.}$$

Exercise 1.6

Find the equations of the pathlines and the streamlines, in the form $r = f(\theta)$, for the velocity field

$$\mathbf{u} = r\cos(\tfrac{1}{2}\theta)\,\mathbf{e}_r + r\sin(\tfrac{1}{2}\theta)\,\mathbf{e}_\theta.$$

Hint: First find the streamlines.

2 The stream function

Section 1 showed that streamlines provide a pictorial method of representing the velocity field of a fluid flow.

Any velocity field $\mathbf{u}$ which represents a fluid flow must satisfy the continuity equation. Starting from this equation, we shall define a scalar field called the *stream function*. This scalar field specifies a fluid flow, and its contours are the streamlines of the flow.

As in Section 1, this section concerns two-dimensional flows (see page 5). The definition of the stream function depends on the flow being two-dimensional.

One advantage of working with the stream function rather than the velocity field is that we have to find one scalar function rather than two (u_1 and u_2). In this section we obtain the stream functions for certain basic flows, and Section 3 explains how to use combinations of these basic flows to model more complicated flows.

2.1 Introducing the stream function

We begin with the continuity equation for the two-dimensional flow of a constant-density fluid. In Cartesian coordinates, this equation is

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0,$$

This version of the continuity equation was derived in *Unit 4*, Exercise 3.8(a). The operator $\boldsymbol{\nabla}$ was introduced in *Unit 4* Subsection 5.1.

where $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$. If we introduce the scalar field $\psi(x, y)$, such that

$$u_1 = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad u_2 = -\frac{\partial \psi}{\partial x}, \tag{2.1}$$

then, for any such function ψ, the continuity equation is automatically satisfied, because

What is more, it can be shown that the equation $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ guarantees the existence of such a scalar field $\psi(x, y)$.

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{u} &= \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) \\
&= \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} \\
&= 0, \qquad \text{by the commutative property of partial differentation.}
\end{aligned}$$

This property is satisfied by all the functions ψ that we deal with. In MST209 *Unit 12*, this was called the Mixed Derivative Theorem.

The scalar field $\psi(x, y)$ is called the **stream function** of $\mathbf{u}$. The following example shows how we can find the stream function from the velocity vector field of a uniform flow, and that the contour lines for the scalar field ψ (that is, the curves $\psi =$ constant) are the streamlines for the velocity field.

Example 2.1

Find the stream function $\psi(x, y)$ for the uniform flow with velocity field

$$\mathbf{u} = U \cos\alpha\,\mathbf{i} + U \sin\alpha\,\mathbf{j}, \qquad \text{where } U, \alpha \text{ are constants.}$$

Show that the contour lines $\psi =$ constant are the streamlines of this flow, as found in Exercise 1.3 on page 15.

Solution

For the uniform flow, Equation (2.1) gives

$$u_1 = \frac{\partial \psi}{\partial y} = U \cos\alpha \qquad \text{and} \qquad u_2 = -\frac{\partial \psi}{\partial x} = U \sin\alpha.$$

Integrate the second equation to give

$$\psi(x,y) = -Ux\sin\alpha + f(y),$$

where f is an arbitrary function. Differentiating this equation with respect to y, we have

$$\begin{aligned}\frac{\partial\psi}{\partial y} &= 0 + f'(y)\\ &= U\cos\alpha,\end{aligned}$$

from the first equation. Integrating $f'(y) = U\cos\alpha$ gives

$$f(y) = Uy\cos\alpha + A,$$

where A is an arbitrary constant. Hence the stream function is

$$\psi(x,y) = -Ux\sin\alpha + Uy\cos\alpha + A.$$

Here we apply the method of *Unit 3* Subsection 1.2.

The lines $\psi =$ constant are defined by the equation

$$-Ux\sin\alpha + Uy\cos\alpha + A = \text{constant} = B, \text{ say.}$$

Write $(B-A)/(U\cos\alpha) = C$ (also a constant), to obtain

$$y - x\tan\alpha = C.$$

We assume that $\cos\alpha \neq 0$ here. If $\cos\alpha = 0$ then

$$\mathbf{u} = \pm U\,\mathbf{j}, \quad \psi = \mp Ux + A,$$

with contours $x =$ constant.

These equations are the same as those of the streamlines found in Exercise 1.3. They are parallel lines making an angle α with the x-axis. Figure 2.1 shows four of these streamlines. (The directions shown are for $U > 0$.) ■

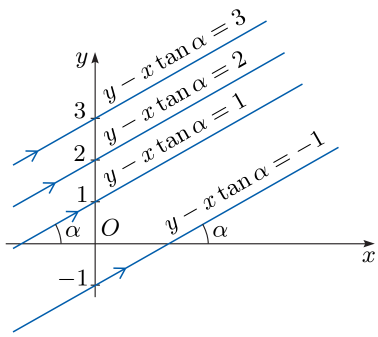


Figure 2.1

Exercise 2.1

Find the stream function $\psi(x,y)$ for the flow with velocity field

$$\mathbf{u} = ay\,\mathbf{i}, \qquad \text{where } a \text{ is a positive constant.}$$

Sketch several contour lines for $\psi(x,y)$.

This flow is called a *shear flow*, because the velocity component in the x-direction increases linearly with y, as shown in Figure 2.2.

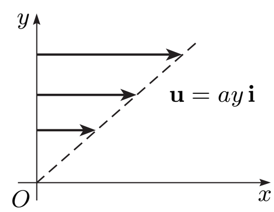


Figure 2.2

For the shear flow in Exercise 2.1, the velocity vector field is parallel to the x-axis, so that the streamlines are lines parallel to the x-axis. As in Example 2.1, the contour lines, $\psi =$ constant, are the streamlines. We now show that, for any velocity field $\mathbf{u}$, the streamlines are always the curves $\psi =$ constant, where $\psi(x,y)$ is the stream function of $\mathbf{u}$.

Figure 2.3 shows part of a streamline. Suppose that the streamline has parametric equations $x = x(s)$ and $y = y(s)$, for some parameter s. Then, if $\mathbf{r}$ is the position vector of any point P on the streamline, $d\mathbf{r}/ds$ is a tangent vector to the streamline. Since at every point of a streamline $\mathbf{u}$ is parallel to the tangent, we have

$$\mathbf{u} \times \frac{d\mathbf{r}}{ds} = \mathbf{0}.$$

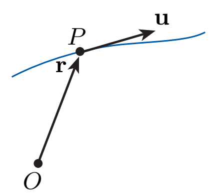


Figure 2.3

Now

$$\mathbf{u} = \frac{\partial \psi}{\partial y}\,\mathbf{i} - \frac{\partial \psi}{\partial x}\,\mathbf{j} \qquad \text{and} \qquad \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\,\mathbf{i} + \frac{dy}{ds}\,\mathbf{j},$$

so that

$$\mathbf{u} \times \frac{d\mathbf{r}}{ds} = \left(\frac{\partial \psi}{\partial y}\frac{dy}{ds} + \frac{\partial \psi}{\partial x}\frac{dx}{ds} \right) \mathbf{k} = \mathbf{0}.$$

Using the Chain Rule for a function of two variables,

$$\left(\frac{\partial \psi}{\partial y}\frac{dy}{ds} + \frac{\partial \psi}{\partial x}\frac{dx}{ds} \right) \mathbf{k} = \frac{d\psi}{ds}\,\mathbf{k} = \mathbf{0}.$$

This version of the Chain Rule was introduced in MST209 *Unit 12*, and put to use in *Units 3* and *4* of this course.

Therefore $d\psi/ds = 0$ along a streamline, and hence ψ is constant along a streamline.

Each streamline is therefore a contour line for the stream function $\psi(x, y)$. This argument applies at each instant of time, since both the streamline and the stream function may be time-dependent for an unsteady flow.

The stream function provides a second method of finding the equations of the streamlines, which is often more convenient than using Procedure 1.2 on page 16.

As you will see in Section 3, the stream function for a complicated flow can be found by adding together the stream functions for several basic flows. The velocity field and the streamlines for the more complicated flow are then obtained directly from this stream function.

The next example shows how the plane polar coordinate form for the stream function can be derived.

Example 2.2

Suppose that $\psi(x, y)$ is the stream function for the velocity field $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$. Show that:

(a) $\boldsymbol{\nabla}\psi$ is perpendicular to $\mathbf{u}$;

(b) $\mathbf{u} = \boldsymbol{\nabla}\psi \times \mathbf{k}$.

(c) Hence obtain the equations defining the stream function $\psi(r, \theta)$, where $\mathbf{u}$ is given in plane polar coordinates.

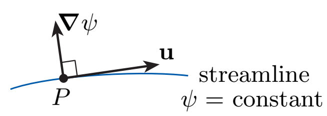


Figure 2.4

Solution

(a) Two vectors are perpendicular when their scalar product is zero and neither is the zero vector. From the defining equations for ψ, we have

$$u_1 = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad u_2 = -\frac{\partial \psi}{\partial x}.$$

Hence

$$\begin{aligned}(\boldsymbol{\nabla}\psi) \cdot \mathbf{u} &= \left(\frac{\partial \psi}{\partial x}\,\mathbf{i} + \frac{\partial \psi}{\partial y}\,\mathbf{j} \right) \cdot (u_1\,\mathbf{i} + u_2\,\mathbf{j}) \\ &= (-u_2\,\mathbf{i} + u_1\,\mathbf{j}) \cdot (u_1\,\mathbf{i} + u_2\,\mathbf{j}) = 0,\end{aligned}$$

and thus $\boldsymbol{\nabla}\psi$ and $\mathbf{u}$ are perpendicular.

An alternative argument is based on the contours for ψ. Since these are streamlines, at any point P on a contour line, the velocity vector $\mathbf{u}$ is parallel to the tangent. One property of the gradient vector $\boldsymbol{\nabla}\psi$ is that it is directed normal to a contour (see *Unit 4* Subsection 1.3). Hence $\boldsymbol{\nabla}\psi$ is perpendicular to $\mathbf{u}$ (see Figure 2.4 above).

(b) The vector product of $\nabla\psi$ with $\mathbf{k}$ is

$$\begin{aligned}\nabla\psi \times \mathbf{k} &= \left(\frac{\partial\psi}{\partial x}\,\mathbf{i} + \frac{\partial\psi}{\partial y}\,\mathbf{j}\right) \times \mathbf{k}\\ &= \frac{\partial\psi}{\partial y}\,\mathbf{i} - \frac{\partial\psi}{\partial x}\,\mathbf{j}\\ &= u_1\,\mathbf{i} + u_2\,\mathbf{j} = \mathbf{u}.\end{aligned}$$

(c) In plane polar coordinates,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\mathbf{e}_\theta,$$

and the equation from part (b), $\mathbf{u} = \nabla\psi \times \mathbf{k}$, becomes

$$\begin{aligned}\mathbf{u} &= u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta\\ &= \left(\frac{\partial\psi}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\mathbf{e}_\theta\right) \times \mathbf{k}\\ &= \frac{1}{r}\frac{\partial\psi}{\partial\theta}\,\mathbf{e}_r - \frac{\partial\psi}{\partial r}\,\mathbf{e}_\theta.\end{aligned}$$

Hence the equations

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad \text{and} \qquad u_\theta = -\frac{\partial\psi}{\partial r} \tag{2.2}$$

define the stream function $\psi(r,\theta)$ in polar coordinates. ■

This is the expression for $\mathbf{grad}\,\psi$ in cylindrical polar coordinates (see *Unit 4* Subsection 1.3), with $\partial\psi/\partial z = 0$ since the flow is two-dimensional.

The cross products between the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_z = \mathbf{k}$ were given in *Unit 4* Subsection 1.1.

The strategy of this solution is worth noting. Starting from a definition in Cartesian coordinates, we derive a vector result, $\mathbf{u} = \nabla\psi \times \mathbf{k}$, which holds in any coordinate system. Then we deduce from this the corresponding form of the definition in plane polar coordinates.

Example 2.3

Find the stream function and the equation of the streamlines for the flow with velocity vector field

$$\mathbf{u} = U\cos\theta\left(1 - \frac{a^2}{r^2}\right)\mathbf{e}_r - U\sin\theta\left(1 + \frac{a^2}{r^2}\right)\mathbf{e}_\theta.$$

Solution

Using Equations (2.2), we have

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = u_r = U\cos\theta\left(1 - \frac{a^2}{r^2}\right), \tag{2.3}$$

$$-\frac{\partial\psi}{\partial r} = u_\theta = -U\sin\theta\left(1 + \frac{a^2}{r^2}\right). \tag{2.4}$$

Integrating Equation (2.4) with respect to r gives

$$\psi(r,\theta) = U\sin\theta\left(r - \frac{a^2}{r}\right) + f(\theta),$$

where f is an arbitrary function. Substituting into Equation (2.3) gives

$$\frac{1}{r}\left[U\cos\theta\left(r - \frac{a^2}{r}\right)\right] + \frac{1}{r}f'(\theta) = U\cos\theta\left(1 - \frac{a^2}{r^2}\right).$$

Thus $f'(\theta) = 0$ and so $f(\theta) = C$, where C is an arbitrary constant. The stream function is then

$$\psi(r,\theta) = U\sin\theta\left(r - \frac{a^2}{r}\right) + C.$$

The streamlines are the lines of constant ψ, given by

$$U\sin\theta\left(r - \frac{a^2}{r}\right) = \text{constant}. \quad ■$$

Exercise 2.2

Find the stream functions for the following velocity fields:

(a) $\mathbf{u} = \dfrac{k}{r}\,\mathbf{e}_\theta$, where k is a constant (a vortex of strength $2\pi k$);

(b) $\mathbf{u} = \dfrac{m}{r}\,\mathbf{e}_r$, where m is a constant (a source of strength $2\pi m$).

The following procedure summarises the method for finding the stream function and the streamlines associated with it.

Procedure 2.1

To find the stream function and the associated streamlines, when the velocity field $\mathbf{u}$ has been determined either as $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ or as $\mathbf{u} = u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta$, proceed as follows.

(a) Solve the differential equations defining the stream function,

$$u_1 = \frac{\partial \psi}{\partial y} \quad \text{and} \quad u_2 = -\frac{\partial \psi}{\partial x} \tag{2.1}$$

or

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = -\frac{\partial \psi}{\partial r}, \tag{2.2}$$

to obtain $\psi(x, y)$ or $\psi(r, \theta)$.

(b) The streamlines are the contour lines of the stream function, ψ, and so are represented by the equation $\psi = \text{constant}$.

Exercise 2.3

Find the stream functions for the following velocity fields:

(a) $\mathbf{u} = Kr\,\mathbf{e}_\theta$, where K is a constant;

(b) $\mathbf{u} = y\,\mathbf{i} - (x - 2t)\,\mathbf{j}$.

Sketch some of the streamlines for times $t = 0$ and $t = 1$ in part (b).

For the flow in part (a), the fluid rotates around the origin as if it were a rigid body.

In each of the examples and exercises, the expression for the stream function contains an arbitrary constant, C. Without any loss of generality, this constant is usually chosen to be zero — for example, the stream function for Example 2.3 is then $\psi(r, \theta) = U \sin\theta\,\left(r - a^2/r\right)$ — and the streamlines have the same defining equation as before. Since the velocity components are given as derivatives of ψ, the value chosen for the constant C has no effect on the velocity vector field $\mathbf{u}$.

2.2 Physical interpretation of the stream function

The last subsection showed that one property of the stream function ψ is that the contour lines, $\psi =$ constant, are the same as the streamlines of the flow. We now go on to demonstrate another physical property of the stream function, namely, that the change in ψ between two points P and Q in the (x, y)-plane is equal to the volume flow rate across a surface of unit depth on *any* curve from P to Q; see Figure 2.5, in which we consider a two-dimensional flow parallel to the (x, y)-plane. The surface $PQRT$ has unit depth parallel to the z-axis, so that $PT = QR = 1$. In Figure 2.6, this surface is seen from above, and in the following analysis the volume flow rate through a curve PQ in the (x, y)-plane implies the volume flow rate through a surface S, perpendicular to the (x, y)-plane, through the curve PQ and of unit depth.

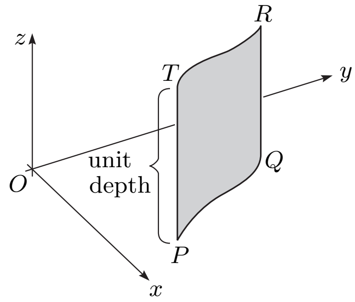


Figure 2.5

The volume flow rate (or flux) of fluid through the surface S is given by

See *Unit 4* Subsection 2.1.

$$V = \int_S \mathbf{u} \cdot \mathbf{n} \, dA,$$

where $\mathbf{n}$ is the unit vector normal to the curve PQ at an arbitrary point B, as shown in Figure 2.6. If the curve PQ is parametrised in terms of the distance s along it from P, then the area element can be taken as

$$\delta A = \delta s \times 1,$$

since the depth of the element is 1; see Figure 2.7. The surface integral for V then becomes

$$V = \int_P^Q \mathbf{u} \cdot \mathbf{n} \, ds.$$

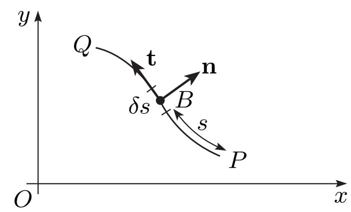


Figure 2.6

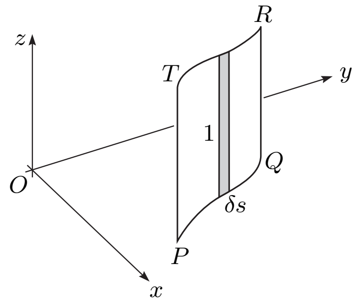


Figure 2.7

We can find an expression for the unit normal, $\mathbf{n}$, in terms of $\mathbf{r}$, the position vector of B, and the unit vector $\mathbf{k}$. Since s is a parameter representing distance along the curve from P, the unit tangent vector $\mathbf{t}$ to the curve PQ at the point B (see Figure 2.6) is

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\,\mathbf{i} + \frac{dy}{ds}\,\mathbf{j},$$

and the unit vector $\mathbf{n} = \mathbf{t} \times \mathbf{k}$ is normal to the curve and is given by

$$\mathbf{n} = \mathbf{t} \times \mathbf{k} = \left(\frac{dx}{ds}\,\mathbf{i} + \frac{dy}{ds}\,\mathbf{j}\right) \times \mathbf{k} = \frac{dy}{ds}\,\mathbf{i} - \frac{dx}{ds}\,\mathbf{j}.$$

Putting $\mathbf{u} = (\partial\psi/\partial y)\,\mathbf{i} - (\partial\psi/\partial x)\,\mathbf{j}$ gives

$$\begin{aligned} V &= \int_P^Q \mathbf{u} \cdot \mathbf{n} \, ds \\ &= \int_P^Q \left(\frac{\partial\psi}{\partial y}\frac{dy}{ds} + \frac{\partial\psi}{\partial x}\frac{dx}{ds}\right) ds \\ &= \int_P^Q \frac{d\psi}{ds} \, ds \\ &= \psi(Q) - \psi(P). \end{aligned}$$

By the Chain Rule,
$$\frac{\partial\psi}{\partial y}\frac{dy}{ds} + \frac{\partial\psi}{\partial x}\frac{dx}{ds} = \frac{d\psi}{ds}.$$

Thus V depends only on the values of ψ at P and Q, and not on the choice of curve PQ. This analysis has established the following result.

If $V > 0$, the net flow is positive from left to right across any curve from P to Q. If $V < 0$, the net flow is positive in the opposite direction.

> The change in value of the stream function ψ between any two points P, Q in the fluid is the volume flow rate of fluid through any curve from P to Q.

Exercise 2.4

Show that the volume flow rate across a streamline is zero. (This result confirms the first 'important property' of Subsection 1.3 that, at any instant of time, there is no fluid crossing any streamline.)

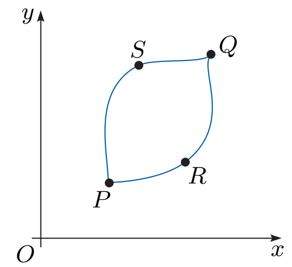


Figure 2.8

It follows from the highlighted result before Exercise 2.4 that the volume flow rates across any two curves joining P to Q are equal. This statement is equivalent to $\nabla \cdot \mathbf{u} = 0$, the continuity equation for a fluid of constant density. Consider the two paths PRQ and PSQ, shown in Figure 2.8. The volume flow rate across PRQ must equal the volume flow rate across PSQ, because $\nabla \cdot \mathbf{u} = 0$ implies that there can be no accumulation of fluid in the region enclosed by $PRQSP$. This provides an alternative form of expression for the continuity equation, which is derived now.

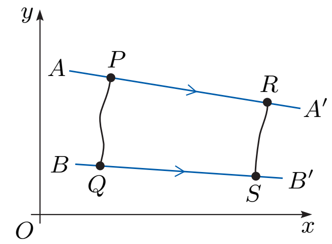


Figure 2.9

Figure 2.9 shows two streamlines, AA' and BB', in the flow of a constant-density fluid. Let P and Q be two points, on AA' and BB' respectively; then the volume flow rate crossing any curve joining Q to P is $\psi(P) - \psi(Q)$. Similarly, the volume flow rate crossing any curve joining S (on BB') to R (on AA') is $\psi(R) - \psi(S)$. Now, P and R lie on the same streamline AA', so that $\psi(P) = \psi(R)$, and similarly $\psi(Q) = \psi(S)$. Hence the quantity $\psi(P) - \psi(Q)$ is equal to $\psi(R) - \psi(S)$. This establishes the following result.

The volume flow rate between two streamlines is constant, and is independent of where it is measured.

There is further discussion of the continuity equation in Section 4.

For more about the stream function, see the *Media Guide*.

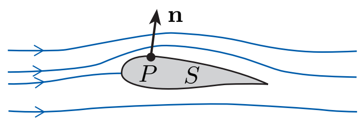
2.3 Flow past boundaries

Exercise 2.4 showed that the volume flow rate across a streamline is zero, and this result reflects the fact that fluid does not flow through a streamline. Experience indicates that fluid cannot flow through solid boundaries; for instance, water does not flow through the sides of a bath. In this subsection, we show that the streamline equation, $\psi =$ constant, provides one method of modelling solid boundaries.

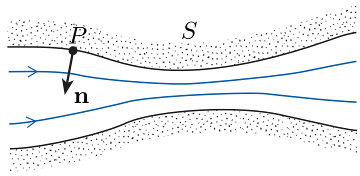
All fluid flows occur in the presence of boundaries; these are often in the form of solid boundaries, but boundaries between two fluids, such as the surface of water open to the atmosphere, are also important. For a solid boundary, there is no flow through the boundary surface S, and so the normal velocity component of the fluid and that of the boundary (which may be moving) are equal. Figure 2.10 shows two flows past solid boundaries:

(a) a flow outside a solid boundary in the shape of an aerofoil (i.e. the cross-section of an aircraft wing);

(b) the flow in a channel of varying cross-section.



(a)



(b)

Figure 2.10

If, at some point P on a solid boundary, the normal unit vector drawn *into* the fluid is $\mathbf{n}$, then we can express the 'no-flow-through' condition at the boundary as

$$\mathbf{u} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n},$$

where $\mathbf{u}$ is the fluid velocity at P and $\mathbf{b}$ is the boundary velocity at P. If the boundary S is at rest, then the normal velocity component of the fluid is zero on the boundary. This is usually called the *normal boundary condition*, and is valid for all real fluids.

Normal boundary condition

For the flow of a fluid past a solid boundary at rest, $u_{\text{n}} = \mathbf{u} \cdot \mathbf{n} = 0$ at each point of the boundary, where u_{n} is the fluid velocity component normal to the boundary.

A second boundary condition may be required, to describe the value of the tangential velocity component on a solid boundary (see Figure 2.11).

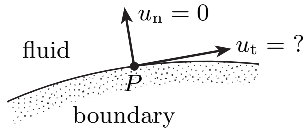


Figure 2.11

If, in the model, the fluid is assumed to be inviscid, that is, if the viscous forces in the fluid are ignored, then there can be a relative tangential velocity between the fluid and the boundary. Hence the tangential velocity component of the fluid, u_{t}, and that of the boundary are not related, so no condition at the boundary is imposed on u_{t} in this case.

If, however, the effects of viscosity are included in the model, then friction effects are present, and there is then no slippage between the boundary and the fluid, so that the tangential velocity component of the fluid *relative* to the boundary is zero. (These conditions apply only when the continuum hypothesis is valid, i.e. *always* in this course.)

This is known as the *no-slip condition* for the flow of viscous fluid past a boundary.

To illustrate these boundary conditions, consider the motion of a boat through otherwise still water, and a fluid particle at a point P near the boat (see Figure 2.12). In the inviscid model, the particle is only pushed aside as the boat moves through the water; it is not dragged along with the boat. The layer of water 'next' to the boat slips past the boat in this case. When viscosity is included in the model, the particle is both pushed aside and pulled forwards by the boat. The layer of water 'next' to the boat is dragged along with the boat.

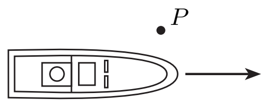


Figure 2.12

If the normal boundary condition is violated, then either the fluid flows into the 'solid' boundary (a permeable membrane or porous boundary), or the fluid flows away from the solid boundary leaving pockets of vapour. This latter phenomenon can occur with high speed propeller and turbine blades and is called *cavitation*. It usually occurs when the boundary is moved very rapidly and the adjacent fluid cannot keep up.

You will see an example of a porous boundary in *Unit 8*.

Example 2.4

Find the boundary condition for an inviscid fluid moving in the presence of each of the following boundaries, S.

(a) S is the x-axis (at rest).

(b) S is parallel to the x-axis and is moving with velocity $U\,\mathbf{i} + V\,\mathbf{j}$.

(In each case, consider the fluid to be above the boundary.)

Recall that the flow is two-dimensional. Hence to say that S is the x-axis means that S is the (x, z)-plane.

Solution

(a) The normal boundary condition

$$\mathbf{u} \cdot \mathbf{n} = 0 \qquad \text{on } S$$

becomes

$$\mathbf{u} \cdot \mathbf{j} = 0 \qquad \text{on } S,$$

since $\mathbf{n} = \mathbf{j}$ (see Figure 2.13). In terms of Cartesian components,

$$\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$$

and so

$$\mathbf{u} \cdot \mathbf{j} = u_2 = 0 \qquad \text{on } y = 0.$$

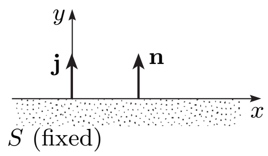


Figure 2.13

(b) For the moving boundary,

$$\mathbf{u} \cdot \mathbf{n} = (U\,\mathbf{i} + V\,\mathbf{j}) \cdot \mathbf{n} \qquad \text{on } S,$$

and again $\mathbf{n} = \mathbf{j}$ (see Figure 2.14). We have

$$\mathbf{u} \cdot \mathbf{j} = u_2 = V \qquad \text{on } S.$$

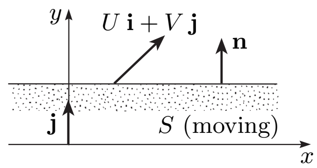


Figure 2.14

The fluid is inviscid, so there is no tangential boundary condition on $\mathbf{u}$ in either case. ■

Exercise 2.5

For each of the following models, find the boundary conditions for the flow of a fluid past a circular cylinder, of radius a, which is at rest with its axis along the z-axis.

(a) The fluid is assumed to be inviscid.

(b) The fluid is viscous.

Hint: Use plane polar coordinates.

In this unit and in *Unit 6*, we consider an inviscid model of a fluid, so it is only the normal boundary condition that can be applied. For the flow past a solid body at rest, using the inviscid model, the normal velocity component of the fluid is zero on the boundary of the body. Thus at any point of the boundary surface, the fluid velocity is parallel to the tangent to the boundary. This can be restated as follows.

> A solid boundary is always a streamline in ideal flow.

As introduced in *Unit 1* Subsection 1.3, ideal flow neglects fluid viscosity.

Conversely, any streamline may be taken to represent a solid boundary. In the following example, we take a given stream function and determine fluid flows past a boundary which one of the streamlines could represent. There are, of course, many possibilities, since any streamline can be considered to be the boundary of a solid body.

Example 2.5

Find the flow past a boundary represented by the stream function

$$\psi(r, \theta) = U \sin\theta \left(r - \frac{a^2}{r} \right).$$

This stream function was obtained in Example 2.3.

Solution

We shall find some of the streamlines and suggest which of these might model the boundaries of a real flow. Consider first the streamline $\psi = 0$. This requires

$$\text{either} \qquad \sin\theta = 0 \qquad \text{or} \qquad r^2 - a^2 = 0;$$

that is,

$$\theta = 0 \text{ or } \pi \qquad \text{or} \qquad r = a \quad (\text{since } r \geq 0 \text{ only}).$$

The streamline $\psi = 0$ therefore consists of three parts (see Figure 2.15):

(i) the circle $r = a$ for all θ;

(ii) the positive x-axis, for which $\theta = 0$;

(iii) the negative x-axis, for which $\theta = \pi$.

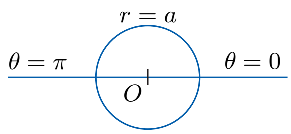


Figure 2.15 The streamline $\psi = 0$, which divides at the points $r = a$, $\theta = 0$ or π, where the velocity is zero.

Consider next the streamline $\psi = Ua$. Then

$$Ua = U\sin\theta\left(r - \frac{a^2}{r}\right) \qquad \text{or} \qquad a = r\sin\theta\left(1 - \frac{a^2}{r^2}\right),$$

which can be written in terms of Cartesian coordinates as

$$\frac{a}{y} = 1 - \frac{a^2}{x^2 + y^2}. \tag{2.5}$$

Recall that $r\sin\theta = y$.

Now separate x and y. Equation (2.5) may be re-expressed as

$$\frac{y/a}{y/a - 1} - \left(\frac{y}{a}\right)^2 = \left(\frac{x}{a}\right)^2 \qquad (y \neq a).$$

This is done by first making $(x^2 + y^2)/a^2$ the subject of the equation, as you can check.

Choosing values for y/a leads to the corresponding values for x/a, as shown in Table 2.1. This streamline is shown as the curve ABC in Figure 2.16. The streamline $\psi = -Ua$ is the reflection of ABC in the (x/a)-axis, since its equation is

$$-\frac{a}{y} = 1 - \frac{a^2}{x^2 + y^2}.$$

This streamline is shown as $A'B'C'$ in Figure 2.16.

Choosing x/a and finding y/a is more cumbersome, except when $x = 0$.

Table 2.1

y/a	x/a
1.04	± 4.99
1.15	± 2.52
1.30	± 1.63
1.62	0

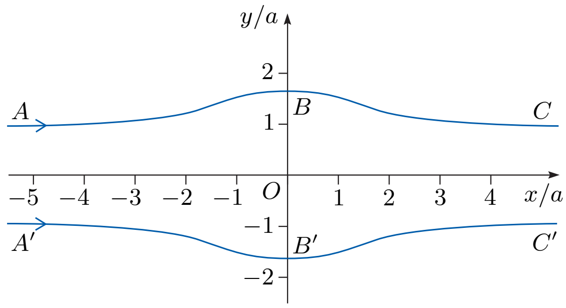


Figure 2.16 The streamlines $\psi = \pm Ua$

We should expect the streamlines $\psi = \pm 2Ua$ to be similar in form to the curves ABC and $A'B'C'$, but to be approximately twice as far from the (x/a)-axis. These streamlines, which may be plotted in a similar way, are shown in Figure 2.17, overleaf.

The expectation of 'twice as far' is based on the fact that $\psi \simeq Uy$ for large r, so that (except close to O) doubling ψ will double y.

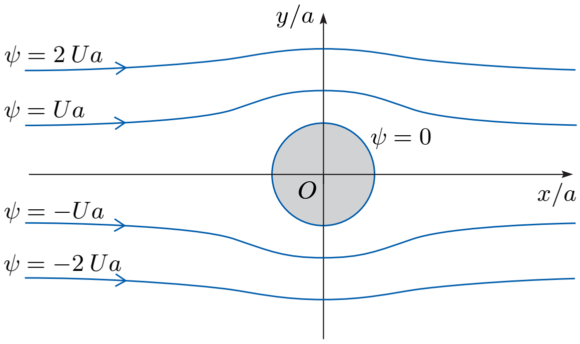


Figure 2.17

One possible flow modelled by this stream function is a uniform flow parallel to the x-axis, past the circular boundary $r = a$. For this flow, we ignore all streamlines inside the circle $r = a$ (see Figure 2.17).

Here 'uniform' refers to the flow far from the circular boundary.

Alternatively, we could take the streamlines $\psi = \pm Ua$ and the circular part of $\psi = 0$ to be solid boundaries, with a flow between the symmetrical pair of boundaries and past the circle $r = a$ (see Figure 2.18). ■

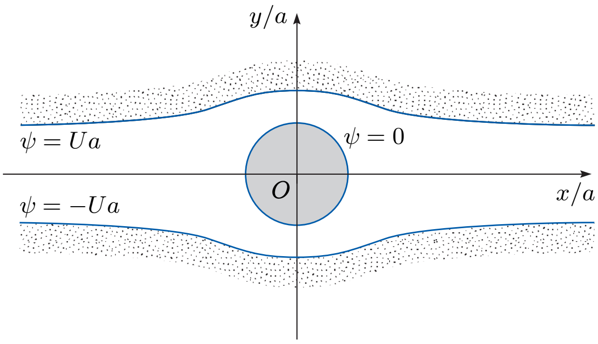


Figure 2.18

Exercise 2.6

Given the stream function $\psi = 2xy$, sketch the streamlines for $\psi = 0, 2, 4, 10$. Suggest a possible boundary for this flow.

End-of-section exercise

Exercise 2.7

(a) Find the stream function $\psi(x, y)$ for the velocity field $\mathbf{u} = -2y\,\mathbf{i} - 2x\,\mathbf{j}$. (Express ψ in a form in which any arbitrary constant is taken as zero.)

(b) Sketch the streamlines for $\psi = 0, 2, 4, 10$.

(c) Suggest a possible boundary for this flow.

3 Modelling by combining stream functions

For two-dimensional flows, the stream function ψ introduced in Section 2 provides a representation of the fluid velocity, and the lines of constant ψ are streamlines. The method of analysis was somewhat mathematical; given a velocity field we could find the stream function, and we found the stream functions for certain basic flows such as a source and a vortex. In order to move towards the modelling of real fluid flows, we try to recognise a real flow as a combination of these simpler basic flows. In this section, we concentrate on finding the stream functions for some of these combinations. By drawing the streamlines, it may then be possible to recognise one particular streamline as a boundary in the flow.

Recall that ψ is a scalar field.

This idea was introduced in Subsection 2.3.

Of course, not all flows can be modelled in this way, because recognising a possible combination of basic flows (probably from one of the flow visualisation techniques) may not be easy.

We shall investigate a few examples of combinations of simple flow patterns which lead to results of practical interest. Again, we are building from a knowledge of the stream functions of the simple flows, and proposing possible applications for the stream functions of the combinations. However, this approach will provide an insight into how we could tackle problems of modelling real fluid flows.

Table 3.1 summarises the stream functions for some of the basic flows.

Table 3.1

Type of flow	Velocity field $\mathbf{u}$	Stream function ψ	Reference
Uniform flow parallel to the x-axis	$U\,\mathbf{i}$	Uy (Cartesian); $Ur\sin\theta$ (polar)	Put $\alpha = 0$ in Example 2.1; Example 2.3 with $a = 0$
Uniform flow at an angle α to the x-axis	$U\cos\alpha\,\mathbf{i} + U\sin\alpha\,\mathbf{j}$	$-Ux\sin\alpha + Uy\cos\alpha$	Example 2.1
Source of strength $2\pi m$ at the origin	$\dfrac{m}{r}\,\mathbf{e}_r \quad (m > 0)$	$m\theta$	Exercise 2.2(b); with $m < 0$, this is a sink of strength $\lvert m\rvert$
Vortex of strength $2\pi k$ at the origin	$\dfrac{k}{r}\,\mathbf{e}_\theta$	$-k\ln r$	Exercise 2.2(a)
Rigid body rotation	$Kr\,\mathbf{e}_\theta$	$-\frac{1}{2}Kr^2$	Exercise 2.3(a)

3.1 The Principle of Superposition

The technique of modelling complicated flows by combining simple basic flows was developed in the nineteenth century by W. Rankine (1820–72). The method relies on an addition property of vector fields:

Rankine, a Scottish engineer, was a pioneer in the field of theoretical thermodynamics.

> for each point at which two vector fields exist, the resultant vector field is the vector sum of the two constituent fields.

Suppose that ψ_1 and ψ_2 are the stream functions associated with any two velocity vector fields, $\mathbf{u}_1$ and $\mathbf{u}_2$ respectively. We can use the result of Example 2.2(b) on page 21 to relate the velocity vectors and the stream functions; we have

$$\mathbf{u}_1 = \boldsymbol{\nabla}\psi_1 \times \mathbf{k} \qquad \text{and} \qquad \mathbf{u}_2 = \boldsymbol{\nabla}\psi_2 \times \mathbf{k}.$$

The resultant velocity vector field $\mathbf{u}$ is the sum of $\mathbf{u}_1$ and $\mathbf{u}_2$, so that

$$\begin{aligned}\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 &= (\boldsymbol{\nabla}\psi_1 \times \mathbf{k}) + (\boldsymbol{\nabla}\psi_2 \times \mathbf{k}) \\ &= (\boldsymbol{\nabla}\psi_1 + \boldsymbol{\nabla}\psi_2) \times \mathbf{k} \\ &= \boldsymbol{\nabla}(\psi_1 + \psi_2) \times \mathbf{k}.\end{aligned}$$

Thus the stream function for the flow field with velocity $\mathbf{u}_1 + \mathbf{u}_2$ is the sum $\psi_1 + \psi_2$. The velocity fields $\mathbf{u}_1$ and $\mathbf{u}_2$ are *superposed* to give the velocity field $\mathbf{u}_1 + \mathbf{u}_2$. This can be summarised as follows.

The Principle of Superposition

If the velocity fields $\mathbf{u}_1$ and $\mathbf{u}_2$ have respective stream functions ψ_1 and ψ_2, then the velocity field $\mathbf{u}_1 + \mathbf{u}_2$ has stream function $\psi_1 + \psi_2$.

Exercise 3.1

Show that $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ satisfies the continuity equation, where $\mathbf{u}_1$ and $\mathbf{u}_2$ are velocity fields of constant-density fluid flows.

The Principle of Superposition can be used to find the stream functions for combinations of sources, sinks and uniform flows. The next example shows this idea applied to a flow pattern that consists of a source and sink pair.

The *Media Guide* session at the end of this section features an interactive exercise applying the Principle of Superposition.

Example 3.1

Find the stream function and streamlines for the flow due to a source of strength $2\pi m$ at A $(-a, 0)$ and a sink of the same strength at B $(a, 0)$.

Solution

Let θ_1 be the polar angle as measured from B as origin, and let θ_2 be the polar angle as measured from A as origin (see Figure 3.1). Then the stream function of the sink at B is $\psi_1 = -m\theta_1$, and the stream function of the source at A is $\psi_2 = m\theta_2$. By the Principle of Superposition, the overall flow, due to the combination of source and sink, has stream function

$$\psi = \psi_1 + \psi_2 = m(\theta_2 - \theta_1).$$

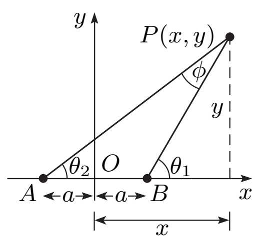


Figure 3.1

Now let P be any point, other than A or B, with coordinates (x, y). From the definitions of θ_1 and θ_2, their values at P will be given by

$$\tan\theta_1 = \frac{y}{x-a} \qquad \text{and} \qquad \tan\theta_2 = \frac{y}{x+a} \qquad \text{(see Figure 3.1)}.$$

Using a standard trigonometric formula, it follows that

$$\begin{aligned}\tan(\theta_2 - \theta_1) &= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2\tan\theta_1} \\ &= \frac{y/(x+a) - y/(x-a)}{1 + y^2/(x^2-a^2)} = -\frac{2ay}{x^2+y^2-a^2}.\end{aligned}$$

The stream function of the combined flow is therefore

$$\psi = m(\theta_2 - \theta_1) = m\arctan\left(-\frac{2ay}{x^2+y^2-a^2}\right).$$

The equation for streamlines is $\psi =$ constant, that is, $\theta_2 - \theta_1 =$ constant. Now $\theta_1 - \theta_2$ is equal to the angle ϕ subtended at P by the line segment AB (see Figure 3.1), and the locus of all such points as P varies (while ϕ is held constant) is a circular arc through A and B. Hence the streamlines are arcs of circles, as shown in Figure 3.2. By symmetry, each circle centre

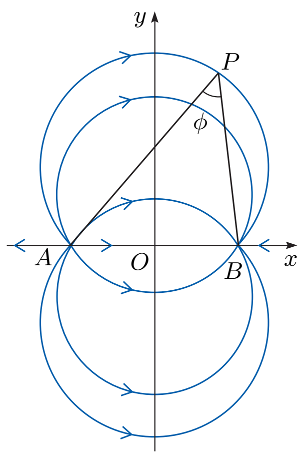


Figure 3.2

is on the y-axis. However, the special cases $\phi = 0$ and $\phi = \pi$ give the x-axis as a further streamline.

An alternative to this geometrical argument is as follows. If $\psi =$ constant then, from the Cartesian expression for the stream function,

$$x^2 + y^2 - a^2 = 2Cy, \qquad \text{or} \qquad x^2 + (y - C)^2 = a^2 + C^2,$$

The constant value of ψ here is $m \arctan(-a/C)$.

where C is a constant. This is the equation of a circle with centre at $(0, C)$, passing through $(\pm a, 0)$. ■

Exercise 3.2

Find the stream function and equation of streamlines (in Cartesian coordinates) caused by placing two sources, each of strength $2\pi m$, at the points A $(-a, 0)$ and B $(a, 0)$. (There is no need to sketch the streamlines.)

The doublet

Now consider a limiting case of the source and sink combination analysed in Example 3.1. Suppose that the source and sink are moved towards each other along the x-axis, by reducing a, while at the same time the strength $2\pi m$ of each is increased. If this is done in such a way that the product ma (a constant multiple of strength × distance apart) is kept constant then, in the limit as $a \to 0$, a new basic flow called a **doublet** is created.

Another name for a doublet is a *dipole*.

The corresponding effect on the streamline pattern in Figure 3.2 is to yield the streamline pattern for a doublet (see Figure 3.3), consisting of all circles with centre on the y-axis that are tangent to the x-axis, plus the x-axis itself.

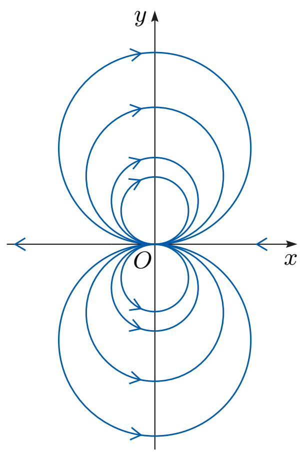


Figure 3.3 Streamline pattern for a doublet at the origin

The line AB joining the source and the sink is called the *axis* of the doublet. This is taken to be positive in the direction from sink to source, i.e. from B to A. The *strength* of the doublet is defined to be $4\pi am$, which is the product of the strength $2\pi m$ of the source or sink and of the distance $2a$ between them before the limit is taken. The direction can be incorporated with the strength to define a *vector strength* $-4\pi am\,\mathbf{i}$.

To find an expression for the stream function of a doublet, we evaluate the limit of the stream function found in Example 3.1 for a source and sink combination, as $a \to 0$ and $m \to \infty$ while the product am remains fixed.

The result from Example 3.1 was

$$\psi = m \arctan\left(-\frac{2ay}{x^2 + y^2 - a^2}\right),$$

which in plane polar coordinates is

$$\psi = m \arctan\left(-\frac{2ar\sin\theta}{r^2 - a^2}\right).$$

Now, for small α, we have $\tan\alpha \simeq \alpha$. Hence, for small enough values of a,

$$\arctan\left(-\frac{2ar\sin\theta}{r^2 - a^2}\right) \simeq -\frac{2ar\sin\theta}{r^2 - a^2}.$$

It follows that

$$\psi \simeq -\frac{2amr\sin\theta}{r^2 - a^2} = -\frac{\lambda r\sin\theta}{r^2 - a^2}, \qquad \text{where } \lambda = 2am.$$

From above, in terms of λ, the doublet has strength $2\pi\lambda$ and vector strength $-2\pi\lambda\,\mathbf{i}$.

Now taking the limit as $a \to 0$ gives

$$\lim_{\substack{a\to 0\\ m\to\infty\\ am \text{ constant}}} \left(-\frac{\lambda r\sin\theta}{r^2-a^2}\right) = -\frac{\lambda r\sin\theta}{r^2} = -\frac{\lambda\sin\theta}{r}.$$

Thus the stream function for a doublet of vector strength $-2\pi\lambda\,\mathbf{i}$ is given by

$$\psi(r,\theta) = -\frac{\lambda\sin\theta}{r}.$$

This is also a good approximation to the stream function for the original source/sink pair when $r \gg a$, that is, far from the origin.

Since both A and B tend to the origin in the limit, the streamlines are now circles through the origin with centres on the y-axis, as shown in Figure 3.3. The streamline pattern is symmetric about the x-axis.

Exercise 3.3

Find the velocity field of the flow caused by a doublet of vector strength $-2\pi\lambda\,\mathbf{i}$ at the origin.

3.2 Combining sources, doublets and uniform flows

The doublet provides another of the basic flows which can be used, in combination with sources, sinks and uniform flows, to model real fluid flows. In this subsection, we consider the combination of

(i) a source and a uniform flow, and

(ii) a doublet and a uniform flow.

Consider first a source and a steady uniform flow. Because both basic flows are steady, the streamlines are the same as the pathlines. So we can sketch the streamlines 'intuitively' by imagining the effect of fluid particles approaching the source along parallel straight lines and fluid particles leaving the origin O along the radial lines. We should expect the uniform flow lines far from the source to be little changed due to the presence of the source. But fluid particles emanating from the source at O into the oncoming uniform flow will be pushed back, as shown in Figure 3.4. At some point on the x-axis, S say, the two separate velocities (due to the uniform flow and the source) will cancel, and the velocity of the fluid will be zero. A point in the fluid where the fluid velocity is zero is called a **stagnation point**.

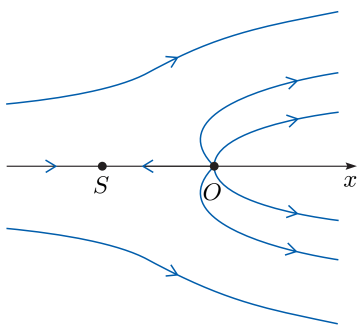


Figure 3.4

The following example shows how the position of a stagnation point can be found using the stream function for a flow.

Example 3.2

Find the position of the stagnation point for the combination of a source of strength $2\pi m$ at the origin and a uniform flow $U\,\mathbf{i}$.

Solution

The stream function for this combination of a source at the origin and uniform flow is given by

$$\begin{aligned}\psi &= m\theta + Uy \\ &= m\theta + Ur\sin\theta.\end{aligned}$$

At a stagnation point, the velocity components are both zero. In polar coordinates, from Equations (2.2), we have

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{1}{r}(m + Ur\cos\theta) = 0 \tag{3.1}$$

and

$$u_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta = 0. \tag{3.2}$$

Solving Equation (3.2) for θ gives $\theta = 0$ or $\theta = \pi$. From Equation (3.1), if $\theta = 0$, then $r = -m/U$, and if $\theta = \pi$, then $r = m/U$. Since r cannot be negative (by definition), the solution $\theta = 0$ can be discounted as physically unrealistic, so that there is only one stagnation point, given by

$$\theta = \pi, \qquad r = \frac{m}{U},$$

or by $(-m/U, 0)$ in Cartesian coordinates. ■

This is consistent with the intuitive guess shown in Figure 3.4.

Exercise 3.4

Find the positions of the stagnation points for the combination of a doublet of vector strength $-2\pi\lambda\,\mathbf{i}$ and a uniform flow $U\,\mathbf{i}$. Show that the stagnation points lie on the circle $r = \sqrt{\lambda/U}$, and that this circle is part of the streamline $\psi = 0$.

We now consider again the flow in Example 3.2. The solution to this example shows that there is a stagnation point at $r = m/U$, $\theta = \pi$; at this point, the fluid velocity is zero. The next example uses the value of the stream function at this point to help sketch the streamline pattern for a source in a uniform flow.

Example 3.3

The stream function for a combination of a source of strength $2\pi m$ at the origin and a uniform flow with velocity $U\,\mathbf{i}$ is $\psi = m\theta + Uy$. By sketching the streamline pattern, show that this combination may be used as a model for the flow past a blunt body.

It is often convenient to use a combination of Cartesian and polar coordinates in this type of problem.

Solution

First find the equation of the streamline that passes through the stagnation point, S. At this point (from Example 3.2), $\theta = \pi$ and $y = 0$, so that

$$\psi = m\pi \qquad \text{at } S.$$

Hence S lies on the streamline

$$m\theta + Uy = m\pi \qquad \text{or} \qquad y = \frac{m}{U}(\pi - \theta). \tag{3.3}$$

Since this equation is satisfied by $\theta = \pi$, $y = 0$, the negative x-axis (NSO in Figure 3.5, on the next page) is part of this streamline.

Now we seek the rest of the streamline given by Equation (3.3). As θ decreases from π, y increases and reaches the value $m\pi/(2U)$ when $\theta = \frac{1}{2}\pi$.

Then y continues to increase as θ decreases. For positive values of θ close to zero, y is close to the value $m\pi/U$. This reflects the fact that, far downstream, the uniform flow dominates the effect of the source, and the streamlines are (approximately) parallel to the x-axis. Put more technically, the line $y = m\pi/U$ is an asymptote for the streamline given by Equation (3.3). The part of the streamline for $0 < \theta \leq \pi$ is shown as SRT in Figure 3.5. Since the uniform flow and source are symmetric about the x-axis, the streamlines in the region $y < 0$ are reflections in the x-axis of the streamlines in the region $y > 0$. Thus the curve $SR'T'$, i.e. the reflection of SRT in the x-axis, is also part of the streamline through S.

Note that the streamline $\psi = m\pi$ has branches which meet at S.

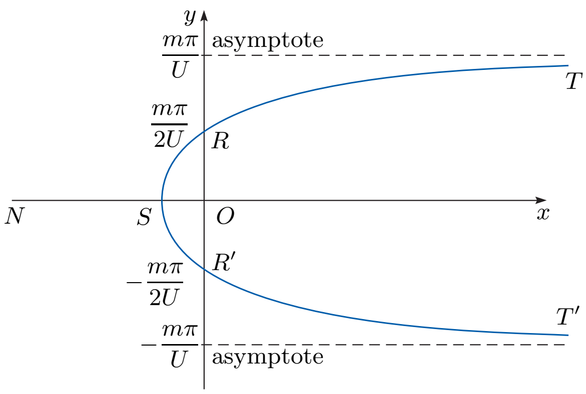


Figure 3.5 The streamline through S

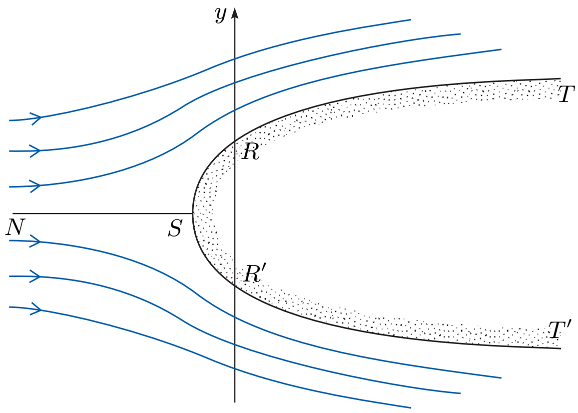


Figure 3.6

To draw some of the other streamlines, we could choose values for ψ, ψ_n say, and then sketch the curves $m\theta + Uy = \psi_n$. However, we shall adopt a more intuitive approach. The curve $TRSR'T'$ can form a bounding streamline, and since fluid cannot cross a streamline, we could consider the fluid outside this boundary as flowing past a solid body. The flow is steady, so the pathlines and the streamlines coincide. Thus to sketch the streamline pattern outside $TRSR'T'$, we imagine the paths of fluid particles starting off along parallel lines for large negative x and approaching the boundary. These particles will be deflected past the boundary and travel along the lines shown in Figure 3.6.

The bounding streamline can be taken as a solid boundary, and this combination of a source and a uniform flow can then be taken as a model for the flow of an incompressible fluid past a blunt body.

We have described the body as blunt, but have not shown that it is smooth, rather than pointed or kinked, at the stagnation point, S. This requires that the tangent to the curve $TRSR'T'$ should be vertical at S. The details of showing this are omitted.

One approach is to show that the curve $TRSR'T'$ always lies between the vertical line through S and the circle through S with centre O. Since both this line and the circle have a vertical tangent at S, the result follows.

In this example, we have sketched the streamline pattern outside the bounding streamline $TRSR'T'$, and the flow lines in the vicinity of the source do not appear, although the source is an essential part of the model. The source is a modelling device for obtaining the shape of the blunt solid boundary. ■

Exercise 3.5

For the flow in Example 3.3, by drawing the paths of fluid particles that emerge from the origin (i.e. from the source), sketch the streamline pattern inside the bounding streamline $TRSR'T'$.

The blunt body shown in Figure 3.6 is known as a *half body*, since it has a nose but no tail. The combination of the uniform flow and source provides the stream function and hence the velocity field for the flow past such a body. This provides a useful model for the flow at the upstream end of symmetrical bodies with a large length-to-width ratio, such as bridge supports. Alternatively, the streamline pattern in the region $y \geq 0$ could represent the flow of water in a river with a rise in the river bed, or the flow of wind over a hillside.

At the downstream end of the body, the streamlines and the solid boundary are almost parallel to the x-axis, so this combination cannot model the flow past a closed body, such as the one shown in Figure 3.7.

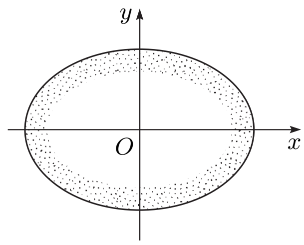


Figure 3.7

Exercise 3.6

What basic flow can be added to the source and uniform flow, to model the flow past a closed body such as the one shown in Figure 3.7?

Next you are asked to look at flow past a special closed shape, the circle.

Exercise 3.7

By sketching some streamlines, show that the combination of a doublet of vector strength $-2\pi\lambda\,\mathbf{i}$ at the origin and a uniform flow $U\,\mathbf{i}$ can be used to model the flow past a circle of radius $\sqrt{\lambda/U}$. (Sketch the streamlines outside this circle only.)

The streamline through the stagnation points is given in the solution to Exercise 3.4.

Figure 3.8 shows the streamline pattern for the combination of a doublet (of vector strength $-2\pi\lambda\,\mathbf{i}$) and a uniform flow, outside the streamline $\psi = 0$. This streamline is a circle of radius $\sqrt{\lambda/U}$ and can be taken to represent a solid boundary. The combination can then be used to model the flow of a fluid past a circular cylinder. The similarity between the streamline pattern in Figure 3.8, obtained from the doublet/uniform flow combination, and the streamline pattern for the flow of air past a cylinder, is rather striking. It is behind the cylinder, in the wake, that the model breaks down.

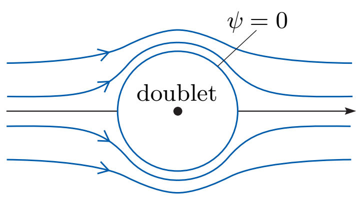


Figure 3.8

In this section, we started with some of the basic flows and suggested flows that combinations of them could represent. In practice, fluid mechanics is concerned with the 'inverse problem'; that is, for a flow past a given boundary, find the stream function. Investigating the flow of air past an aircraft wing or the flow of water past a hydrofoil are examples of such a problem.

We can model problems of this type by combining sources and sinks with a uniform flow. The streamline pattern for a source in a uniform flow can model the ideal flow past a blunt object. If we introduce a sink on the x-axis, the effect is to pull the streamlines back in, and a combination of a source at A $(-a, 0)$, a sink at B $(a, 0)$ and a uniform flow, models the flow past a closed body (see Figure 3.9). This is a similar streamline pattern to that shown in Figure 3.8, but the solid boundary, $\psi = 0$, is now not a circle but is more elliptical in shape. (The shape created in this way is known as a *Rankine oval*.) Adding further sources and sinks (indicated by + and − respectively in Figure 3.10, overleaf) can produce a flatter boundary, with the shape of the cross-section of an aircraft wing.

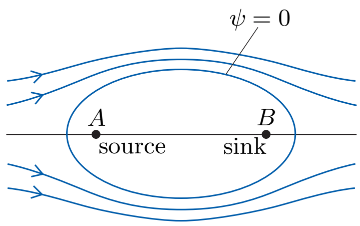


Figure 3.9 Rankine oval

The stream function for the flow can be written down in terms of those for the sources and sinks and that for the uniform flow, and from this stream function we can deduce the streamline pattern and the velocity field. This will show the features of the flow near a solid boundary. Furthermore, as you will see in Section 5, a knowledge of the velocity field allows us to predict the pressure distribution and hence the net force on a solid boundary, due to the flow of a fluid past it.

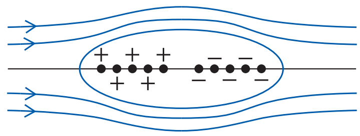


Figure 3.10

To conclude this section, you are invited to visually explore the flow effects that can be produced with various different basic flows.

Carry out the activities for this section in the Media Guide.

End-of-section exercise

Exercise 3.8

Four sources, of equal strength $2\pi m$, are placed with one at each of the points $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$.

(a) Write down the stream function for this combination of sources, in Cartesian coordinates.

(b) Show that the origin is a stagnation point.

(c) Show that the x- and y-axes are streamlines.

(d) Use your intuition to sketch the streamline pattern, and show that this combination could model the flow produced by a source at (1,1) in a corner (see Figure 3.11).

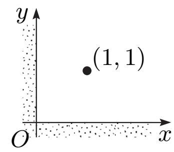


Figure 3.11

4 *Description of fluid motions*

In describing fluid motions we have used the terms ‘steady’, ‘uniform’ and ‘incompressible’. Subsection 4.1 specifies more precisely what is meant by these terms, and gives examples of their use.

In order to prepare for the derivation of *Euler’s equation of motion*, in Section 5, it is necessary to specify flow properties (such as velocity and density) as functions of position and time, and to distinguish two different types of time derivative of such functions. These matters are addressed in Subsection 4.2.

We also look more closely at the continuity equation, in Subsection 4.3, and derive two special cases of it.

The continuity equation was derived in *Unit 4* Subsection 3.3.

4.1 Steady and uniform flows

The flow of a real fluid is usually very complicated, and a complete solution of the equations for the associated model is seldom possible. In order to make progress, we are forced to modify the original model by making further simplifying assumptions about

(i) the properties of the fluid, and

(ii) the type of flow.

Unit 1 discussed two simplifying assumptions about fluid properties, which allow the successful modelling of many flow situations. These were to ignore the compressibility and the viscosity of the fluid. Throughout Block 2 (*Units 5–8*), it is usually assumed that the fluids under discussion are liquids, and as such are taken to be incompressible; in most cases, this means that the density is constant. This model is discussed further in Subsection 4.3.

Gases may also be modelled as incompressible under certain conditions, but liquids are much less compressible than gases.

In *Units 5* and *6*, we assume that the viscosity is negligible; you will see the limitations of the inviscid model in *Unit 7*, and in *Unit 8* we revise the model to take viscosity into account. Viscous effects are important for all flows in the vicinity of solid boundaries, and also for liquids whose coefficient of viscosity is high (for example, lubricating oil, which is a hundred times more viscous than water at 20°C).

The incompressible and inviscid assumptions concern properties of the fluid. Assumptions about the type of flow are usually expressed as conditions on the time and spatial (partial) derivatives of the flow variables (for example, velocity and density). In *Unit 1*, we described flows in which conditions at every point are independent of time, t, as 'steady'. Mathematically, this means, for example, that $\partial\mathbf{u}/\partial t = \mathbf{0}$ and $\partial\rho/\partial t = 0$, where $\mathbf{u}$ is the velocity and ρ is the density. We define a **steady flow** to be one for which the partial derivatives with respect to time of all flow and fluid properties are zero at each point in the region of flow. Flows in which changes with time do occur are called *non-steady* (or *unsteady*).

If, at a particular instant of time, the velocity vector field $\mathbf{u}$ does not change from point to point, the flow is said to be **uniform**. Thus uniform flows have the form $\mathbf{u} =$ constant vector or $\mathbf{u} = \mathbf{u}(t)$, and so, for a uniform flow,

$$\frac{\partial\mathbf{u}}{\partial x} = \frac{\partial\mathbf{u}}{\partial y} = \frac{\partial\mathbf{u}}{\partial z} = \mathbf{0}.$$

If changes in $\mathbf{u}$ with position do occur, then the flow is called *non-uniform.* Often the term 'uniform' is given a looser meaning than that above. For example, $\mathbf{u} = u_1(x)\,\mathbf{i}$ is *uniform in the y-direction* and $\mathbf{u} = u_1(y)\,\mathbf{i}$ is *uniform in the x-direction.*

For example, the shear flow of Exercise 2.1 is uniform in the x-direction.

Uniform flows can exist only for ideal fluids (with zero viscosity), but they can be a good model for fluids with low viscosities in regions far from boundaries. For example, the flow of a river is not uniform across its width — it is slower near the banks and the river-bed because of the effects of viscosity — but in the region at the middle of the river and well above its bed, the flow is approximately uniform. It may be possible to approximate the flow in a river as a uniform flow for the whole width and depth of the river. Figure 4.1 shows the velocity profile taken along a horizontal line across a river.

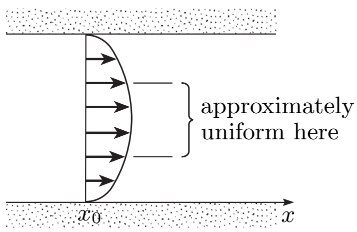


Figure 4.1 Velocity vectors at $x = x_0$ for flow in a river.

Table 4.1 summarises, with examples, each of the four possible combinations of steady, non-steady, uniform and non-uniform flows.

Table 4.1

Flow type	Implication for $\mathbf{u}$	Example
A: Steady, uniform	$\mathbf{u} = \mathbf{u}_0$, a constant, no time dependence, no spatial dependence	$\mathbf{u} = U \cos\alpha\, \mathbf{i} + U \sin\alpha\, \mathbf{j}$ (uniform stream at angle α)
B: Steady, non-uniform	$\mathbf{u} = \mathbf{u}(x, y, z)$, no time dependence	$\mathbf{u} = U y\, \mathbf{i}$ (shear flow)
C: Non-steady, uniform	$\mathbf{u} = \mathbf{u}(t)$, no spatial dependence	A uniformly accelerating flow (most uncommon)
D: Non-steady, non-uniform	$\mathbf{u} = \mathbf{u}(x, y, z, t)$	The flow past an aerofoil

Exercise 4.1

Which of the flow types in Table 4.1 (A, B, C or D) is the most appropriate candidate to model each of the following flows?

(a) Flow in a long horizontal pipe of constant cross-section.

(b) The flow from a dripping tap.

(c) The flow down a long inclined channel.

(d) The flow in a hurricane.

(e) The flow from a rotary fan.

4.2 Rate of change following the motion

In *Unit 1*, when developing the continuum model, a fluid particle was defined as the limit of a fluid element as the size of the element tends to zero. In order to describe the motion of a fluid particle, a frame of reference is defined so that the position of the particle at any time can be specified, as (x, y, z, t) for example. This specification then provides a means of forming mathematical relationships between the position, velocity and acceleration of a particle at any time.

This approach should be familiar to you from solid particle mechanics.

In solid mechanics, each particle has a position vector $\mathbf{r}(t)$, whose first two time derivatives give the velocity and acceleration of the particle. In fluid mechanics, the fluid particles can move relative to each other (unlike the particles of a rigid body), and this factor makes the position vector approach rather unwieldy. Instead, flows are described by specifying the mathematical form of the flow parameters as functions of the space variables x, y, z and time, t. (From here on, the Cartesian coordinate system is used for reference purposes.) In other words, the whole flow can be described, throughout space and time, by specifying the appropriate scalar and vector fields, for example, $\rho(x, y, z, t)$ and $\mathbf{u}(x, y, z, t)$.

In Cartesian coordinates, $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$.

These functions are scalar and vector fields.

At different times, the point (x, y, z) is occupied by different fluid particles.

Imagine a stream running steadily from a hot spring. The hot water cools as it travels downstream. Let $\Theta(x, y, z, t)$ be the scalar field which gives the temperature at any point and time in the flow. The partial derivative $\partial\Theta/\partial t$ has the following physical interpretation. Suppose that thermometer A is held in the water at a fixed point P. The rate of change of temperature with time, as observed from thermometer A, will be given by $\partial\Theta/\partial t$, since x, y and z are constant at P. This is the *local rate of change* of temperature, the rate of change at the fixed point P.

Now consider thermometer B, moving downstream with the flow; the temperature measured by this thermometer changes with time, but it also changes with position. However, the position of thermometer B itself depends on time, so in effect this temperature is dependent on only one variable, the time t, with each of x, y and z being a function of t. The rate of change now is that obtained *following the motion* of thermometer B. The rate of change of temperature with time observed by the two thermometers will differ, in general.

Suppose, in particular, that the temperature of the stream does not vary at any fixed point, and depends only on the distance x downstream from the spring. Then $\Theta = \Theta(x)$, so $\partial\Theta/\partial t = 0$ and *locally* the temperature does not change. However, for thermometer B, with path $x = x(t)$, there will be a rate of temperature change

$$\frac{d\Theta}{dt} = \frac{d\Theta}{dx}\frac{dx}{dt},$$

depending on the spatial derivative $\Theta'(x)$ and the speed dx/dt of thermometer B. This rate of change is due to the motion of the thermometer, and is known as the *convective rate of change*. This application of the simplest form of the Chain Rule is now generalised.

The Chain Rule for functions of two variables, used earlier in this unit, extends following the same pattern to functions of four variables. Consider a scalar field which is a function of four variables, $f(x, y, z, t)$. If each of these variables, x, y, z and t, is a function of one parameter, s say, then f can also be considered as a function of the single variable s, i.e. $f(s)$.

This is the version of the Chain Rule introduced in MST209 *Unit 12*.

(Here we adopt a customary abuse of notation, using the same symbol f to stand both for the function of four variables and for a function of one variable. However, it should be clear from the context which version of f is intended at each stage.)

Thus in Equation (4.1), f stands for $f(s)$ on the left-hand side and for $f(x, y, z, t)$ on the right-hand side. We also write x for $x(s)$, y for $y(s)$, etc.

The Chain Rule in this case is

$$\frac{df}{ds} = \frac{dx}{ds}\frac{\partial f}{\partial x} + \frac{dy}{ds}\frac{\partial f}{\partial y} + \frac{dz}{ds}\frac{\partial f}{\partial z} + \frac{dt}{ds}\frac{\partial f}{\partial t}. \qquad (4.1)$$

Now suppose that the position vector is a function of time, $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$. The corresponding velocity is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}. \qquad (4.2)$$

Replacing the parameter s by t throughout Equation (4.1) gives

$$\frac{df}{dt} = \frac{dx}{dt}\frac{\partial f}{\partial x} + \frac{dy}{dt}\frac{\partial f}{\partial y} + \frac{dz}{dt}\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t}. \qquad (4.3)$$

The expression for the last term follows from the fact that $dt/dt = 1$.

The rate of change df/dt here is equal to the sum of the local rate of change of $f(x, y, z, t)$, which is $\partial f/\partial t$, and the rate of change of $f(x, y, z, t)$ due to the motion, which is given by the other terms on the right-hand side of Equation (4.3).

Exercise 4.2

Consider the stream described above that runs from a hot spring. Suppose that the stream is straight and of uniform width. Using a coordinate system that has origin O at the source of the stream, at its centre, let x be the distance downstream from the source, and y be measured across the stream, as shown in Figure 4.2. If O is on the water surface, then z

denotes depth below the surface. The temperature Θ of the water is modelled by the function

$$\Theta = 50\,(1 + 0.05\cos(0.1t))\,e^{-0.01x}(1 - 0.02y^2)(1 - 0.04z) \qquad (\text{in } {}^\circ\text{C}).$$

(a) For a thermometer held fixed at the point $(5, 0.2, 0.1)$, what is the rate of change of temperature?

(b) For a thermometer moving across the stream, on the path given by $\mathbf{r}(t) = 5\,\mathbf{i} + 2t\,\mathbf{j} + 0.1\,\mathbf{k}$, what is the rate of change of temperature at the point $(5, 0.2, 0.1)$?

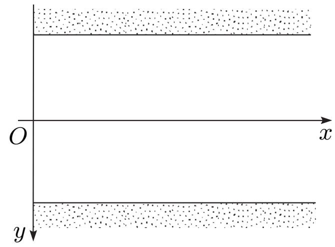


Figure 4.2

The right-hand side of Equation (4.3) can be written in operator form as

$$\left(\frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z}\right) f.$$

Using the operator $\boldsymbol{\nabla}$ and Equation (4.2), this is

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right) f.$$

The vector differential operator $\boldsymbol{\nabla}$ and scalar differential operator $\mathbf{a}\cdot\boldsymbol{\nabla}$ were introduced in *Unit 4* Section 5.

Rather than applying this for any given motion, we restrict attention now to the motion of fluid particles, that is, motion along pathlines. In this case, the vector $\mathbf{v}$ above is replaced by the fluid velocity vector, $\mathbf{u}$. So, for **differentiation following the motion** of fluid particles, we have

$$\frac{df}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\right) f. \tag{4.4}$$

Note here that

$$u_1 = \frac{dx}{dt},\ u_2 = \frac{dy}{dt},\ u_3 = \frac{dz}{dt}.$$

The operator $\partial/\partial t + \mathbf{u}\cdot\boldsymbol{\nabla}$ is referred to as the **total derivative** operator, and is sometimes written as D/Dt. This notation will not be used in this course, since it will be clear from the context whether d/dt refers to differentiating a function of one variable, or differentiating a function of several variables each of which depends on the same single variable. In Equation (4.4), the two terms on the right-hand side have the following interpretation:

This is also called the *material derivative* operator.

$\dfrac{\partial f}{\partial t}$ is the *local* rate of change;

$(\mathbf{u}\cdot\boldsymbol{\nabla})\,f$ is the *convective* rate of change, following the motion.

The total derivative is therefore the sum of the local and convective rates of change. It can be applied to any scalar field, such as pressure, density or chemical concentration.

For more explanation of the rates of change discussed here, see the *Media Guide*.

Exercise 4.3

Suppose that a fluid particle is in a one-dimensional motion for which $x = t^2$, and that the density of the fluid is $\rho(x,t) = x^2 t + \cos t$. Evaluate $d\rho/dt$, by

(a) using differentiation following the motion, then substituting for x in terms of t;

(b) substituting for x in the expression for ρ, then differentiating with respect to t.

Exercise 4.4

Simplify Equation (4.4) for the total derivative of the scalar field $f(x,y,z)$ in a flow with velocity field $\mathbf{u}$, in each of the following cases:

(a) $\mathbf{u} = u_1(x,y,z)\,\mathbf{i}, \qquad f(x,y,z) = y + z;$

(b) steady flow.

The total derivative of a scalar field was introduced above. The extension to taking the total derivative of a vector field is fairly straightforward, at least in Cartesian coordinates, because the total derivative is a linear operator, and the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are all constant.

Consider a vector field

$$\mathbf{F}(x,y,z,t) = F_1(x,y,z,t)\,\mathbf{i} + F_2(x,y,z,t)\,\mathbf{j} + F_3(x,y,z,t)\,\mathbf{k},$$

in which each of x, y and z is a function of t, and apply the Chain Rule to each component of $\mathbf{F}$:

$$\frac{dF_i}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\right) F_i, \qquad \text{where } i = 1,2,3.$$

Any vector field can be expressed in terms of three (scalar) Cartesian component fields, to each of which the total derivative can be applied. Note that

$$(\mathbf{u}\cdot\boldsymbol{\nabla})\,(F_1\,\mathbf{i}) = (\mathbf{u}\cdot\boldsymbol{\nabla})\,F_1\,\mathbf{i},$$

since $\mathbf{i}$ is a constant vector, and similarly for $F_2\,\mathbf{j}$, $F_3\,\mathbf{k}$.

Now combine the components in vectorial form, to obtain

$$\begin{aligned}
\frac{d\mathbf{F}}{dt} &= \frac{dF_1}{dt}\,\mathbf{i} + \frac{dF_2}{dt}\,\mathbf{j} + \frac{dF_3}{dt}\,\mathbf{k} \\
&= \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\right) F_1\,\mathbf{i} + \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\right) F_2\,\mathbf{j} + \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\right) F_3\,\mathbf{k} \\
&= \frac{\partial F_1}{\partial t}\,\mathbf{i} + \frac{\partial F_2}{\partial t}\,\mathbf{j} + \frac{\partial F_3}{\partial t}\,\mathbf{k} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,F_1\,\mathbf{i} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,F_2\,\mathbf{j} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,F_3\,\mathbf{k} \\
&= \frac{\partial \mathbf{F}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{F} \\
&= \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\right) \mathbf{F}.
\end{aligned}$$

Note that if ϕ is a scalar field, then $(\mathbf{u}\cdot\boldsymbol{\nabla})\,\phi = \mathbf{u}\cdot(\boldsymbol{\nabla}\phi)$, but for a vector field $\mathbf{F}$ always write $(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{F}$, since $\mathbf{u}\cdot(\boldsymbol{\nabla}\mathbf{F})$ has no meaning.

The velocity vector field $\mathbf{u}$ is the one whose total derivative is used most often, and $d\mathbf{u}/dt$ is the *acceleration* of a fluid particle. Hence, the **acceleration of a fluid particle** is given by

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u}. \tag{4.5}$$

Exercise 4.5

Consider a river with a section of rapids in which the flow can be modelled as one-dimensional with velocity field $\mathbf{u} = u_1\,\mathbf{i}$, where

$$u_1(x) = U\left(1 + e^{-x^2}\right) \qquad (-5 < x < 5)$$

and x is distance along the river, $x = 0$ being at the centre of the rapids.

(a) Sketch u_1 against x, and find both $\partial\mathbf{u}/\partial t$ and $d\mathbf{u}/dt$. Evaluate du_1/dt at $x = \pm 1, \pm 0.5, \pm 0.25, 0$.

(b) Relate the derivatives $\partial\mathbf{u}/\partial t$ and $d\mathbf{u}/dt$ to physical quantities, as estimated by a fixed observer on the bank who watches
   (i) a trail of sticks flowing past the centre of the rapids;
   (ii) a single stick flowing through the rapids.

Exercise 4.5 demonstrates that, even in steady flow, fluid particles can have an acceleration (due to the convective term).

The operator $\mathbf{u}\cdot\boldsymbol{\nabla}$ was introduced from consideration of Cartesian coordinates. However, this operator is independent of the coordinate system chosen, and if another system is used, then $\mathbf{u}$ and $\boldsymbol{\nabla}$ are expressed in that coordinate system. For example, using cylindrical polar coordinates, we have

$$\mathbf{u} = u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta + u_z\,\mathbf{e}_z \qquad \text{and} \qquad \boldsymbol{\nabla} = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z},$$

This is the cylindrical polar version of **grad**, as in *Unit 4* Subsection 1.3.

so that the convective rate of change is given by

$$\mathbf{u}\cdot\boldsymbol{\nabla} = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}. \tag{4.6}$$

The spherical polar coordinate expression may be obtained similarly. The next exercise illustrates the use of cylindrical polar coordinates.

Exercise 4.6

Find the acceleration of a fluid particle moving in a rotating flow, with velocity field given by

$$\mathbf{u} = \frac{K(t)}{r}\,\mathbf{e}_\theta \qquad (r \neq 0),$$

where K is a function of one variable and r, θ, z are cylindrical polar coordinates. (Note that the unit vector $\mathbf{e}_\theta$ is not constant.)

Recall that

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j},$$
$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j},$$

so that

$$\frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\mathbf{e}_r.$$

4.3 A model for inviscid incompressible flows

In *Unit 1*, it was said that a fluid is *incompressible* if the volume of a given mass of the fluid cannot be reduced by applying a compressional force. For example, any fluid for which the density ρ is given by $\rho = \rho_0$ (constant) is incompressible. The mathematical definition of incompressibility is as follows: a fluid is **incompressible** if the rate of change of density *following the motion* is zero, that is, if

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,\rho = 0.$$

In general, liquids can be modelled as incompressible at constant temperature.

This definition expresses the fact that as a fluid particle proceeds along a pathline, the density at each point it occupies is the same, i.e. each pathline is a line of constant density.

This definition is useful to find the form of the continuity equation for an incompressible fluid. The continuity equation is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0; \qquad (4.7)$$

Unit 4 Subsection 3.3

expanding the second term gives

$$\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\, \rho + \rho\, \boldsymbol{\nabla} \cdot \mathbf{u} = 0.$$

See Equation (5.2) of *Unit 4*.

In terms of the derivative following the motion, this becomes

$$\frac{d\rho}{dt} + \rho\, \boldsymbol{\nabla} \cdot \mathbf{u} = 0.$$

Now if the fluid is incompressible, then $d\rho/dt = 0$, and so we arrive at the following result.

For a fluid of constant density, this was the outcome of *Unit 4*, Exercise 3.8(a).

The continuity equation for an incompressible fluid is $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$.

Clearly, the reverse of this result is also true: if the continuity equation is satisfied and $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, then the fluid is incompressible.

Note that this equation is expressed in terms of the velocity vector field, suggesting that we may talk about *incompressible flows* as well as incompressible fluids. In other words, an incompressible flow is a flow in which the fluid behaves as if it were incompressible.

Example 4.1

Show that the flow due to a two-dimensional source of strength $2\pi m$ is incompressible, except at the source point itself.

Solution

The velocity vector field for such a source, with the origin chosen to be at the source point, is

$$\mathbf{u} = \frac{m}{r}\, \mathbf{e}_r \qquad (r \neq 0),$$

in cylindrical polar coordinates. We have

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta) + \frac{\partial}{\partial z}(u_z).$$

The expression for div in cylindrical polar coordinates was given in *Unit 4* Subsection 3.1. See also the back of the *Handbook*.

Thus, in this case,

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{m}{r}\right) = 0 \qquad (r \neq 0).$$

This flow is incompressible everywhere except at $r = 0$. ■

Exercise 4.7

Which of the following vector fields could be the velocity field for an incompressible flow?

(a) $\mathbf{u} = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$

(b) $\mathbf{u} = x\,\mathbf{i} + 2y\,\mathbf{j} + 0\,\mathbf{k}$

(c) $\mathbf{u} = \dfrac{K}{r^2}\,\mathbf{e}_\theta,$ where K is constant (cylindrical polar coordinates)

If the density is $\rho = \rho_0$ (constant) then, for all velocity fields $\mathbf{u}$,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u_1\frac{\partial \rho}{\partial x} + u_2\frac{\partial \rho}{\partial y} + u_3\frac{\partial \rho}{\partial z} = 0,$$

since all partial derivatives of ρ are zero everywhere, and so the fluid is incompressible. But there are other possibilities for ρ and $\mathbf{u}$ which satisfy the incompressible version of the continuity equation, $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, as the next example shows.

Example 4.2

Show that the flow for which $\rho = \rho(y)$ and $\mathbf{u} = u(y)\,\mathbf{i}$ satisfies the continuity equation (4.7) and is an incompressible flow.

Solution

We have

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 + \frac{\partial}{\partial x}\left(\rho(y)\, u(y)\right) = 0,$$

so the continuity equation is satisfied. Also

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \frac{\partial}{\partial x}\left(u(y)\right) = 0,$$

showing that the flow is incompressible. ■

The flow in Example 4.2 is an example of a *stratified flow*. (Density stratification occurs, for example, in river estuaries due to salinity changes, and in the atmosphere due to temperature changes, layers of autumn mist being a case in point.) Each fluid particle at a given height has the same density, and because there is no y-component of velocity, each particle stays at a constant height. This is an incompressible flow in stratified layers, as illustrated in Figure 4.3.

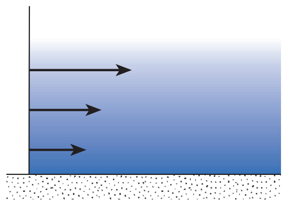


Figure 4.3 Stratified flow in which velocity increases with height, and density decreases with height

Applicability of the incompressible model

The incompressible fluid model is applicable, in particular, to any (real) fluid whose density may be modelled as constant at all points and at all times. For most liquids, this condition is satisfied provided that the temperature is constant. There is a small density variation with temperature for most liquids; for example, water decreases in density by about 4% as the temperature increases from 0°C to 100°C. The density variation of a liquid with pressure is comparatively small; for example, for water an increase of 1 atmosphere causes an increase in density of less than 0.005%.

However, even for gases, certain types of flow past a body at moderate speed can be modelled well as incompressible flows. In such flows, the pressure and density variations are relatively small. For example, density variations in air are less than 1% for flow speeds of up to about $50\,\mathrm{m\,s^{-1}}$.

Recall that $50\,\mathrm{m\,s^{-1}} = 180\,\mathrm{kph} \simeq 112\,\mathrm{mph}$.

The speed of sound in a fluid is a measure of how much the fluid density can change with pressure, since sound waves are transmitted in the fluid by density changes due to pressure, the more compressible fluids taking longer to transmit the pulses forming the wave.

At 20°C, the speed of sound for air is $343\ \mathrm{m\,s^{-1}}$; for pure water, it is $1482\ \mathrm{m\,s^{-1}}$. Water is denser and much less compressible than air.

The incompressible model breaks down for compressible fluids moving at high speeds (that is, speeds comparable to or greater than the speed of sound for the fluid). Compressible flows can be significantly different in kind from incompressible flows. For example, small pressure disturbances are propagated through a compressible fluid as wave motions, travelling at the speed of sound for that fluid. This contrasts with the incompressible fluid model, in which small disturbances are transmitted instantaneously to the whole region occupied by the fluid. The incompressible model implies that the speed of sound in the fluid is infinite.

A good estimate of the speed of sound in a gas is given by $\sqrt{\gamma p_0/\rho_0}$, where p_0, ρ_0 are the background pressure and density, and γ is a constant for the gas ($\gamma \simeq 1.4$ for air). This formula is derived in MS324 Block I *Chapter 1*.

Alternative forms of the continuity equation

To conclude this section, we now turn attention to the form of the continuity equation for two special types of steady incompressible flow. As you know, the mathematical expression of this equation is $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. We now find forms of the equation which do not involve differentiation.

In Subsection 2.2, you saw that there is no flow across a streamline. Imagine now the family of all streamlines which pass through a simple closed curve (see Figure 4.4); such a set of streamlines is called a **streamtube**. Since we are considering steady flow, each streamline is a pathline, and so we may consider the streamtube to be a model of a fixed boundary for the flow.

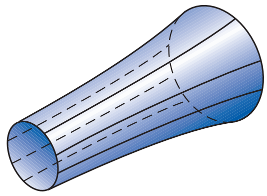


Figure 4.4 A streamtube

Consider the special case of a constant-density fluid and a streamtube which is such that across any plane normal section of it, the fluid velocity is constant and normal to the section (see Figure 4.5). Let the speed at sections 1 and 2 be u_1 and u_2, respectively. Then, since the flow is steady, the law of conservation of mass says that the mass of fluid crossing section 1 per unit time equals that at section 2; that is,

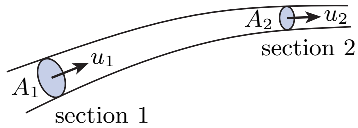


Figure 4.5

$$\rho\, u_1 A_1 = \rho\, u_2 A_2,$$

where A_1, A_2 are the areas of sections 1 and 2, and ρ is the constant density. Hence the volume flow rates are equal, and

$$u_1 A_1 = u_2 A_2. \tag{4.8}$$

It can be shown that Equation (4.8) is a consequence of the form $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ of the continuity equation, by applying Gauss' (Divergence) Theorem, from *Unit 4*, over the streamtube in Figure 4.5.

This form of the continuity equation does not involve differentiation, and can be used to model the steady flow of low-viscosity liquids (e.g. water) in pipes where the flow is approximately constant across a cross-section.

Exercise 4.8

A straight circular pipe carrying water has a cross-section which converges from an area of 0.3 m^2 to 0.15 m^2. If the speed of the water at the larger section is 1.8 $\mathrm{m\,s^{-1}}$, estimate the speed at the smaller section.

The next special case is that of unidirectional flow, in which $\mathbf{u}$ depends only on the space coordinate in the direction of flow (for example, $\mathbf{u} = u(x)\,\mathbf{i}$). In this case, any streamtube of unit thickness into the page, imagined from the real flow, will have rectangular normal sections (see Figure 4.6).

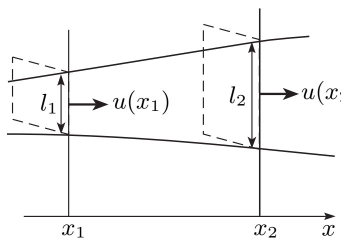


Figure 4.6

Exercise 4.9

(a) If the distance between the streamlines is l_1 at x_1 and l_2 at x_2 (see Figure 4.6), to what does Equation (4.8) reduce in this case?

(b) Figure 4.7 shows a section of a stationary river bore. Explain how to model this flow by means of a streamtube, and use your answer to part (a) to calculate the speed u.

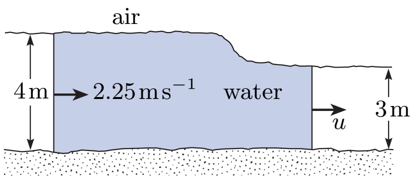


Figure 4.7

As you may have realised, the derivation of Equation (4.8) involved an approximation, in that if the streamtube expands or contracts, the fluid velocity across a plane section cannot be exactly normal to it at all points. In fact Equation (4.8) is exact if u_1 and u_2 are taken to be velocity components normal to the sections, but these will be close to the corresponding speeds if the rate of streamtube expansion or contraction at the sections is small. Similar remarks apply to the result of Exercise 4.9(a); the flow illustrated in Figure 4.6 must have some vertical component for the streamlines to diverge, but this is assumed small enough to ignore.

End-of-section exercises

Exercise 4.10

Classify the flows given by the following velocity fields as steady or non-steady, and as uniform or non-uniform.

(a) $\mathbf{u} = 3\,\mathbf{i} + 10\,\mathbf{j}$

(b) $\mathbf{u} = (3x + 4y)\,\mathbf{i} + 10\,\mathbf{j}$

(c) $\mathbf{u} = \dfrac{2t}{r}\,\mathbf{e}_\theta \qquad (r \neq 0)$

(d) $\mathbf{u} = 3\,\mathbf{e}_r + \dfrac{2}{r}\,\mathbf{e}_\theta \qquad (r \neq 0)$

(e) $\mathbf{u} = (3t + 1)\,\mathbf{i} + 4\,\mathbf{j}$

Exercise 4.11

Find the acceleration following the motion for each of the velocity fields below.

(a) $\mathbf{u} = Uy\,\mathbf{i}$, where U is constant

(b) $\mathbf{u} = U(t)y\,\mathbf{i}$

(c) $\mathbf{u} = U\left(1 + \dfrac{a^2}{r^2}\right)\sin\theta\,\mathbf{e}_\theta$, where U is constant (polar coordinates)

Exercise 4.12

Which of the following velocity fields could represent incompressible flows?

(a) $\mathbf{u} = U\cos\alpha\,\mathbf{i} + U\sin\alpha\,\mathbf{j} \qquad (U, \alpha \text{ constants})$

(b) $\mathbf{u} = -\dfrac{Ky}{x^2 + y^2}\,\mathbf{i} + \dfrac{Kx}{x^2 + y^2}\,\mathbf{j} + 2xy\,\mathbf{k} \qquad (K \text{ constant})$

(c) $\mathbf{u} = U\cos\theta\left(1 - \dfrac{a^2}{r^2}\right)\mathbf{e}_r - U\sin\theta\left(1 + \dfrac{a^2}{r^2}\right)\mathbf{e}_\theta \qquad (U, a \text{ constants})$

(d) $\mathbf{u} = \dfrac{K}{r}\,\mathbf{e}_\theta \qquad (r \neq 0, K \text{ constant})$

Exercise 4.13

Consider the steady flow of a constant-density fluid through a pipe of radius a (constant) which splits into two pipes, each of radius $\frac{1}{2}a$, as shown in Figure 4.8, all motion being horizontal. Express the speeds at section 2 in the narrower pipes in terms of the speed u_1 at section 1, mentioning any assumptions that you make about the nature of the flow.

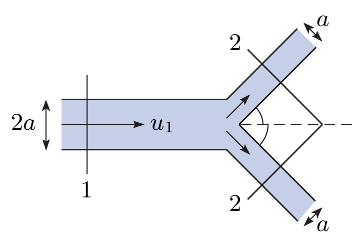


Figure 4.8

5 Euler's equation

Section 4 provided a mathematical framework in which the properties of a fluid in motion can be described as functions of position and time. The continuity equation provides a relationship between density and velocity changes, and takes on the simple form $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ for an incompressible fluid. Any vector field satisfying this equation can be used to model such a fluid in motion.

Many problems in fluid mechanics involve the flow of a fluid past a solid boundary, and it is often important to determine the force produced on a solid body by the fluid flowing past it. A simple example was described in *Unit 1*, where we investigated the force on a lock gate due to the water in a canal. In this case the fluid was at rest. In Section 3 of this unit, we found the stream function, and the streamline pattern, for the flow of a constant-density fluid past a cylinder. From this stream function the velocity of the fluid at any point can be found. The effect on the cylinder of the fluid flowing past it can be described by the net force on the cylinder due to the fluid. Other examples where the force is important are the aerodynamic forces on an aircraft (the lift and drag), the drag force on a motor car and the force on a bend in a pipe caused by fluid flowing within it. All of these arise from a momentum change of the fluid.

See Exercises 3.4 and 3.7.

In this section, an equation of motion is developed which relates the change in momentum to the forces acting in a fluid. We continue with the inviscid model, so that forces due to viscosity will not be included in the equation of motion.

5.1 Derivation of Euler's equation

The linear momentum conservation equation, sometimes called the equation of motion, is an application of Newton's Second Law to a region moving with the fluid and containing a fixed mass of fluid. To obtain an equation of motion that is valid at each point in the fluid, the region is shrunk to a point, as was done in the derivation of the continuity equation in *Unit 4*. We can do this because, in the continuum model, every region enclosing a point, however small, contains some fluid. We equate the rate of change of linear momentum of the region to the net force on the region.

The mass is fixed because the moving region always contains the same set of fluid particles.

Rate of change of linear momentum

Consider a region B of fluid. Divide this region into N fluid volume elements, so that the ith volume element has volume δV_i and contains the point P_i, for $i = 1, 2, \dots, N$ (see Figure 5.1).

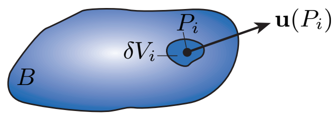


Figure 5.1

The linear momentum of the ith element, $\delta \mathbf{p}_i$, is given approximately by the product of the fluid velocity at P_i and the mass δM_i of the element, that is,

Linear momentum was defined in MST209 *Unit 19*. It is customary to denote the momentum vector by $\mathbf{p}$. This is unconnected with the (scalar) pressure p, introduced later in the argument.

$$\delta \mathbf{p}_i \simeq \mathbf{u}(P_i)\,\delta M_i.$$

The total linear momentum of the fluid within region B is the sum

$$\mathbf{p} = \sum_{i=1}^{N} \delta \mathbf{p}_i \simeq \sum_{i=1}^{N} \mathbf{u}(P_i)\,\delta M_i.$$

The rate of change of linear momentum (following the motion) is

$$\frac{d\mathbf{p}}{dt} \simeq \frac{d}{dt}\left(\sum_{i=1}^{N} \mathbf{u}(P_i)\,\delta M_i\right) \simeq \sum_{i=1}^{N} \frac{d\mathbf{u}}{dt}(P_i)\,\delta M_i.$$

In this differentiation, each δM_i is constant because the volume element always contains the same fluid as it moves with the fluid.

Writing the mass in terms of the volume for each element, we have

$$\frac{d\mathbf{p}}{dt} \simeq \sum_{i=1}^{N} \frac{d\mathbf{u}}{dt}(P_i)\,\rho(P_i)\,\delta V_i.$$

Here we put $\delta M_i \simeq \rho(P_i)\,\delta V_i$, where ρ is the density of the fluid.

Taking smaller and smaller elements, the changes in ρ and $d\mathbf{u}/dt$ throughout each element are reduced, and the sum on the right-hand side becomes a closer approximation to the rate of change of linear momentum of the fluid in region B. In the limit as each element shrinks to a point, the summation becomes a volume integral, and so the rate of change of linear momentum of B is

$$\frac{d\mathbf{p}}{dt} = \int_B \rho\,\frac{d\mathbf{u}}{dt}\,dV. \tag{5.1}$$

Two types of force

Next, we find an expression for the net force on B. The forces acting on a fluid element may be of two types:

(i) surface forces due to the action of the rest of the fluid on the fluid element under consideration;

(ii) body forces which are forces whose origin is external to the fluid, like gravitational and electromagnetic forces.

The different types of force acting on a fluid element were introduced in *Unit 1* Subsection 3.4.

In the discussion of surface forces in *Unit 1*, you saw that surface forces are in general of two kinds — forces normal to the boundary surface of the fluid element, which are called pressure forces, and forces acting in the tangent plane to the surface, which are called viscous forces and are due to the viscosity of the fluid. These viscous forces do not occur in the inviscid model. In this unit, we assume that all the surface forces are normal to the surface of the fluid element. *Unit 8* will introduce viscous forces into the model.

Surface forces

Consider the surface S of the region B as being split into N small surface elements, so that the ith surface element has area δA_i and contains the point Q_i, for $i = 1, 2, \ldots, N$ (see Figure 5.2). The surface force acting on the ith surface element, exerted by the surrounding fluid, is given by

$$\delta\mathbf{F}_i \simeq -p(Q_i)\,\mathbf{n}(Q_i)\,\delta A_i,$$

Pressure and the surface force due to pressure were discussed in *Unit 1* Subsection 3.2.

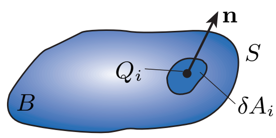


Figure 5.2

where the pressure p is a time-dependent scalar field and $\mathbf{n}$ is the outward unit normal vector, shown in Figure 5.2. The negative sign expresses the fact that the rest of the fluid exerts an inward pressure at each point Q_i on the surface S. In all applications, the pressure p is positive or zero.

The net surface force acting on the surface S is

$$\mathbf{F}_S = \sum_{i=1}^{N} \delta\mathbf{F}_i \simeq \sum_{i=1}^{N} \left(-p(Q_i)\,\mathbf{n}(Q_i)\,\delta A_i\right).$$

Taking smaller and smaller surface elements, the limit of the sum as each element shrinks to a point is the surface integral

$$\mathbf{F}_S = \int_S (-p\,\mathbf{n})\,dA. \tag{5.2}$$

We next convert this surface integral to a volume integral (since the body forces on region B will also be given by a volume integral). Let $\mathbf{e}$ be an arbitrary constant unit vector. Then

$$\int_B \operatorname{div}(-p\,\mathbf{e})\,dV = \int_S (-p\,\mathbf{e}) \cdot \mathbf{n}\,dA \qquad \text{(by Gauss' Theorem)}$$

$$= \mathbf{e} \cdot \int_S (-p\,\mathbf{n})\ dA \qquad \text{(since } \mathbf{e} \text{ is constant).}$$

See *Unit 4* Subsection 3.2.

Also,

$$\int_B \operatorname{div}(-p\,\mathbf{e})\,dV = \int_B (\mathbf{e} \cdot \mathbf{grad}(-p) + (-p)\ \operatorname{div}\mathbf{e})\ dV$$

$$= \mathbf{e} \cdot \int_B \mathbf{grad}(-p)\,dV \qquad \text{(since } \mathbf{e} \text{ is constant).}$$

See Equation (5.2) of *Unit 4*, with div for $\boldsymbol{\nabla}\cdot$ and **grad** for $\boldsymbol{\nabla}$.

The last two results feature the same left-hand side, so the two right-hand sides must equal each other, that is,

$$\mathbf{e} \cdot \int_S (-p\,\mathbf{n})\ dA = \mathbf{e} \cdot \int_B \mathbf{grad}(-p)\,dV.$$

Since $\mathbf{e}$ is arbitrary, this means that

$$\int_S (-p\,\mathbf{n})\ dA = \int_B \mathbf{grad}(-p)\,dV,$$

so that, from Equation (5.2),

$$\mathbf{F}_S = \int_B \mathbf{grad}(-p)\,dV = \int_B (-\boldsymbol{\nabla} p)\ dV. \tag{5.3}$$

The same argument works with any scalar field ϕ in place of $-p$, giving a corollary to Gauss' Theorem:

$$\int_S \phi\,\mathbf{n}\,dA = \int_B \mathbf{grad}\,\phi\,dV.$$

Body forces

Consider now the body forces. These are forces which act at every point in a fluid element, and it is convenient to model such forces by vector fields per unit mass. For example, the force of gravity on a body of mass m is $-mg\,\mathbf{k}$; we could model the acceleration due to gravity by the vector field $\mathbf{F}_{\text{grav}} = -g\,\mathbf{k}$, and then the force on the body becomes $m\,\mathbf{F}_{\text{grav}}$. If the total body force per unit mass is denoted by $\mathbf{F}$, then the force acting on the fluid element at P_i (see Figure 5.1) is

$$\delta\mathbf{F}_i \simeq \mathbf{F}(P_i)\,\rho(P_i)\,\delta V_i \qquad (i = 1, 2, \ldots, N).$$

$\mathbf{F}$ is the vector sum of all possible body forces per unit mass. If $\mathbf{F}_{\text{grav}}$ and $\mathbf{F}_{\text{mag}}$ represent the body forces per unit mass of gravity and a magnetic field, then

$$\mathbf{F} = \mathbf{F}_{\text{grav}} + \mathbf{F}_{\text{mag}}.$$

The net body force acting on the fluid in the region B is therefore given by the volume integral

$$\mathbf{F}_B = \int_B \rho\,\mathbf{F}\,dV. \tag{5.4}$$

Total force and Newton's Second Law

The total force $\mathbf{F}_R$ acting on the region B is the sum of the net surface and body forces. Using Equations (5.3) and (5.4), we have

$$\mathbf{F}_R = \mathbf{F}_S + \mathbf{F}_B = \int_B (-\boldsymbol{\nabla} p)\ dV + \int_B \rho\,\mathbf{F}\,dV = \int_B (-\boldsymbol{\nabla} p + \rho\,\mathbf{F})\,dV.$$

Equating this total force on the region B to the rate of change of linear momentum, in Equation (5.1), gives

$$\int_B \rho\,\frac{d\mathbf{u}}{dt}\,dV = \int_B (-\boldsymbol{\nabla} p + \rho\,\mathbf{F})\,dV$$

or, writing this as one volume integral,

$$\int_B \left(\rho\,\frac{d\mathbf{u}}{dt} + \boldsymbol{\nabla} p - \rho\,\mathbf{F}\right)\ dV = \mathbf{0}.$$

This is the same as applying Newton's Second Law in solid mechanics. In MST209 *Unit 19* this was expressed as

$$\mathbf{F}^{\text{ext}} = \dot{\mathbf{P}}.$$

The equivalent here is

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}_R.$$

Since this equation is derived for an arbitrary region of fluid B, the usual assumption of smoothness of the integrand enables us to write

The idea of 'dropping the integral sign' because B is an arbitrary region was discussed in the derivation of the continuity equation in *Unit 4*.

$$\rho \frac{d\mathbf{u}}{dt} = -\boldsymbol{\nabla} p + \rho \mathbf{F}. \tag{5.5}$$

This is known as **Euler's equation**, and is satisfied at every point in an inviscid fluid.

Equation (5.5) is also called *Euler's momentum equation*. *Unit 1* referred to Euler's *equations* (plural); it may be regarded as one vector equation or three scalar equations.

Exercise 5.1

Show that, for a fluid at rest where the only body force is due to gravity, Euler's equation reduces to the basic equation of fluid statics,

$$\frac{dp}{dz} = -\rho g, \qquad \text{where } p = p(z).$$

This equation was derived in *Unit 1* Subsection 4.1.

Euler's equation and the continuity equation together provide a description for a fluid in motion. The acceleration term on the left-hand side of Euler's equation can be written in terms of the local rate of change of $\mathbf{u}$ at a fixed point and the convective rate of change. We then have

See Equation (4.5).

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho\,(\mathbf{u} \cdot \boldsymbol{\nabla})\,\mathbf{u} = -\boldsymbol{\nabla} p + \rho \mathbf{F}. \tag{5.6}$$

Here $\partial \mathbf{u}/\partial t$ is the local time rate of change of $\mathbf{u}$; in steady flows, this is zero. The term $(\mathbf{u} \cdot \boldsymbol{\nabla})\,\mathbf{u}$ is non-linear in the velocity and is the convective acceleration. As you saw in Exercise 4.5, this convective acceleration can be non-zero even in steady flows.

Euler's equation (5.5) or (5.6) relates the pressure, p, and the total body force per unit mass, $\mathbf{F}$, to the velocity vector field, $\mathbf{u}$. This equation is fundamental to much of the rest of the course. If viscosity is neglected (as here), then Euler's equation must be satisfied at each point in a fluid and at all times.

Dividing Equation (5.6) by ρ gives

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\,\mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla} p + \mathbf{F}. \tag{5.7}$$

The acceleration following the motion, seen on the left-hand side, is equal to the total force per unit mass seen on the right-hand side.

The forces causing the acceleration consist of the body forces, such as weight (due to gravity), together with the surface force caused by pressure. A uniform pressure by itself does not cause acceleration since, on any element, the pressure on one face is matched by an equal pressure on an opposite face, and so no motion occurs. It is natural to expect the *gradient* of the pressure, i.e. the spatial rate of change of p, to be the accelerating surface force.

Essentially, Euler's equation gives a means of finding the pressure, p, when the body forces per unit mass, $\mathbf{F}$, and the velocity field, $\mathbf{u}$, are known. The possibly daunting form of Euler's equation suggests that further modelling assumptions need to be made in order to make progress with its solution.

The velocity field $\mathbf{u}$ must satisfy the continuity equation. In general, Euler's equation is also needed to determine $\mathbf{u}$ in full.

Exercise 5.2

Find the form of Euler's equation for the irrotational steady flow of a fluid of constant density ρ_0 in a conservative force field with scalar potential Ω.

Hint: To simplify $(\mathbf{u} \cdot \boldsymbol{\nabla})\,\mathbf{u}$, use the identity

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\,\mathbf{u} = \boldsymbol{\nabla}(\tfrac{1}{2}u^2) - \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}), \qquad \text{where } u = |\mathbf{u}|.$$

If $\mathbf{F}$ is conservative, then there is a scalar potential field Ω such that $\mathbf{F} = \boldsymbol{\nabla}\Omega$ (see MST209 *Unit 24*, but with Ω in place of $-U$).

For the vector identity given here, see *Unit 4*, Exercise 5.5.

The formula obtained in Exercise 5.2 provides some hope for progress. Euler's equation cannot, in general, be integrated. However, for a conservative force field, Euler's equation can be expressed as

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) = -\frac{1}{\rho}\,\boldsymbol{\nabla} p + \boldsymbol{\nabla}\Omega - \boldsymbol{\nabla}(\tfrac{1}{2}u^2), \tag{5.8}$$

and now both sides can be integrated along the streamlines to give a form of energy equation. This equation and its application to some flow problems form the subject of *Unit 6*.

5.2 Applications of Euler's equation

In one sense, much of the rest of the course uses the continuity equation and Euler's equation to solve problems in fluid mechanics. However, viscosity has been neglected, and this can drastically affect the behaviour of a fluid in the region near a body and, by convection, downstream of it. Viscosity will be included in the model in *Unit 8*.

The strategy for solving fluid flow problems that is developed in this unit depends on first finding the velocity field $\mathbf{u}$. One way of doing so is to model the flow field and boundaries by the superposition of several basic flow types, e.g. uniform flow, source, doublet, vortex, using perhaps the streamlines or experience to suggest suitable basic flow types whose parameters can be adjusted to fit the particular flow circumstances. This may involve use of the stream function to compare the model's streamlines with the flow-visualised streamlines. From the velocity field, Euler's equation is used to determine the pressure.

Use of the stream function takes the continuity equation into account.

Example 5.1

A particular flow of a constant-density fluid is given by the velocity field

$$\mathbf{u} = 2yt\,\mathbf{i} - 2x\,\mathbf{j},$$

and the body force per unit mass is

$$\mathbf{F} = 2y\,\mathbf{i}.$$

Find an expression for the pressure at a general point (x, y).

Solution

Euler's equation, in the form (5.7), is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\,\boldsymbol{\nabla} p + \mathbf{F}.$$

With $\mathbf{u} = 2yt\,\mathbf{i} - 2x\,\mathbf{j}$ and $\mathbf{F} = 2y\,\mathbf{i}$, this becomes

$$2y\,\mathbf{i} + \left(2yt\frac{\partial}{\partial x} - 2x\frac{\partial}{\partial y}\right)(2yt\,\mathbf{i} - 2x\,\mathbf{j}) = -\frac{1}{\rho}\,\boldsymbol{\nabla} p + 2y\,\mathbf{i},$$

which reduces to

$$-4xt\,\mathbf{i} - 4yt\,\mathbf{j} = -\frac{1}{\rho}\,\boldsymbol{\nabla} p. \tag{5.9}$$

The x-component of this equation gives

$$\frac{\partial p}{\partial x} = 4x\rho t$$

which, since ρ is a constant, integrates to give

$$p(x, y, t) = 2x^2\rho t + f(y, t), \tag{5.10}$$

where f is an arbitrary function of two variables. The y-component of Equation (5.9) then gives

$$\frac{\partial p}{\partial y} = 4y\rho t = \frac{\partial f}{\partial y},$$

which integrates to give

$$f(y, t) = 2y^2\rho t + g(t),$$

where g is an arbitrary function. Hence, from Equation (5.10), the pressure at (x, y) and time t is

$$p(x, y, t) = 2(x^2 + y^2)\rho t + g(t).$$ ■

Exercise 5.3

The stream function for a source of strength $2\pi m$ at $(0, b)$, outside a wall along the x-axis (see Figure 5.3), is

$$\psi = m\left(\arctan\left(\frac{y-b}{x}\right) + \arctan\left(\frac{y+b}{x}\right)\right).$$

Find the velocity at the wall, and deduce the pressure at any point of the wall. Assume that the fluid is inviscid and of constant density, that the flow is two-dimensional and irrotational, that body forces can be neglected, and that the pressure is p_0 at $x = \pm\infty$.

Hint: Use the result of Exercise 5.2 and the remark after that exercise about integrating along a streamline.

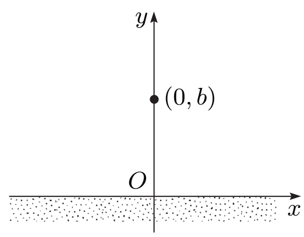


Figure 5.3

End-of-section exercise

Exercise 5.4

A two-dimensional flow of a constant-density, inviscid fluid is given by the velocity field

$$\mathbf{u} = -2x\,\mathbf{i} + (2y + 5t)\,\mathbf{j},$$

and the body force per unit mass is

$$\mathbf{F} = 5\,\mathbf{j}.$$

Find $\boldsymbol{\nabla} p$, using Euler's equation in the form (5.6). Then integrate to find p.

Solutions to the exercises

Section 1

Solution 1.1

From Equation (1.3), we have $d\mathbf{r}/dt = \mathbf{u} = U\,\mathbf{i}$, or

$$\frac{dx}{dt} = U, \qquad \frac{dy}{dt} = 0.$$

Integrating these equations gives

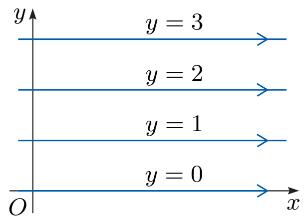
$$x = Ut + C, \qquad y = D,$$

where C and D are arbitrary constants.

The pathline through (x_0, y_0) at time $t = 0$ is given by

$$x = Ut + x_0, \qquad y = y_0.$$

So each pathline is a line parallel to the x-axis, the particle being at $(Ut + x_0, y_0)$ at time t. Since $U > 0$, each fluid particle moves steadily to the right.



Solution 1.2

(a) From Equations (1.3) and (1.4), we have

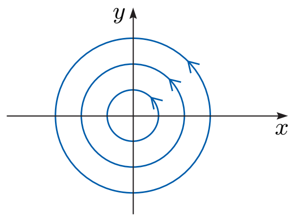
$$\frac{dr}{dt} = 0, \qquad r\frac{d\theta}{dt} = \frac{k}{r} \qquad \text{or} \qquad \frac{d\theta}{dt} = \frac{k}{r^2}.$$

Integrating gives $r = C$, an arbitrary constant, and

$$\theta = \frac{kt}{C^2} + D \qquad (\text{since } r = C),$$

where D is an arbitrary constant.

These pathlines are circles with centre at the origin; particles move steadily around these circles, the speed being slower further away from the origin, because $u_\theta = k/r$ decreases as r increases. Since $\dot{\theta} > 0$, the fluid moves in an anticlockwise direction.



(b) From Equations (1.3) and (1.4), we have

$$\frac{dx}{dt} = x(3t+1), \qquad \frac{dy}{dt} = 2y \qquad (x > 0,\ y > 0),$$

or

$$\frac{1}{x}\frac{dx}{dt} = 3t + 1, \qquad \frac{1}{y}\frac{dy}{dt} = 2.$$

Integrating these gives (since $x > 0,\ y > 0$)

$$\ln x = \tfrac{3}{2}t^2 + t + C, \qquad \ln y = 2t + D,$$

where C and D are arbitrary constants. Taking exponentials of both sides in each case,

$$x = A\exp\left(\tfrac{3}{2}t^2 + t\right), \qquad \text{where } A = e^C,$$

$$y = B\exp(2t), \qquad \text{where } B = e^D.$$

From the second equation, $t = \frac{1}{2}\ln(y/B)$, giving

$$x = A\exp\left[\tfrac{3}{8}\left(\ln\left(\frac{y}{B}\right)\right)^2 + \tfrac{1}{2}\ln\left(\frac{y}{B}\right)\right].$$

Solution 1.3

From Equation (1.5), we have

$$\frac{dy}{dx} = \frac{u_2}{u_1} = \frac{U\sin\alpha}{U\cos\alpha} = \tan\alpha.$$

Integrating gives

$$y = (\tan\alpha)\,x + C,$$

where C is an arbitrary constant.

If (x_0, y_0) lies on a streamline, then

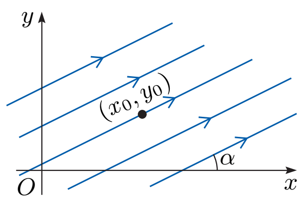
$$y_0 = (\tan\alpha)\,x_0 + C,$$

so that, on subtraction,

$$y - y_0 = (x - x_0)\tan\alpha.$$

(If $\alpha = \pm\frac{1}{2}\pi$, then $\mathbf{u} = \pm U\,\mathbf{j}$ and the streamlines are given by $x = x_0$.)

The streamlines are (at all times) the parallel lines inclined at angle α to the x-axis. The figure below shows directions on the streamlines for the case $U > 0$.



Solution 1.4

(a) Since $u_r = 0$, the streamlines are given by $r =$ constant. Thus the streamlines are circles with centre at the origin. (Note that, because this flow is steady, the streamlines are identical to the pathlines, as found in Solution 1.2(a).)

(b) From Equation (1.5),

$$\frac{dy}{dx} = \frac{2y}{x(3t+1)} \quad (x \neq 0) \qquad \text{or} \qquad \frac{1}{y}\frac{dy}{dx} = \frac{2}{x(3t+1)}.$$

With t constant ($\neq -\frac{1}{3}$), we integrate with respect to x to obtain (since $x > 0, y > 0$)

$$\ln y = \frac{2}{3t+1}\ln x + C,$$

where C is an arbitrary constant, so that

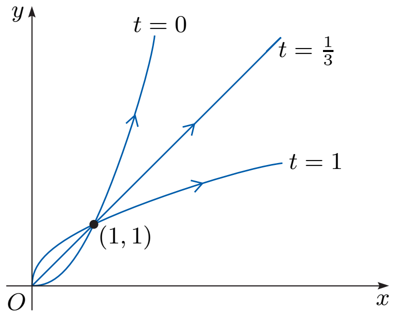
$$y = Ax^{2/(3t+1)} \quad (t \neq -\tfrac{1}{3}), \qquad \text{where } A = e^C.$$

If $(x_0, y_0) = (1, 1)$ then

$$1 = A \times 1^{2/(3t+1)} = A, \qquad \text{since } 1^{2/(3t+1)} = 1.$$

Thus $A = 1$ for the streamline through (1,1) at any time t.

The streamlines $y = x^2$, $y = x$, $y = x^{1/2}$ (for $t = 0$, $t = \frac{1}{3}$, $t = 1$, respectively) are shown in the following figure. (Note that the equations differ from those of the pathlines, given in Solution 1.2(b).)



Solution 1.5

From Equations (1.3) and (1.4), the pathlines are given by

$$\frac{dx}{dt} = a\cos(\omega t), \qquad \frac{dy}{dt} = a\sin(\omega t).$$

Integrating gives

$$x = \frac{a}{\omega}\sin(\omega t) + C, \qquad y = -\frac{a}{\omega}\cos(\omega t) + D,$$

where C and D are arbitrary constants.

When $t = 0$, $(x, y) = (0, 0)$, so that

$$C = 0 \qquad \text{and} \qquad D = \frac{a}{\omega}.$$

The required pathline is

$$x = \frac{a}{\omega}\sin(\omega t), \qquad y = \frac{a}{\omega}\left(1 - \cos(\omega t)\right),$$

which, on eliminating t, gives

$$x^2 + \left(y - \frac{a}{\omega}\right)^2 = \frac{a^2}{\omega^2}.$$

This is a circle with radius a/ω and centre $(0, a/\omega)$.

From Equation (1.5), the streamlines are given by

$$\frac{dy}{dx} = \frac{a\sin(\omega t)}{a\cos(\omega t)} = \tan(\omega t),$$

which, on integrating, gives

$$y = x\tan(\omega t) + E,$$

where E is an arbitrary constant. When $t = 0$, $(x, y) = (0, 0)$, so that $E = 0$. The required streamline is $y = 0$.

Solution 1.6

Since this is a steady flow, the streamlines and the pathlines are the same. We shall determine their equations from the differential equation (1.6) for the streamlines:

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{u_r}{u_\theta} = \frac{r\cos(\frac{1}{2}\theta)}{r\sin(\frac{1}{2}\theta)}.$$

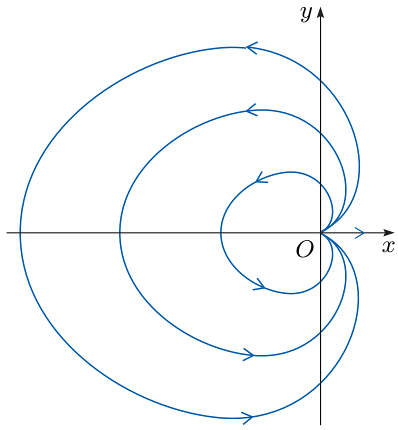
Integrating with respect to θ, we obtain

$$\ln r = 2\ln|\sin\tfrac{1}{2}\theta| + C \qquad (\theta \neq 0),$$

where C is an arbitrary constant. Taking exponentials of both sides gives

$$r = A\sin^2\left(\tfrac{1}{2}\theta\right), \qquad \text{where } A = e^C.$$

(These streamlines, shown in the following figure, are heart-shaped curves known as *cardioids*. Note that $\theta = 0$, the positive x-axis, which was excluded above, is also a streamline.)



Section 2

Solution 2.1

From Equations (2.1), we have

$$u_1 = ay = \frac{\partial\psi}{\partial y}, \qquad u_2 = 0 = -\frac{\partial\psi}{\partial x}.$$

Integrating the first equation with respect to y gives

$$\psi(x, y) = \tfrac{1}{2}ay^2 + f(x),$$

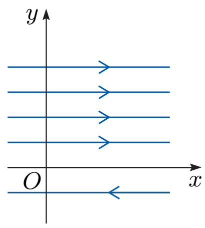
where f is an arbitrary function. Hence

$$\frac{\partial\psi}{\partial x} = f'(x) = 0,$$

so $f(x) = C$, where C is a constant, giving

$$\psi(x, y) = \tfrac{1}{2}ay^2 + C.$$

The figure below shows the contour lines $\psi =$ constant, which are the lines $y =$ constant. (Since $a > 0$, the flow is from left to right for $y > 0$ and from right to left for $y < 0$. There is no flow along $y = 0$.)



Solution 2.2

(a) From Equations (2.2), we have

$$u_r = 0 = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = \frac{k}{r} = -\frac{\partial\psi}{\partial r}.$$

Integrating the second equation with respect to r gives

$$\psi(r, \theta) = -k\ln r + f(\theta),$$

where f is an arbitrary function. Hence

$$\frac{\partial\psi}{\partial\theta} = f'(\theta) = 0,$$

so $f(\theta) = C$, where C is a constant, giving

$$\psi(r, \theta) = -k\ln r + C.$$

(b) From Equations (2.2), we have

$$u_r = \frac{m}{r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = 0 = -\frac{\partial\psi}{\partial r}.$$

Integrating the first equation with respect to θ gives

$$\psi(r,\theta) = m\theta + f(r),$$

where f is an arbitrary function. Hence

$$\frac{\partial\psi}{\partial r} = f'(r) = 0,$$

so $f(r) = C$, where C is a constant, giving

$$\psi(r,\theta) = m\theta + C.$$

Solution 2.3

(a) From Equations (2.2), we have

$$u_r = 0 = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = Kr = -\frac{\partial\psi}{\partial r}.$$

Integrating the second equation with respect to r gives

$$\psi(r,\theta) = -\tfrac{1}{2}Kr^2 + f(\theta),$$

where f is an arbitrary function. Hence

$$\frac{\partial\psi}{\partial\theta} = f'(\theta) = 0,$$

so $f(\theta) = C$, where C is a constant, giving

$$\psi(r,\theta) = -\tfrac{1}{2}Kr^2 + C.$$

(b) From Equations (2.1), we have

$$u_1 = y = \frac{\partial\psi}{\partial y}, \qquad u_2 = -(x-2t) = -\frac{\partial\psi}{\partial x}.$$

Integrating the second equation with respect to x gives

$$\psi(x,y) = \tfrac{1}{2}x^2 - 2tx + f(y),$$

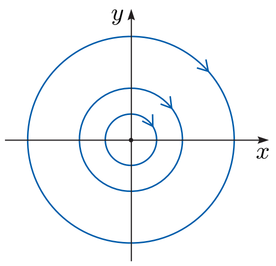
where f is an arbitrary function. (Note that t is kept fixed throughout; hence we write $f(y)$ rather than $f(y,t)$ here.) Now

$$\frac{\partial\psi}{\partial y} = f'(y) = y,$$

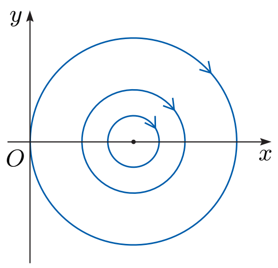
so that $f(y) = \tfrac{1}{2}y^2 + C$, where C is a constant. Thus

$$\psi(x,y) = \tfrac{1}{2}(x^2+y^2) - 2tx + C \qquad \text{(with } t \text{ constant)}.$$

If $t = 0$, then $\psi(x,y) = \tfrac{1}{2}(x^2+y^2) + C$, and the streamlines $\psi =$ constant are circles with centre at the origin. Since $u_1 = y$ (or since $u_2 = -x$), the flow is in the clockwise direction.



If $t = 1$, then $\psi(x,y) = \tfrac{1}{2}(x^2+y^2) - 2x + C$, and the streamlines are circles with centre $(2,0)$. Again, the flow is clockwise.



Solution 2.4

Let P and Q be two points on a streamline, and let PQ be the segment of the streamline between P and Q. Then $\psi(P) = \psi(Q)$, since ψ is constant along a streamline. Therefore V, the volume flow rate across PQ, is given by

$$V = \int_P^Q \frac{d\psi}{ds}\,ds = \psi(Q) - \psi(P) = 0.$$

Solution 2.5

(a) The unit vector normal to the surface is $\mathbf{n} = \mathbf{e}_r$. Hence the normal boundary condition is

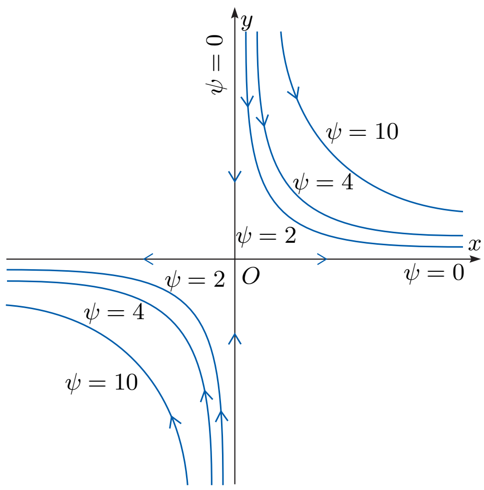
$$\mathbf{u}\cdot\mathbf{n} = u_r = 0 \qquad \text{on } r = a.$$

(b) In addition to the normal boundary condition from part (a), the tangential component of $\mathbf{u}$ must be zero on the cylinder; that is,

$$\mathbf{u}\cdot\mathbf{e}_\theta = u_\theta = 0 \qquad \text{on } r = a.$$

Solution 2.6

When $\psi = 0$, the streamline is $xy = 0$, which describes the x- and y-axes. If $\psi = 2$, then $y = 1/x$; if $\psi = 4$, then $y = 2/x$; if $\psi = 10$, then $y = 5/x$. These streamlines are shown below.



The velocity vector field is

$$\mathbf{u} = \frac{\partial\psi}{\partial y}\,\mathbf{i} - \frac{\partial\psi}{\partial x}\,\mathbf{j} = 2x\,\mathbf{i} - 2y\,\mathbf{j},$$

which gives the directions of flow, as shown.

One possible flow would be past a right-angled corner in the first quadrant, which is modelled by choosing as boundaries the streamlines

$$x = 0,\ y \ge 0 \qquad \text{and} \qquad y = 0,\ x \ge 0.$$

Solution 2.7

(a) From Equations (2.1), we have

$$u_1 = -2y = \frac{\partial\psi}{\partial y}, \qquad u_2 = -2x = -\frac{\partial\psi}{\partial x}.$$

Integrating the second equation with respect to x gives

$$\psi(x,y) = x^2 + f(y),$$

where f is an arbitrary function. Hence

$$\frac{\partial\psi}{\partial y} = f'(y) = -2y,$$

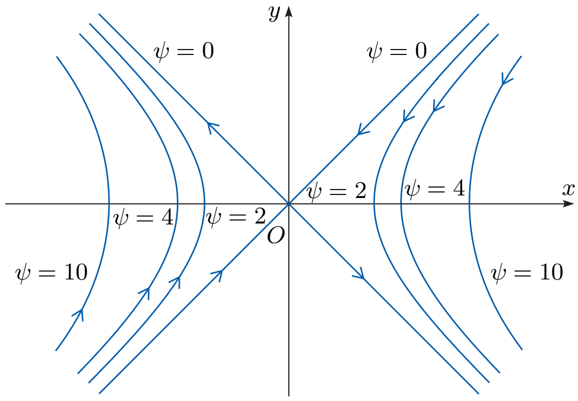
so that

$$f(y) = -y^2 + C, \qquad \text{where } C \text{ is a constant.}$$

Thus

$$\psi(x,y) = x^2 - y^2 \qquad (\text{taking } C = 0).$$

(b) When $\psi = 0$, the streamline is $x^2 - y^2 = 0$, which describes the pair of straight lines $x = y$, $x = -y$. If $\psi = 2$, then $x^2 - y^2 = 2$; if $\psi = 4$, then $x^2 - y^2 = 4$; if $\psi = 10$, then $x^2 - y^2 = 10$. These are hyperbolas, as shown below.



(c) One possible flow would be in the region for which $x > |y|$, with boundary $x = y$ and $x = -y$ $(x > 0)$.

Section 3

Solution 3.1

The continuity equation here is $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$. Now $\mathbf{u}_1$ and $\mathbf{u}_2$ must each satisfy this equation, so

$$\boldsymbol{\nabla} \cdot \mathbf{u}_1 = 0 \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \mathbf{u}_2 = 0.$$

Hence,

$$\boldsymbol{\nabla} \cdot (\mathbf{u}_1 + \mathbf{u}_2) = \boldsymbol{\nabla} \cdot \mathbf{u}_1 + \boldsymbol{\nabla} \cdot \mathbf{u}_2 = 0,$$

showing that $\mathbf{u}_1 + \mathbf{u}_2$ satisfies the continuity equation.

Solution 3.2

The method required is very similar to that of Example 3.1. By the Principle of Superposition, the overall flow, due to the combination of two sources, has stream function

$$\psi = m(\theta_1 + \theta_2),$$

where θ_1, θ_2 are defined in Example 3.1. As before,

$$\tan\theta_1 = \frac{y}{x-a}, \qquad \tan\theta_2 = \frac{y}{x+a},$$

so that

$$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

$$= \frac{y/(x-a) + y/(x+a)}{1 - y^2/(x^2 - a^2)} = \frac{2xy}{x^2 - y^2 - a^2}.$$

The stream function of the combined flow is therefore

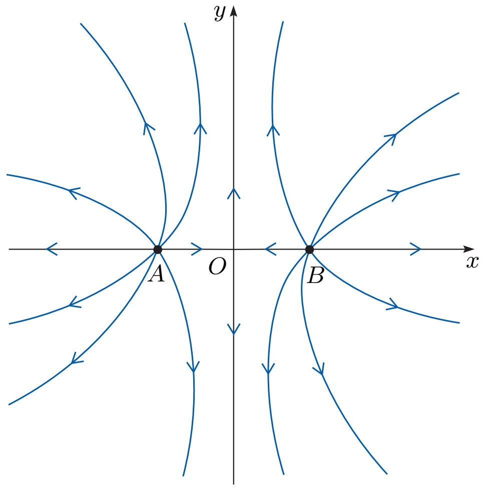
$$\psi(x,y) = m \arctan\left(\frac{2xy}{x^2 - y^2 - a^2}\right).$$

The equation for streamlines is $\psi = \text{constant}$, that is,

$$\frac{2xy}{x^2 - y^2 - a^2} = \text{constant} \quad \text{or} \quad x^2 - y^2 - a^2 = 2Cxy,$$

where C is constant. The two axes, $x = 0$ and $y = 0$, are also streamlines.

(Apart from the two axes, these streamlines are rectangular hyperbolas ('rectangular' meaning that the two asymptotes are at right angles) passing through $(\pm a, 0)$. The angle of these asymptotes depends on the choice of C. Some streamlines are sketched below.)



Solution 3.3

The stream function for the doublet is

$$\psi(r,\theta) = -\frac{\lambda \sin\theta}{r}.$$

From Equations (2.2), the corresponding velocity components are

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = -\frac{\lambda\cos\theta}{r^2}, \qquad u_\theta = -\frac{\partial\psi}{\partial r} = -\frac{\lambda\sin\theta}{r^2}.$$

Hence the velocity field is given by

$$\mathbf{u} = -\frac{\lambda}{r^2}\left(\cos\theta\,\mathbf{e}_r + \sin\theta\,\mathbf{e}_\theta\right).$$

(The speed of the flow is $|\mathbf{u}| = \lambda/r^2$.)

Solution 3.4

The stream function for the combination of a doublet of vector strength $-2\pi\lambda\,\mathbf{i}$ and uniform flow $U\,\mathbf{i}$ is

$$\psi(r,\theta) = Ur\sin\theta - \frac{\lambda\sin\theta}{r}.$$

From Equations (2.2), the velocity $\mathbf{u} = u_r\,\mathbf{e}_r + u_\theta\,\mathbf{e}_\theta$ is given by

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta - \frac{\lambda\cos\theta}{r^2}, \tag{S.1}$$

$$u_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta - \frac{\lambda\sin\theta}{r^2}. \tag{S.2}$$

At a stagnation point, $u_r = u_\theta = 0$, so from Equation (S.1), $\theta = \pm\pi/2$ or $r = \sqrt{\lambda/U}$; and from Equation (S.2), $\theta = 0$ or π, or $r = \sqrt{-\lambda/U}$ (which is not a real number and therefore is not an admissible value for r). Hence the stagnation points have polar coordinates $r = \sqrt{\lambda/U}$, $\theta = 0$ and $r = \sqrt{\lambda/U}$, $\theta = \pi$. These both lie on the circle $r = \sqrt{\lambda/U}$ (at the points where the circle crosses the x-axis).

The streamline

$$\psi(r,\theta) = Ur\sin\theta - \frac{\lambda\sin\theta}{r} = 0$$

is composed of two parts:

$$\theta = 0 \text{ and } \theta = \pi, \qquad \text{which make up the } x\text{-axis,}$$

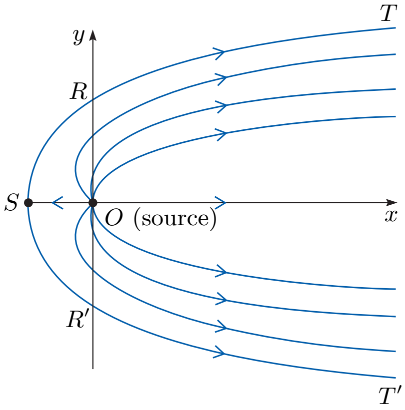
and

$$r = \sqrt{\lambda/U},$$

which is the circle on which the stagnation points lie.

Solution 3.5

The streamline pattern is below.



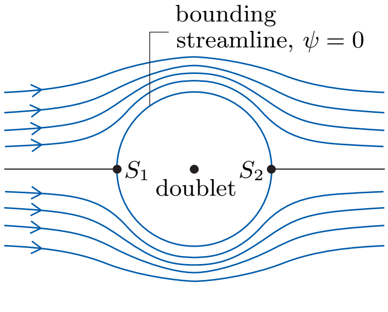
A particle emerging into the region $x < 0$ will be turned back by the uniform stream, and will remain within the boundary $TRSR'T'$. (Alternatively, we may regard the particle as being turned back owing to the presence of the boundary.) Particles emerging into the region $x > 0$ will be swept along by the uniform stream.

Solution 3.6

A sink or a doublet (with vector strength $-2\pi\lambda\,\mathbf{i}$), placed on the positive x-axis, will have this effect.

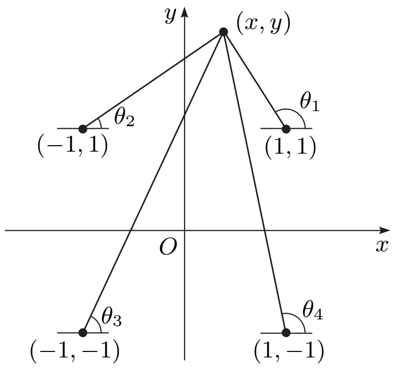
Solution 3.7

The stream function is $\psi(r,\theta) = Ur\sin\theta - \lambda\sin\theta/r$. From Solution 3.4, part of the streamline $\psi = 0$ is the circle $r = \sqrt{\lambda/U}$, so we can take this circle as a boundary. A particle from upstream with $y > 0$ will be deflected upwards by the doublet as it approaches the circle $r = \sqrt{\lambda/U}$, and then will be drawn back to the line of its original path again by the doublet. (Thinking of the doublet as a source and sink combination aids intuition here.) Since the flow is symmetric about the x-axis, the resulting pattern is as shown below. (Here S_1 and S_2 are the two stagnation points of the flow, as found in Solution 3.4.)



Solution 3.8

(a) Let θ_1, θ_2, θ_3 and θ_4 be defined as in the figure below.



The stream function is then

$$\begin{aligned}\psi &= m(\theta_1+\theta_2+\theta_3+\theta_4)\\ &= m\left(\arctan\left(\frac{y-1}{x-1}\right)+\arctan\left(\frac{y-1}{x+1}\right)\right.\\ &\qquad\left.+\arctan\left(\frac{y+1}{x+1}\right)+\arctan\left(\frac{y+1}{x-1}\right)\right).\end{aligned}$$

(b) Noting that (from a standard derivative and the Composite Rule)

$$\begin{aligned}\frac{\partial}{\partial x}\left(\arctan\left(\frac{y}{x}\right)\right) &= \frac{1}{1+(y/x)^2}\,\frac{\partial}{\partial x}\left(\frac{y}{x}\right)\\ &= \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right) = -\frac{y}{x^2+y^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial y}\left(\arctan\left(\frac{y}{x}\right)\right) &= \frac{1}{1+(y/x)^2}\,\frac{\partial}{\partial y}\left(\frac{y}{x}\right)\\ &= \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2},\end{aligned}$$

the partial derivatives of ψ are

$$\begin{aligned}\frac{\partial\psi}{\partial x} &= m\left(\frac{-(y-1)}{(x-1)^2+(y-1)^2}-\frac{y-1}{(x+1)^2+(y-1)^2}\right.\\ &\qquad\left.-\frac{y+1}{(x+1)^2+(y+1)^2}-\frac{y+1}{(x-1)^2+(y+1)^2}\right),\\ \frac{\partial\psi}{\partial y} &= m\left(\frac{x-1}{(x-1)^2+(y-1)^2}+\frac{x+1}{(x+1)^2+(y-1)^2}\right.\\ &\qquad\left.+\frac{x+1}{(x+1)^2+(y+1)^2}+\frac{x-1}{(x-1)^2+(y+1)^2}\right)\end{aligned}$$

If $x = y = 0$, then

$$u_1 = \frac{\partial\psi}{\partial y} = 0 \quad\text{and}\quad u_2 = -\frac{\partial\psi}{\partial x} = 0,$$

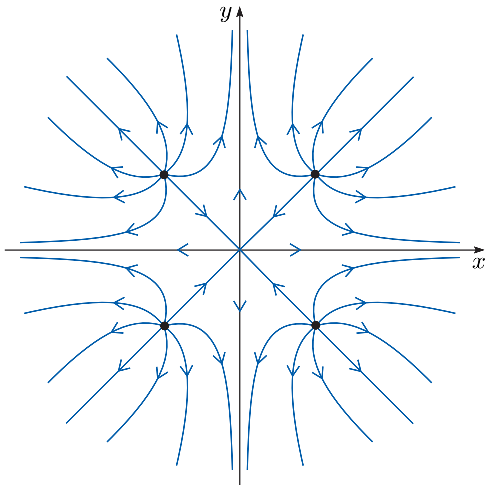
so the origin is a stagnation point.

(c) If $x = 0$, then

$$\begin{aligned}\psi &= m\left(\arctan\left(-(y-1)\right)+\arctan(y-1)\right.\\ &\qquad\left.+\arctan(y+1)+\arctan\left(-(y+1)\right)\right) = 0,\end{aligned}$$

since $\tan(-\theta) = -\tan\theta$. Thus $\psi = 0$ for all y. Similarly, if $y = 0$, then $\psi = 0$ for all x. Hence the x- and y-axes are parts of the streamline $\psi = 0$ (in fact, there are no other parts).

(d) The streamline pattern is as follows.



Since the x- and y-axes are streamlines, we can take the non-negative x- and y-axes as boundaries. The streamlines in the region $x > 0$, $y > 0$ then model the flow due to a source in a corner.

Section 4

Solution 4.1

(a) The flow (if not turbulent) can be taken as steady. In the inner region of the pipe (away from the wall) the flow is more or less uniform: type A. Close to the wall, the velocity varies with distance from the wall: type B.

(b) The drops from the tap do not form a continuum, so overall none of the classifications is relevant, but each drop can be considered to be a flow of type D.

(c) The fluid will accelerate down the inclined channel, and so the flow is steady but non-uniform: type B.

(d) Ignoring wind gusts and sideways movement of the hurricane, the flow could be modelled as steady but non-uniform: type B. If sideways movement is included, then type D is appropriate.

(e) A rotary fan produces almost steady flow, and it can be modelled as such: type B.

Solution 4.2

(a) The rate of temperature change for a fixed thermometer is the local rate of change of temperature, which is

$$\frac{\partial \Theta}{\partial t} = -0.25\sin(0.1t)\, e^{-0.01x}(1-0.02y^2)(1-0.04z).$$

Evaluated at the point $(5, 0.2, 0.1)$, this gives

$$\frac{\partial \Theta}{\partial t} \simeq -0.24\sin(0.1t).$$

(b) In this case $\mathbf{v} = 2\,\mathbf{j}$, and so, from Equation (4.2), $dx/dt = 0$, $dy/dt = 2$, and $dz/dt = 0$. Hence Equation (4.3) gives

$$\frac{d\Theta}{dt} = 2\frac{\partial \Theta}{\partial y} + \frac{\partial \Theta}{\partial t}.$$

The expression for $\partial\Theta/\partial t$ is as in part (a). Also

$$\frac{\partial \Theta}{\partial y} = 50(1+0.05\cos(0.1t))\, e^{-0.01x}(-0.04y)(1-0.04z).$$

Evaluated at the point $(5, 0.2, 0.1)$, we have

$$\frac{d\Theta}{dt} \simeq -0.24\sin(0.1t) - 0.76(1+0.05\cos(0.1t)).$$

This thermometer reaches $(5, 0.2, 0.1)$ at time $t = 0.1$, for which $d\Theta/dt \simeq -0.80$. (At this time, the fixed thermometer gives $\partial\Theta/\partial t \simeq -0.0024$.)

Solution 4.3

(a) The total derivative is

$$\begin{aligned}\frac{d\rho}{dt} &= \frac{\partial \rho}{\partial t} + u_1\frac{\partial \rho}{\partial x}\\ &= x^2 - \sin t + \frac{dx}{dt}(2xt).\end{aligned}$$

Putting $x = t^2$, this becomes

$$\frac{d\rho}{dt} = t^4 - \sin t + 2t\,(2t^2 \times t) = 5t^4 - \sin t.$$

(b) Putting $x = t^2$ gives

$$\rho = (t^2)^2 t + \cos t = t^5 + \cos t.$$

The (ordinary) derivative of this is

$$\frac{d\rho}{dt} = 5t^4 - \sin t,$$

which, as expected, is the same as in part (a).

Solution 4.4

(a) Since $\partial f/\partial t = 0$, $\partial f/\partial x = 0$ and $u_2 = u_3 = 0$, Equation (4.4) becomes

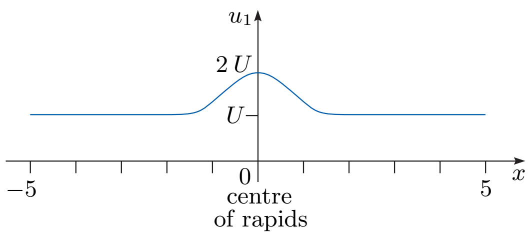
$$\frac{df}{dt} = 0.$$

(b) For steady flow, the time partial derivative $\partial f/\partial t$ is zero. Hence Equation (4.4) becomes

$$\frac{df}{dt} = (\mathbf{u}\cdot\boldsymbol{\nabla})f = u_1\frac{\partial f}{\partial x} + u_2\frac{\partial f}{\partial y} + u_3\frac{\partial f}{\partial z}.$$

Solution 4.5

(a) The graph of u_1 against x is shown in the figure that follows. (From a speed of approximately U at $x = -5$, the river speeds up to $2U$ at $x = 0$, and then slows down almost to U at $x = 5$.)



The local (time) rate of change of $\mathbf{u}$ is

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial u_1}{\partial t}\,\mathbf{i} = \mathbf{0}$$

everywhere, since u_1 depends on x only.

The total rate of change of $\mathbf{u}$ is

$$\frac{d\mathbf{u}}{dt} = \frac{du_1}{dt}\,\mathbf{i} = \left(\frac{\partial u_1}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})u_1\right)\mathbf{i}$$
$$= \left(0 + u_1\frac{\partial}{\partial x}(u_1)\right)\mathbf{i}$$
$$= U\left(1+e^{-x^2}\right)U\left(-2xe^{-x^2}\right)\mathbf{i}$$
$$= -2U^2x\,e^{-x^2}\left(1+e^{-x^2}\right)\mathbf{i}.$$

(So du_1/dt is an odd function, being positive when x is negative, and vice versa.)

The values of du_1/dt are as follows.

x	-1	-0.5	-0.25	0	0.25	0.5	1
du_1/dt $(\div U^2)$	1.006	1.385	0.911	0	-0.911	-1.385	-1.006

(b) (i) A stationary observer on the bank, watching a trail of sticks pass through the centre of the rapids, would note that each stick had the same velocity at the point $x=0$ (i.e. $\partial u_1/\partial t = 0$), and so there is no local acceleration of water at that point of the river.

(ii) The observer on the bank would see the stick accelerate into the rapids (as shown by the table in part (a), the maximum speed being attained when $x=0$, at which point $du_1/dt=0$), and decelerate out.

Solution 4.6

Using Equation (4.5), the total derivative is

$$\frac{d\mathbf{u}}{dt} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u}.$$

Since $\partial\mathbf{e}_\theta/\partial t = 0$, we have

$$\frac{\partial\mathbf{u}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{K(t)}{r}\right)\mathbf{e}_\theta = \frac{1}{r}\frac{dK(t)}{dt}\mathbf{e}_\theta = \frac{K'(t)}{r}\mathbf{e}_\theta.$$

Also, from Equation (4.6), $(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u}$ is equal to

$$\left(\frac{K(t)}{r}\,\mathbf{e}_\theta\cdot\left(\mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\right)\left(\frac{K(t)}{r}\,\mathbf{e}_\theta\right)$$
$$= \frac{K(t)}{r^2}\frac{\partial}{\partial\theta}\left(\frac{K(t)}{r}\,\mathbf{e}_\theta\right)$$
$$= -\frac{(K(t))^2}{r^3}\,\mathbf{e}_r, \qquad \text{since } \frac{\partial\mathbf{e}_\theta}{\partial\theta} = -\mathbf{e}_r.$$

Hence we obtain

$$\frac{d\mathbf{u}}{dt} = \frac{K'(t)}{r}\,\mathbf{e}_\theta - \frac{(K(t))^2}{r^3}\,\mathbf{e}_r.$$

(Note that, although the velocity $\mathbf{u}$ has only one non-zero component, the acceleration has both a radial and a transverse component.)

Solution 4.7

In each case, we check whether $\boldsymbol{\nabla}\cdot\mathbf{u}$ is zero.

(a) $\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(-2z) = 0,$

so this could be an incompressible flow.

(b) $\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(0) = 3,$

so this cannot be an incompressible flow.

(c) $\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(u_\theta) + \frac{\partial}{\partial z}(u_z)$

$$= \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{K}{r^2}\right) = 0, \qquad \text{since } \mathbf{u} = \frac{K}{r^2}\,\mathbf{e}_\theta,$$

so this could be an incompressible flow.

Solution 4.8

Using Equation (4.8), we have

$$0.3\times 1.8 = 0.15\times u,$$

where u is the speed at the smaller section. Hence $u = 3.6\,\mathrm{m\,s^{-1}}$.

(This calculation confirms intuition: a contraction leads to a higher speed. *Unit 6* will show that it also implies a *lower* pressure.)

Solution 4.9

(a) Equation (4.8), $u_1A_1 = u_2A_2$, becomes

$$u(x_1)(l_1\times 1) = u(x_2)(l_2\times 1), \qquad \text{or}$$
$$l_1u(x_1) = l_2u(x_2).$$

(This means that the speed is greater when the streamlines are closer.)

(b) Model the air/water boundary and the river bed by streamlines, and assume that the speed of flow at any vertical section is independent of the depth of the river. Then, from part (a), we have

$$4\times 2.25 = 3\times u, \qquad \text{so} \qquad u = \tfrac{9}{3} = 3\,\mathrm{m\,s^{-1}}.$$

Solution 4.10

(a) Steady, uniform

(b) Steady, non-uniform

(c) Non-steady, non-uniform

(d) Steady, non-uniform

(e) Non-steady, uniform

Solution 4.11

In each case, the acceleration following the motion is given by

$$\frac{d\mathbf{u}}{dt} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u}.$$

(a) If $\mathbf{u} = Uy\,\mathbf{i}$, then

$$\frac{d\mathbf{u}}{dt} = (Uy\,\mathbf{i}\cdot\boldsymbol{\nabla})(Uy\,\mathbf{i}) = Uy\frac{\partial}{\partial x}(Uy)\,\mathbf{i} = \mathbf{0}.$$

(b) If $\mathbf{u} = U(t)y\,\mathbf{i}$, then

$$\frac{d\mathbf{u}}{dt} = \frac{\partial}{\partial t}(U(t)y\,\mathbf{i}) + (U(t)y\,\mathbf{i}\cdot\boldsymbol{\nabla})(U(t)y\,\mathbf{i})$$
$$= U'(t)y\,\mathbf{i} + U(t)y\frac{\partial}{\partial x}\,(U(t)y)\;\mathbf{i}$$
$$= U'(t)y\,\mathbf{i}.$$

(c) If $\mathbf{u} = U(1 + a^2/r^2)\sin\theta\,\mathbf{e}_\theta = u_\theta\,\mathbf{e}_\theta$ then, from Equation (4.6), we have

$$\frac{d\mathbf{u}}{dt} = \left(\frac{u_\theta}{r}\frac{\partial}{\partial\theta}\right)(u_\theta\,\mathbf{e}_\theta)$$

$$= U\left(1 + \frac{a^2}{r^2}\right)\frac{\sin\theta}{r}\,U\left(1 + \frac{a^2}{r^2}\right)\left(\cos\theta\,\mathbf{e}_\theta + \sin\theta\frac{\partial\mathbf{e}_\theta}{\partial\theta}\right)$$

$$= U^2\left(1 + \frac{a^2}{r^2}\right)^2\frac{\sin\theta}{r}\,(\cos\theta\,\mathbf{e}_\theta - \sin\theta\,\mathbf{e}_r)\,.$$

Solution 4.12

If $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ then the flow is incompressible.

(a) Here we have

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{\partial}{\partial x}(U\cos\alpha) + \frac{\partial}{\partial y}(U\sin\alpha)$$

$$= 0, \qquad \text{since } U \text{ and } \alpha \text{ are constants.}$$

This is a possible incompressible flow.

(b) Here we have

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{\partial}{\partial x}\left(-\frac{Ky}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{Kx}{x^2+y^2}\right) + \frac{\partial}{\partial z}(2xy)$$

$$= -\frac{-Ky(2x)}{(x^2+y^2)^2} + \frac{-Kx(2y)}{(x^2+y^2)^2} + 0 = 0.$$

This is a possible incompressible flow.

(c) Here we have

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(u_\theta)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(rU\cos\theta\left(1 - \frac{a^2}{r^2}\right)\right)$$

$$+ \frac{1}{r}\frac{\partial}{\partial\theta}\left(-U\sin\theta\left(1 + \frac{a^2}{r^2}\right)\right)$$

$$= \frac{1}{r}\left(U\cos\theta + U\cos\theta\frac{a^2}{r^2}\right)$$

$$+ \frac{1}{r}(-U\cos\theta)\left(1 + \frac{a^2}{r^2}\right) = 0.$$

This is a possible incompressible flow.

(d) Here we have

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{K}{r}\right) = 0 \qquad (r \neq 0).$$

This is a possible incompressible flow.

Solution 4.13

By symmetry (since the flow is horizontal) the speeds at each section 2 will be the same, u_2 say. By the continuity equation (4.8), we have

$$u_1A_1 = 2(u_2A_2) \qquad \text{so} \qquad u_2 = \frac{u_1A_1}{2A_2}.$$

But $A_1 = \pi a^2$, $A_2 = \frac{1}{4}\pi a^2$; hence $A_1/A_2 = 4$, and

$$u_2 = 2u_1.$$

Section 5

Solution 5.1

In this case, $\mathbf{u} = \mathbf{0}$ and $\mathbf{F} = -g\,\mathbf{k}$, so that Euler's equation (5.5), $\rho\,d\mathbf{u}/dt = -\boldsymbol{\nabla}p + \rho\,\mathbf{F}$, becomes

$$\mathbf{0} = -\boldsymbol{\nabla}p - \rho g\,\mathbf{k}.$$

Therefore

$$\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial z} = -\rho g.$$

From the first two equations, $p = p(z)$ and so $\partial p/\partial z = dp/dz$, giving the required result.

Solution 5.2

Euler's equation (5.6) is

$$\rho\,\frac{\partial\mathbf{u}}{\partial t} + \rho\,(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u} = -\boldsymbol{\nabla}p + \rho\,\mathbf{F}.$$

Using the hint, we have

$$\rho\,\frac{\partial\mathbf{u}}{\partial t} + \rho\,\boldsymbol{\nabla}\left(\tfrac{1}{2}u^2\right) - \rho\,\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u})$$

$$= -\boldsymbol{\nabla}p + \rho\,\mathbf{F}. \tag{S.3}$$

Steady flow implies $\partial\mathbf{u}/\partial t = \mathbf{0}$.

Irrotational flow implies $\boldsymbol{\nabla}\times\mathbf{u} = \mathbf{0}$.

Conservative force field implies the existence of a scalar field Ω for which $\mathbf{F} = \boldsymbol{\nabla}\Omega$.

Constant density implies $\rho = \rho_0$.

Thus Equation (S.3) becomes

$$\rho_0\boldsymbol{\nabla}\left(\tfrac{1}{2}u^2\right) = -\boldsymbol{\nabla}p + \rho_0\boldsymbol{\nabla}\Omega,$$

which may be written as

$$\boldsymbol{\nabla}\left(\tfrac{1}{2}u^2 + \frac{p}{\rho_0} - \Omega\right) = \mathbf{0}.$$

Solution 5.3

The given stream function is

$$\psi = m\left(\arctan\left(\frac{y-b}{x}\right) + \arctan\left(\frac{y+b}{x}\right)\right).$$

The velocity at the wall $y = 0$ must have y-component zero, so $\mathbf{u}(x,0) = u_1(x,0)\,\mathbf{i}$. Now by Equations (2.1),

$$u_1 = \frac{\partial\psi}{\partial y} = m\left(\frac{x}{x^2+(y-b)^2} + \frac{x}{x^2+(y+b)^2}\right),$$

so that

$$u_1(x,0) = \frac{2mx}{x^2+b^2}.$$

Since the flow is steady and irrotational with $\mathbf{F} = \mathbf{0}$, we have, from Solution 5.2,

$$\boldsymbol{\nabla}\left(\tfrac{1}{2}u^2 + \frac{p}{\rho}\right) = \mathbf{0}.$$

Integrating along the boundary wall (a streamline),

$$\tfrac{1}{2}u_1^2 + \frac{p}{\rho} = \text{constant} \qquad \text{on } y = 0.$$

When $x = \pm\infty$, we have $p = p_0$ and $u_1 = 0$; hence

$$\tfrac{1}{2}\rho u_1^2 + p = p_0 \qquad \text{on } y = 0.$$

Thus the pressure distribution on the wall is

$$p(x,0) = p_0 - \tfrac{1}{2}\rho u_1^2(x,0) = p_0 - \frac{2\rho m^2x^2}{(x^2+b^2)^2}.$$

Solution 5.4

Equation (5.6) gives

$$\rho\,\frac{\partial \mathbf{u}}{\partial t} + \rho\,(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u} = -\boldsymbol{\nabla} p + \rho\,\mathbf{F}.$$

Now $\mathbf{F} = 5\,\mathbf{j} = \partial\mathbf{u}/\partial t$, and so

$$\begin{aligned}\boldsymbol{\nabla} p &= \rho\left(\mathbf{F} - \frac{\partial \mathbf{u}}{\partial t} - (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right)\\ &= -\rho\left(-2x\frac{\partial}{\partial x} + (2y+5t)\frac{\partial}{\partial y}\right)(-2x\,\mathbf{i} + (2y+5t)\,\mathbf{j})\\ &= -\rho\,(4x\,\mathbf{i} + 2(2y+5t)\,\mathbf{j}).\end{aligned}$$

In terms of components, we have

$$\frac{\partial p}{\partial x} = -4\rho x, \qquad \frac{\partial p}{\partial y} = -2\rho\,(2y+5t).$$

Integrating the first equation gives

$$p(x,y,t) = -2\rho x^2 + f(y,t),$$

where f is an arbitrary function of two variables.
Hence

$$\frac{\partial p}{\partial y} = \frac{\partial f}{\partial y} = -2\rho\,(2y+5t),$$

from which

$$f(y,t) = -2\rho y\,(y+5t) + g(t),$$

where g is an arbitrary function. Therefore

$$p(x,y,t) = -2\rho x^2 - 2\rho y\,(y+5t) + g(t).$$

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