Mental mathematics

Chris Bills

What is mental mathematics?

Try to do the following without drawing or writing anything, think for several minutes about the problem then reflect on what was going on in your mind when you were thinking about it. You may find it helpful to close your eyes after you have read it.

Task 15.1

Imagine a quadrilateral. Now imagine a circle that has each of the vertices of the quadrilateral on its circumference.

What can you say about the position of the centre of the circle?

Unless your mind was totally inactive, or full of thoughts unrelated to the problem, then what you have been engaged in is mental mathematics and metacognition (thinking about your own thinking). It could be that you were able to visualise a quadrilateral mentally and circumscribe a circle and to 'see' them in your head sufficiently vividly to be able to answer the initial question. Many people do not have clear 'pictures in the head'; you may have been bringing to mind memories of diagrams you had drawn or had seen. You may instead (or as well as) have been trying to recall any circle theorems that might help or trying to remember if you had ever done anything like this before. Mixed up with these images and thoughts may well have been feelings of frustration and despondency if you were unsuccessful or a sense of euphoria if you enjoyed working on the problem.

Even if you were unable to say anything about the position of the circumcentre of the quadrilateral you may well have been engaged in mathematical activity – specialising, conjecturing, generalising, predicting, verifying, refuting, proving – as you turned the problem over in your mind. Alternatively, you may have stopped after you had done as asked and simply imagined the situation for a particular quadrilateral. This problem, and your reactions to it, are illustrative of what is involved in mental mathematics and the problems associated with doing mathematics without any aids to memory. You may wish at this stage to try the problem on paper or using dynamic geometry software. If you do not feel the need to do so, simply think about how much less (or more) mental mathematics is involved when physical diagrams are allowed. It is likely that before you draw you imagine what you are going to draw. As you draw the second edge of the quadrilateral, for instance, you will probably start to imagine the rest of the quadrilateral to decide in advance whether it will be possible to put a circle through the vertices. You might have decided that it would be much simpler to draw the circle first then imagine various quadrilaterals in relation to the centre before drawing a few illustrative examples. Looking at your drawings may have generated more mental mathematical activity.

We necessarily engage in private mathematical thinking when we are working with more public 'signs' (diagrams, words, symbols) so that 'mental mathematics' is interspersed with sign production in order to generate and communicate our mathematical activity. For the purposes of this chapter, however, I shall use the term 'mental mathematics' for the mathematical thinking that we can engage in without physical aids.

What is a mental image?

Try a non-geometric problem in your head.

> **Task 15.2**
>
> *x* squared subtract *x* is equal to *x* multiplied by three. What is the value of *x*?

If you have found a solution (or two) you have used an image. It is possible that again you had a vivid mental visualisation, a picture in your mind of the equation written out in your own writing or like a teacher's board work, printed in a book or on a computer screen. It is just possible that as you worked on the equation you 'saw' the subsequent lines of symbols appearing as you thought through each step in the procedure for solving equations. You may have supplemented this with (or used instead) instructions to yourself such as 'collect the unknowns', 'add the same to both sides', 'take the *x* outside the bracket', etc. These words are just as much images as are the visualisations because they 're-present' previous experiences to your mind. You may for instance 'know' that equations of the form $x^2 - a = 0$ give the solutions 0, and because this is an image formed from the numerous examples you have done of this type, not because you learned this as a general rule. Even if you do not need to verbalise it, it is part of your thought process and is an image that you use subconsciously.

The word 'image' is used most commonly for quasi-sensory experiences. Try thinking of some food or drink you really like, or hate, to test if you have images. Long years of chocolate consumption may have left you with vivid sight, taste and smell images as well as emotional images such as comfort or guilt. Just *one* acquaintance with chilli may have left you with equally vivid images. Where images are concerned it is quality not just quantity of previous experiences that matter. Not everyone, however, has quasi-sensory images. If you did not visualise the quadrilateral, equation, or chocolate (and could not now even if you concentrated very hard on trying to 'see' it) then you are not alone. Even people with strong imagery, however, would find it easier to form a variety of mental images for familiar everyday objects than for mathematical objects.

In order to communicate what was in my head I have to describe or draw something visual, or say or write words or symbols. To this extent the visual and verbal are the most easily communicable images. We need to be aware, however, that in transforming our thoughts into communicable form we might well transform the thought. I might say or draw something that I think is more acceptable to my audience than that which was actually in my head. Or what is in my head may not be expressible in any common language so what I express is my attempt at a translation into visual, verbal or written form. Thus, mental images may not be in the format that we use to describe them, but those pictorial or verbal communications are an approximation to what was in our minds.

Try this in your head.

> **Task 15.3**
>
> Calculate negative three subtract negative eight

I gave that problem, orally, in seminar and was told by one of the group that she had pictured the numbers on a number line, but had panicked when she couldn't remember what to do. Her mind had gone blank – the visualisation on its own was no use without images of the procedure.

Your own thinking may have involved imagining the question written in mathematical symbols with or without a number line, you may have experienced head or body movements as you re-enacted turning and walking backwards, you may have seen sandcastles and holes, thermometers, staircases or any other pictures that had been used by teachers to illustrate this type of problem to you. You may have found yourself muttering 'an enemy of an enemy is a friend' or 'two minuses make a plus'. It is likely that you have had a variety of experiences of integers that have left you with images. In reading my list of possibilities some more may have come to your mind, but when you first did the calculation it was probably your preferred image that was used. If you just 'knew' the answer and were not aware of any thinking it may be because the imagistic thinking has been condensed into a subconscious procedure that, though based on some physical representation or verbally expressed rule, happens automatically

I shall use 'mental image' to imply that what comes to mind for the purposes of thinking has been influenced by previous physical or mental activity – what we have seen or done or talked about or thought about before. Images are representations of previous experiences.

What is a verbal image?

While many people do not claim to have mental visual images the language they use to describe their thinking shows the influence of their previous experiences. I have asked children between the ages of 7 and 9 years to perform mental calculations and have then asked, 'What was in your head when you were thinking of that?'. Their responses revealed the influence of previous activities. Some used language associated with counting objects, for instance:

- 17 + 8 I added ten onto seventeen and took away two and that gave me twenty-five.
- 48 + 23 Forty with twenty is sixty, forty-eight add twenty comes to sixty-eight and then you *add* three *on*.

Others used words associated with manipulation of materials and numbers:

- 97 + 10 I *move the seven* and then I knew which one, I know got to add a ten on, I can't add a ten so I *put the one in front* of it and then a *nought in the middle*.
- 30 + ? = 80 I know that three add five is eight so just *turn it into* tens.

Others used the language of the written algorithms:

- 48 + 23 Eight and the three made eleven, so I *carried a figure* and *put it under there* then four and the two
- 65 – 29 Five take away nine, but you can't do so you *cross out* the six, *make that* five, and *make that* fifteen.

All these pupils had the same variety of classroom experiences yet they each had different ways of expressing their process of calculation. It is this use of language associated with previous activities that I shall refer to as a 'verbal image'. The variety of images suggests that they are the pupils' own constructions, influenced by one or many of the physical representations used by the teacher, and not just learned rules. Many of the words are common in the classroom, but different pupils show preference for different images. These responses were recorded before the teaching of specified mental calculation strategies was commonplace in primary schools.

What are mental calculation strategies?

The National Numeracy Strategy: Framework for teaching Mathematics, implemented in all English primary schools from 1999 with the extension for Years 7–9 piloted in 2000, stresses that an ability to calculate mentally lies at the heart of numeracy, that mental methods should be emphasised and that written algorithms should develop out of mental methods. In the guidance for teacher, 'Teaching mental calculation strategies' (QCA 1999), it is made clear that:

… children should learn number facts 'by heart' and be taught to develop a range of mental strategies for quickly finding from known facts a range of related facts that they cannot recall rapidly.

And furthermore:

For much work at Key Stages 1 and 2, a mental approach to calculation is often the most efficient and needs to be taught explicitly.

(p. 3)

It lists the facts that children should be able to recall, the mental strategies they should be able to use and the mental calculations they should be able to perform in each year. For instance, by Year 3 children should recall addition and subtraction facts for all numbers to 20, they should use the strategy of adding or subtracting a 'near multiple often' to or from a two-digit number (e.g. 36 + 42 = 36 + 40 + 2), 'bridge through a multiple of ten then adjust' (e.g. 36 + 42 = 36 + 4 + 38) or 'use knowledge of number facts and place value to add or subtract pairs of numbers' (e.g. 36 + 42 = 30 + 40 + 6 + 2) so that they should be able to decide the most efficient method to calculate mentally. As outcomes of teaching, Year 7 pupils should, for example: add or subtract 0.1 and 0.01 to or from any number, multiply and divide a number by 1, 10, 100, 1000, 0.1, 0.01, derive quickly decimal complements in to one or two decimal places (e.g. 1 = 0.8 + 0.2, 1 = 0.41 + 0.59).

The key to the pupil's development of this range of strategies is thought to be classroom discussion of different strategies. Teachers use questions such as, 'How did you work that out?', 'Who did it another way?', 'Which is the easier/easiest way?' to stress that efficiency, the speed and ease with which strategies lead to a correct answer, is importance. In addition QCA (1999) recommends informal recording and the use of tools such as number lines and hundred squares to develop understanding of number. It explicitly states that mental calculation is not the same as mentally picturing a written algorithm. Paper and pencil can, it suggests, be used to support mental calculation particularly through the use of diagrams that encourage the development of mental imagery An empty number line, for instance, can provide a visual representation to demonstrate to others the way in which the calculation is being performed

The emphasis on, and frequent practice of, mental calculation in primary schools is likely to result in some pupils forming images that were not previously common. Sharing ideas through class discussion and through pupils communicating different strategies to one another may well lead to mental visual and verbal images that go beyond images related to counting or to written algorithms.

Is a mental visual image seen?

Task 15.4

Imagine a number line. Look at the number thirty-six. Now add twenty.

Much of the research on mental visual imagery has demonstrated that it has many of the characteristics of visual perception – the same area of the brain is in use when people are thinking about an image as when they are looking at something, the time taken to scan or rotate mental visual images is consistent with the time taken with pictures and physical objects. It seems reasonable then to talk of ‘seeing’ a picture in the mind, but it is clearly not like a photograph. When we have an image of something we have previously seen it is our own construction and is thus dependent on what we noticed about the object when we were looking at it and how much we can remember. It is also dependent on our ability to recreate it in our mind. If I choose to imagine a number line I might have a hazy idea of a straight horizontal line with some vertical markers on it and a zero. As I think more about it I can imagine bolder markings for the tens and longer lines for the fives because I have seen number lines like that and noticed those features. As I ‘scan’ along the line from 36 to 56 I may imagine the numbers along the way because I know what they are.

This dependence on ‘having seen’, and noticed what has been seen, is an important aspect of visual imagery. If a child is asked to get a picture of a four-sided shape in their head, they most of all visualise a square or a rectangle and it is always in the stereotypical orientation, with sides vertical and horizontal, that has been used as illustration by teachers. The pernicious effect of ‘fixed’ images that we bring to mind when asked to imagine objects is illustrated by the following example.

Task 15.5

Imagine a wooden cube. Imagine suspending it by a piece of string attached at one vertex. Lower it onto a bed of damp sand so that a vertex just touches the surface and then push the cube down so that the vertex makes a hole in the sand. Remove the cube and look at the hole. What shape is it?

About half of the people who visualise this situation say that the hole is a square based pyramid. The most likely reason for this is that the only polyhedron they have ever seen in this orientation (i.e. vertex uppermost) is a regular octahedron. If you reasoned that the hole had to be a triangular based pyramid and made no attempt at visualising you were still reliant on having previous experience of cubes and to this extent you were relying on your image of a cube.

All this seems to imply that we can only imagine what we have already seen yet surely the power of imagination is that it is creative? This is true, but I can only imagine a two-headed elephant because I have seen a normal elephant and with this as a base my imagination can transform it into something I have never seen. I can imagine the vertical calculation for 346 subtract 172 not because I have seen it before, but because I have seen calculations *like* it before. To this extent an image is ‘seen’ if there is a memory of having seen something like it. We need not feel that we cannot mentally visualise something simply because we do not have a vivid, video-screen type picture in the mind. I shall refer to a mental visual image as simply an awareness of spatial relationship, in whatever way they are manifested in the mind.

Is a mental visual image useful?

I have given the number line problem in Task 15.4 to 9-year-old children. The majority has said they could ‘see’ the number line and the 36, but did not use their image for the addition. This is not surprising since, for most of us, a considerable mental effort is required to keep a picture in mind and we have difficulty doing the calculation and creating the image at the same time. Only if the calculation is performed without conscious effort might we be convinced we have seen it happen on the number line. For many people the image seems epiphenomenal– it is created to go with the calculation, but does not guide thoughts, it is an optional extra.

Seen simply as an aid to mental calculation, a mental image would seem likely to be more of a hindrance than a help if we are using up mental processing capacity in creating it. It is not unusual for young children to visualise objects to count when given a mental addition question, and for other mental calculations to be performed by visualising the written algorithm, or at least performing procedures as if the written calculation were being performed. These forms of image might even inhibit the development of more efficient strategies for mental calculation.

The value of visual imagery, on paper or in the head, thus lies in its potential to facilitate and generate mathematical thought. Visualisation is an important aspect of mathematical activity and if visual thinking can occur without recourse to paper or electronic means of production then it seems likely to be beneficial. So as teachers we need to consider how we might encourage pupils to develop the ability to use mental visual images. The National Curriculum suggests that at Key Stage 1:

pupils should be taught to: describe properties of shapes that they can see or visualise

and

pupils should be taught to: observe, visualise and describe positions.

At Key Stage 2:

pupils should be taught to: visualise and describe 2-D and 3-D shapes … , visualise 3-D shapes from 2-D drawings … , visualise and describe movements, visualise and predict the position of a shape following a rotation, reflection or translation.

The clue to how this might be achieved lies in the suggestion in the ‘Breadth of study’ for Key Stage 3 which recommends:

practical work with geometrical objects to develop their ability to visualise these objects and work with them mentally.

There appears to be no suggestion that all pupils should be trained to form mental visual images so we can only teach pupils how to create physical visualisations, on paper or computer screen, and encourage them to visualise mentally as a result.

Can mental imagery be ‘taught’?

It is clear that the National Curriculum, both for pupils and trainee teachers, and the Mathematics Framework are written on the assumption that children can be encouraged to use mental visual imagery as a result of experiences with physical visual images. There is also the assumption that a facility with mental calculation will develop as a result of frequent practice and that an understanding of different strategies will result from classroom discussion. It is tempting to think that visual representations such as the number line and verbal imagery developed from it (e.g. ‘bridging through multiples of ten’ and ‘jumping forward and backward to landmarks’) will be as useful to pupils as they are for teachers. There is a well-known paradox, however, that a representation may only be understood if one already possesses the concept that it is supposed to represent. Simply encouraging pupils to visualise a number line is of no practical use unless the concepts it can illustrate are understood.

The ‘constructivist’ view of learning is that each individual constructs their own picture of reality from their own experiences and ‘social’ constructivists emphasise the role of the community of the classroom in providing these experiences. Whilst learners do not construct a copy of a pre-existing mathematical reality, or work on an ‘internal representation’ of a ‘physical representation’ of number, their images are based on their classroom experiences and particularly their experience with the representations used by their teachers. The images that are used in subsequent thinking are re-presentations of those experiences. The teacher needs to be aware, however, that not all pupils will form images at the same pace and that some may not form images at all. It is also likely that many pupils will not use the same image as the teacher.

In order to promote the construction of mental images that can be used in future thinking, classroom activities need to provide the experiences, and encouragement for reflection on them, that might lead to formation of those images. Geometry provides obvious examples for mental visual images, but also illustrates how physical visualisations can lead to mental visualisations that restrict rather than generate mathematical thinking. Try this.

Task 15.6

Imagine a triangle and a straight line drawn near it. Now reflect the triangle in the line.

That task could be meaningless if you have not had the usual classroom experiences. If you have had some experience of drawing reflections on plane shapes you may have been able to form a picture in your mind or at least have a vague recall of what it looked like when you did draw it. The 'usual classroom experiences' however often lead to errors since reflection is invariably in a vertical or horizontal line. The mental image that is formed as a result often leads to the common error, when working without aids, of drawing the reflection as if the line of reflection were vertical or horizontal, even when the line is at an angle to the vertical. To combat this the learners' attention needs to be focused on the process of reflection, they need to work with many orientations of shape and line, and they need to notice the features of the relationships between a shape and its reflection. They need to use this awareness to construct the 'mathematical object', reflection, not simply as a stereotypical visualisation, but as an understanding of its characteristics. The teacher's role is thus to educate this awareness and this is done through discussing examples and counter- examples and encouraging pupils to communicate their own thinking about what a reflection is.

Can mental mathematics be taught?

There is a concern that the attempt to teach mental calculation strategies will be as unsuccessful for some pupils as previous attempts to teach written algorithms. If rules of manipulation of numbers on paper, or of manipulation of quantities in the mind, appear arbitrary to pupils then they are likely to be misapplied. It is too easy to make the assumption that if those who are unsuccessful at mathematics are shown what it is that successful mathematicians do then they, too, can become successful. The mental calculation strategies that are now being actively taught are the efficient strategies that people have previously developed for themselves. They were developed as a result of an understanding of quantity and the way in which combinations and decompositions of quantities relate to the numerals that are used to express them. Those pupils who have not made sense of the connection between the signified (quantity) and the signifiers (number words and numerals) are unlikely to make sense of any of the ways of manipulating the symbols, even if they can do the manipulations. Similarly, mental images of algebraic manipulation, for instance, can only be formed if the learner has noticed what is happening when the process is performed physically, when it is described or when the result is seen.

There is another aspect of pupils' individual cognitive abilities. The mental image may manifest itself as visual or verbal or both and the extent to which thinking involves either or both of these modalities may be due to individual differences in thinking styles as well as differences in experiences. Presmeg (1986) showed that few able high school students are 'visual thinkers', i.e. prefer to use methods involving diagrams (physical or mental images) rather than 'analytic' methods (words and symbols). She noted that there was often little encouragement from teachers for visual methods and so they were not valued by students. She also discovered that visual thinkers seemed to be most successful in the classes of teachers who used both visual and non-visual methods rather than in the classes of teachers who were either strongly visual or strongly non-visual. Preference for visual methods was also found to be independent of the student's mathematical attainment (Presmeg, 1995).

What are the implications for teaching and learning?

Task 15.7

Think of a number between one hundred thousand and one million. Now round it to the nearest ten thousand.

Did you 'see' it or say it to yourself or have some other way of thinking about it? Does it matter whether it was a visual or verbal image or both or neither? If the number in your head was just a collection of words would it help to visualise before writing it or rounding it? If you can visualise it do you need to verbalise before writing or rounding? Could you even do this in your head? Most people are on the limit of their ability to hold a number in mind when it is has six digits so you may have found yourself writing it with your finger on the table. If you could be trained to keep larger numbers in mind would it be worth the effort?

When teachers ask pupils to workout in their heads the mean of a small set of data, imagine the graph of $y = (x - 2)(x + 3)$ and say where it intersects the x-axis, mentally rotate and reflect a triangle to decide if two drawn triangles are congruent, or any other task without physical aids available, the teacher and the pupils will be using a variety of images. If the teacher and pupils are only interested in the answer then it does not matter what was in their heads when they were doing it. If they are interested in the way they think and whether they might think in different ways then that variety does matter and is worth exploring. If mental mathematics is to be any more than learning efficient mental calculation strategies or training for mental visualisation then there needs to be more discussion about mental imagery in all modalities and how it relates to the learning of mathematics. Being more aware of the way we think is a first step toward taking more control of our thinking.

On a more mundane level, an awareness of the variety of images that others use may help with communication. If the teacher assumes that all pupils have the same thing in mind then communication will not be possible. Consider for example 12 ÷ 3. How did you read it and what images come to mind when you think about it? You may have said '12 shared by three', '12 divided by three', '12 shared into three (parts or lots)', '12 divided into three (parts or lots)', '12 shared into threes', '12 divided into threes', 'How many threes in 12?', 'Threes into 12 go … ', etc. You may or may not have visualised something and if you did it may or may not have been a picture appropriate to the way you said it. Now imagine a class of children working on 128 ÷ 32. Many will have a preferred, fixed verbal image for division, 'shared by' perhaps, the teacher may be talking about 'how many of these in this'. It is likely that there will be, at least initially, some confusion. Try this.

Task 15.8

Four point six multiplied by seven.

Many people can perform that calculation entirely as words in their heads, the partition is transparent in the language and there is not too much to hold in the mind. Other people need to visualise it. Is either group behaving in a more desirable way?

It has been argued that children who rely on physical materials and particularly on counting strategies for calculation are disadvantaged in not developing more efficient mental skills. It is also suggested that a reliance on visual thinking can delay the beginnings of abstract thinking. Teachers, concerned with pupils' cognitive development need to be aware of these possibilities and encourage flexible thinking through discussion and through a change in emphasis away from the answer toward the processes for getting answers. Imagery in all its guise needs to be recognised and included in that discussion.

References

Presmeg, N. (1986) 'Visualisation in high school mathematics', *For Learning of Mathematics*, 6(3): 42–6.

Presmeg, N. C. (1995) 'Preference for visual methods and international study', in L. Meira and D. Carraher (eds) *18th International Conference for the Psychology of Mathematics Education*, vol. 3, Recife, Brazil: Programme Committee: 58–65.

QCA (1999) *The National Numeracy Strategy. Teaching mental calculation strategies*, QCA, London.