Teaching mathematics

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Introduction and guidance

Introduction and guidance

This free course is designed for non-specialist mathematics teachers of 8 to 14-year olds, teaching assistants and parents and draws on the established provision of mathematics education at the Open University. Course content introduces the underpinning pedagogical theory for the learning of mathematics topics including pre-requisite understanding, how learners access the important concepts, common misconceptions, and small steps that challenge learners. It includes learning activities which can be undertaken in the classroom and links to existing free resource banks. You will read about learning, try some mathematics and consider effective approaches to facilitate mathematics learning.

In a 24-hour course it is not possible to cover all of the middle school curriculum, nor is it possible to provide a full account of mathematics pedagogy. However, the aim for the course is to provide some ideas for effective approaches to teaching mathematics and a brief account of current philosophy behind the teaching of mathematics. This includes teaching for learners' understanding, which underpins the idea of Mastery being promoted in many schools.

After completing this course, you will be able to:

- develop understanding of mathematical content in the middle school years, ages 8 to 14 years
- begin to understand how learners learn mathematics
- identify effective strategies and approaches for teaching mathematical topics
- become more confident in the mathematics classroom.

Moving around the course

Working through the individual sections of this course you will meet points where you will be asked to do some mathematics. In this case you will see the word 'Activity'. When you have completed the maths, there will usually be a solution given. Click on the button labelled '**Reveal answers'** to check your working.

Sometimes the Activity will prompt you to think over what you have read, or you will be asked to reflect on your understanding of the learning and teaching of mathematics. After this there will usually be some further suggestions or context. Click on the button labelled '**Reveal discussion'** to access these suggestions or context.

In the 'Summary' at the end of each week, you will find a link to the next week. If at any time you want to return to the start of the course, click on 'Full course description'. From here you can navigate to any part of the course.

It's also good practice, if you access a link from within a course page (including links to the quizzes), to open it in a new window or tab. That way you can easily return to where you've come from without having to use the back button on your browser.

Introduction and guidance

The Open University would really appreciate a few minutes of your time to tell us about yourself and your expectations for the course before you begin, in our optional start-of-course survey. Participation will be completely confidential and we will not pass on your details to others.

What is a badged open course?

While studying *Teaching mathematics* you have the option to work towards gaining a digital badge.

Badged courses are a key part of The Open University's mission *to promote the educational well-being of the community.* The courses also provide another way of helping you to progress from informal to formal learning.

To complete a course you need to be able to find about 24 hours of study time, over a period of about 8 weeks. However, it is possible to study them at any time, and at a pace to suit you.

Badged courses are all available on The Open University's OpenLearn website and do not cost anything to study. They differ from Open University courses because you do not receive support from a tutor. But you do get useful feedback from the interactive quizzes.

What is a badge?

Digital badges are a new way of demonstrating online that you have gained a skill. Schools, colleges and universities are working with employers and other organisations to develop open badges that help learners gain recognition for their skills, and support employers to identify the right candidate for a job.

Badges demonstrate your work and achievement on the course. You can share your achievement with friends, family and employers, and on social media. Badges are a great motivation, helping you to reach the end of the course. Gaining a badge often boosts confidence in the skills and abilities that underpin successful study. So, completing this course should encourage you to think about taking other courses.

How to get a badge

Getting a badge is straightforward! Here's what you have to do:

- read each week of the course
- score 50% or more in the two badge quizzes in week 4 and week 8.

For all the quizzes, you can have three attempts at most of the questions (for ‘true or false?’ type questions you usually only get one attempt). If you get the answer right first time you will get more marks than for a correct answer the second or third time. Therefore, please be aware that it is possible to get all the questions right but not score 50% and be eligible for the badge on that attempt. If one of your answers is incorrect you will often receive helpful feedback and suggestions about how to work out the correct answer.

For the badge quizzes, if you're not successful in getting 50% the first time, after 24 hours you can attempt the whole quiz, and come back as many times as you like.

We hope that as many people as possible will gain an Open University badge – so you should see getting a badge as an opportunity to reflect on what you have learned rather than as a test.

If you need more guidance on getting a badge and what you can do with it, take a look at the OpenLearn FAQs. When you gain your badge you will receive an email to notify you and you will be able to view and manage all your badges in My OpenLearn within 24 hours of completing the criteria to gain a badge.

Get started with Week 1.

Week 1: Teaching and learning: whole numbers and decimals

1 How children develop number sense

Figure 2 Set of numbers from 0 to 9

This section should help you appreciate that understanding how children develop number sense is helpful to understanding how they will learn about the number system and calculation in the middle school years. It can also help you understand where potential problems stem from.

1.1 Learning to count and counting effectively

Young children learn about quantity as they interact with their environment. They recognise that it is possible to have one of something, or to have more of something. They can judge whether there is more of one item, say, sweets, than there is of another item, say, biscuits, by comparison (for example, by using one-to-one correspondence).

Small children also learn to count, usually helped by the adults and older children in their life. For example, it is common to count the number of stairs you are climbing, or the number of biscuits you have. Counting seems to be an easy skill to learn but it is actually more complex than many people think.

Activity 1 How do children learn to count?

Allow 5 minutes

Take a minute to think about what counting entails and the skills needed to be able to count.

Discussion

Did you think of the following?

- Children need to learn the names for the numbers and the order in which they come. In the English language this is one, two, three, four, etc.
- When counting objects, children learn to refer to each object (perhaps by pointing) and to say the next number in the sequence. This is called **one–to–one correspondence**.
- The last number in the counting sequence gives the number of objects being counted (one, two, three, four, so there are four objects). This is known as **cardinality**. The cardinal number is the last number in the counting sequence and this gives you the quantity of the objects. Clearly, you need to start with the number 1 if you are to do this accurately.
- Finally, children learn that the order in which the objects are counted does not matter. They still get the same final number. This is the **conservation** principle.

1.2 Making the connection between quantity and number

Young children learn about quantity in a natural, intuitive way and are able to understand that you can have more of one thing than another. Quantity is a measure of how much you have of something. In order to be able to indicate the quantity, you need to use numbers (and a measure, so that you can say you have a number of kilograms or metres of something; but, for our current purposes, you are thinking of discrete objects which can be counted).

Children are taught to count and to name the numbers used in counting. The next stage of their development in understanding number is to put quantity and number together. Specifically, children need to be able to link the final number in the counting sequence with the size of the quantity of objects which they have counted. This is not an easy concept to develop and can take several years in primary school..

Children learn that adding 1 to a number gives the next number in the sequence. However, it is more difficult to grasp the effect of adding 2 to a number.

Early calculation skills which are taught in primary school

a. Counting forwards from 1 and backwards to 1.
b. Starting from different numbers and then counting forwards and backwards.
c. Addition by counting. For example, you want to add 7 biscuits to 4 biscuits. In order of sophistication, you could do this in three ways:

- **Count all** – you could count out 7 biscuits, then you could count out 4 biscuits. Then, after you have put the two groups together, you could count the total number of biscuits.
- **Count on** – you could keep the number 4 in our head and then count on 7 from there.
- **Count on from larger number** – you could save yourself some effort by beginning with the larger number, i.e. 7, and count on 4 from there.

This all seems very easy, but you need to remember that young children do not have the ready familiarity with numbers that you have. To try to imagine how tricky it can be for a young child to use counting skills in addition, try the following activity (without converting the letters into numbers!).

Activity 2 Adding numbers when you are unfamiliar with their names

Allow 5 minutes

Suppose the counting sequence is Y, J, A, P, K, T, B, M.

- Count from A to B.
- What is J + K?
- Count back from M to P.
- Count back from B to J.
- What is T – A?
- What is K – Y?

Discussion

From A to B is P.

J + K = B

From M to P is P

From B to J is K

T – A = A

K – Y = P

How did you do?

1.3 Part-whole relationships and number bonds

You can make the number 8 in different ways: , , etc. Children can learn these by manipulating physical objects such as bricks and tiles. Starting with 8 objects, a child can split these objects into two or more groups(Figure 3).

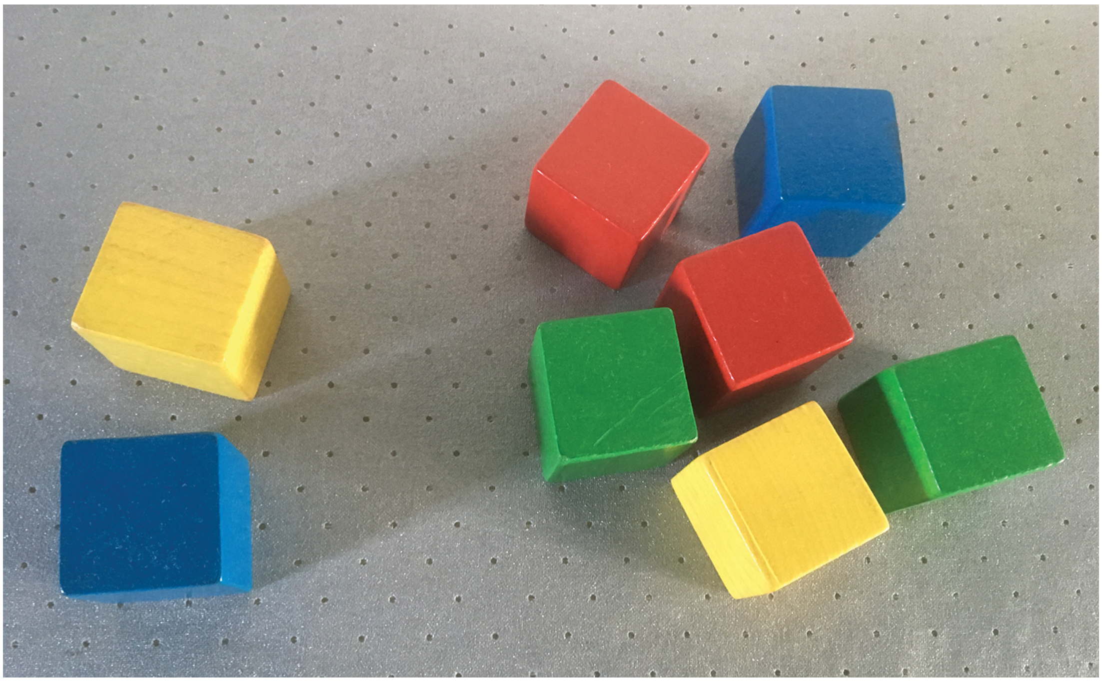
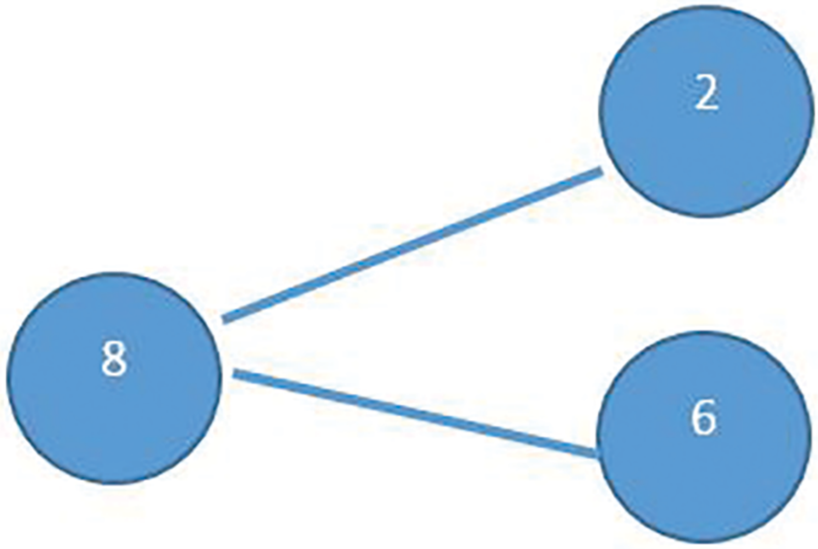


Figure 3 Using blocks to demonstrate number bonds to 8

This can also be done pictorially using a part-whole diagram or a bar diagram (Figure 4).



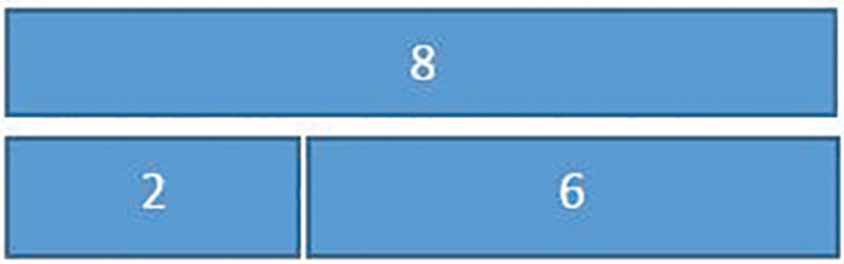


Figure 4 Diagrams which illustrate

Part-whole relationships are important for learning about number as they develop the understanding of how numbers are made up of other numbers. That is, numbers can be partitioned into smaller constituents in the same way that 8 can be made up of 2 and 6.

The **number bonds** refer to pairs of numbers which sum to a given number. The important sums are 10, 20, 100, etc. Learning the number bonds to 10 is an important first step in being able to calculate. For example, adding 6 + 7 can be done as 6 + 4 + 3 if it is known that 6 and 4 are numbers which sum to 10. This particular method is known as **bridging** because you are making the number up to 10 and then adding the remaining part of the number to go past it.

In this way, addition is facilitated by using the number bonds to 10. So, knowing that 4 is added to 6 to make 10, and then knowing the part-whole relationships for 7, i.e. that 7 is also 4 + 3.

1.4 Understanding that addition and subtraction are inverses of each other

Children first learn to apply inverse relationships to quantities before they can apply them to numbers.

For example, five-year old children understand that if you have 6 biscuits and you add 3 more biscuits and then take those same 3 biscuits away, then you have the same number of biscuits that you started with.

However, if you were to take away 3 different biscuits, the child may not realise that the result is the same number of biscuits that you started with. This suggests that the child is thinking in a very concrete way about the biscuits themselves, rather than abstractly thinking about numbers.

When children eventually understand that addition and subtraction are inverses of each other, this is a big step towards them developing number sense.

Multiplying using additive reasoning

Children learn how to multiply in the first instance by using repeated addition. For example, if there are 3 flowers in each of 4 vases, children can work out the total by counting 3 for each vase using a one-to-one correspondence. This underpins the concept of multiplication and paves the way for developing secure methods of multiplication calculations later on.

If children do not understand the concept of multiplication as repeated addition then learning their multiplication tables will not really help later in their school career. The basic foundational concept is important.

Laying the foundations for future mathematics learning

If children are able to understand that addition and subtraction are inverses of each other, and if they can use one-to-one correspondence to solve early multiplication problems, this gives them a secure foundation for later learning of mathematics, not just in arithmetic. It is important that children who struggle with this in the early school years are given extra support to make these developmental strides.

1.5 Moving on from counting and working out relationships between quantities

Suppose I have 7 biscuits and you have 4 biscuits. I have 3 more than you.

Here, the number 3 gives the relationship between the two different quantities of biscuits, between the number of my biscuits and the number of your biscuits. It is harder for children to work out problems like this, which involve relationships, than problems that just involve quantities. It is not so much the number crunching that is the difficulty but the conceptual understanding of relationships which is difficult. Children can count biscuits but they cannot count the relationship between the numbers of biscuits.

Skills which help children to grasp the concept of relationships between quantities are: adding and subtracting, understanding that these are inverses of each other and part-whole relationships.

Moving from using manipulatives to pictures and on to abstract numbers

When children first begin learning about numbers in the early years, their teachers will give them concrete materials to manipulate and count. These are referred to as **manipulatives**. Using manipulatives such as counters, bricks and specialist educational equipment is necessary for young children to develop their number sense.

As children become more proficient at using manipulatives, the teachers will begin to link the manipulatives to pictures, and to use pictorial images to represent number. As the children become more proficient at using images, they will be carefully introduced to more abstract ways of representing a number problem.

Whenever children need to make sense of new knowledge, teachers will often return to the chosen manipulatives or pictures and start the process of abstraction again.

2 Developing mental methods

Being able to calculate mentally helps learners to develop a secure understanding of numbers and the number system, allows learners to develop their own intuitive methods and provides a firm foundation for written methods of calculation.

The practice of doing mental calculations is an important skill in everyday life. It also helps to develop fluency and confidence with numbers. How can you help your learners to develop effective mental methods? What follows are examples that you can use or modify for use with your learners.

2.1 Using number bonds to 10, 20, 100

Learning number bonds is an important skill which supports learners to be able to answer more difficult questions. For example, learning how to quickly calculate number bonds to 100 can help with bridging over 100.

A quick way of calculating 100 – 38 is to make the tens number up to 90 and the units number up to 10.

So

You can also say that 62 is the complement to 38.

Note that we have not used an ‘equals’ sign because these are not strictly mathematical sentences but they mirror what the learner may be saying in their head or aloud.

Similar quick ways can be used to find numbers that sum to 1000. For example, to work out 1000 – 159:

So or

This method also works for reckoning the change from £10.00.

2.2 Mental addition and subtraction strategies

Using number bonds can help with mental addition calculations because numbers can be split up to make the tens or hundreds number and then the extra can be added on.

Activity 3 Adding large numbers mentally

Allow 5 minutes

Try these questions and reflect on your methods.

a.

b.

c.

Discussion

Did you partition the numbers into smaller numbers? For example, for question (a) did you work out 40 + 30 and then work out 3 + 2?

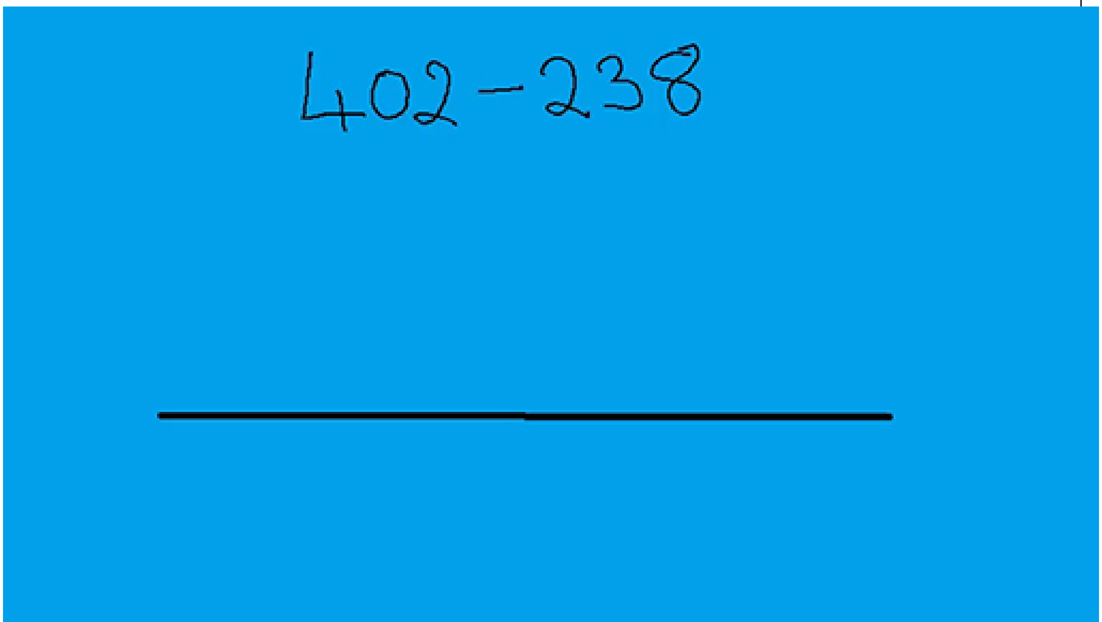
Did you bridge across a multiple of ten? For example, in question (b) did you mentally change 47 + 38 into 47 + 3 (to make 50) and then add 35?

Once a learner can calculate what to add to 38 to make 100 they can answer the question 100 – 38 and move on to calculations such as 402 – 238 by bridging through 300.

This mental arithmetic can be illustrated using an empty number line. The actions on the number line can act as a way to record the stages in a mental calculation. This is shown in the following video.

Video content is not available in this format.

Video 1 Using the empty number line to subtract 238 from 402



We can use this method to answer questions such as £4.02 – £2.38 or 4.03 m – 2.38 m, taking us into the realm of using measures. This method also helps with time calculations where learners need to remember that time is not a decimal system. For example, a train journey which starts at 2.38 pm and goes on until 4.02 pm takes 22 minutes + 1 hour + 2 minutes which comes to 1 hour 24 minutes (bridging through 3 pm and 4 pm).

Activity 4 Subtraction problems for you to solve

Allow 5 minutes

Try answering the following in your head using whichever method you prefer.

a.

b.

c.

You may check your answers using a calculator.

2.3 Mental multiplication and division strategies

Doubling and halving are easy for most people, so make use of these tactics:

- doubling and doubling again is the same as multiplying by 4
- doubling and doubling and doubling again is the same as multiplying by 8.

Use the inverse of doubling (i.e. the opposite of doubling) which is halving:

- halving is the same as dividing by 2
- halving and halving again is the same as dividing by 4
- halving and halving and halving again is the same as dividing by 8.

Once we begin talking about halving being the same as dividing by 2 this connects to fractions, decimals and percentages. There will be more of this in Week 2 but for now:

- half of 100% is 50% → to find 50% of an amount just half it
- half of 50% is 25% → to find 25% of an amount half and half it again.

Activity 5 Finding halves and quarters

Allow 5 minutes

Try the following questions.

Discussion

Listed below are some useful mental multiplication strategies:

Multiply by 5 by multiplying by 10 then halving

Multiply by 20 by multiplying by 10 then doubling (or vice versa)

Multiply by 50 by multiplying by 100 and halving (or vice versa)

Multiply by 11 or 9 by multiplying by 10 and adding or subtracting

Multiply by 21 or 19 by multiplying by 20 and adding or subtracting

Understanding division problems

Many learners (and adults) have problems with division because they do not exactly understand what it means. A straightforward division can be ambiguous without the context of the problem. Learners need to understand the difference between sharing and grouping which are the two main types of division problem. For example:

- can mean there are 12 sweets **shared** between 3 people, so they each get 4 sweets.
- 12 has been divided (or shared) into 3 equal parts of 4.
- 12 ÷ 3 can mean that I have 12p, so how many 3p sweets can I buy? In this case, you can count up in threes to find how many sweets you can buy. You are counting up in **groups of 3,** (i.e. 3, 6, 9, 12). So there are four 3s in 12.

Division by partitioning can be a useful strategy

Partitioning the number into a multiple of the divisor plus the remainder is a useful way if a learner is familiar with their multiplication tables. For example, 84 ÷ 7:

e.g. $84 \div 7$

$$\begin{aligned} 84 \div 7 &= (70 + 14) \div 7 \\ &= 10 + 2 \\ &= 12 \end{aligned}$$

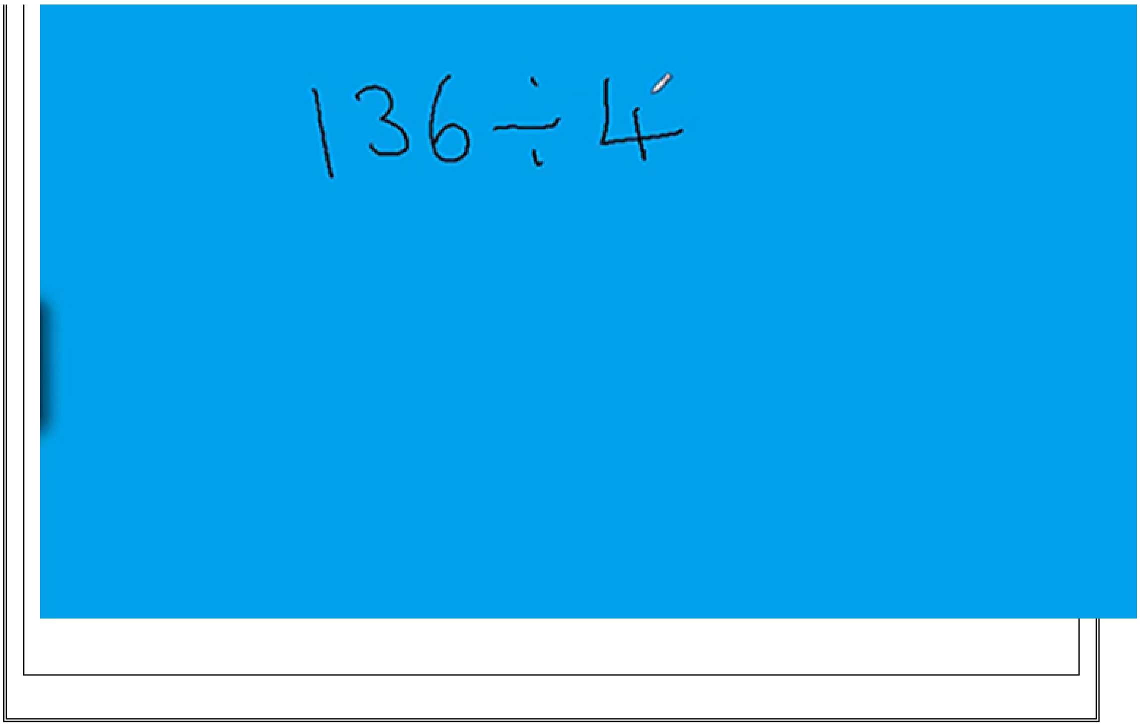
Activity 6 Division by partitioning

Allow 10 minutes

Calculate 136 ÷ 4 using the partition method. Then watch Video 2 for a demonstration of one way to do this.

Video content is not available in this format.

Video 2 Dividing 136 by 4 using partitioning



Multiplication tables

While there is no doubt that knowing the multiplication tables is a useful skill, learning by rote without an understanding of the concept of multiplication and of division is less than useful. Mental maths, which includes multiplication and division, helps learners to form the appropriate concepts.

Many successful mathematicians do not know their tables by rote. However, they do know how to work them out as needed.

Developing mental arithmetic with your learners

Practise frequently with your learners. The beginning and end of lessons are often good times to practise mental arithmetic – maybe spend two or three minutes doing this.

Starter activities could include working out mental arithmetic questions as part of a quiz, an activity, etc. Make it into a game.

Do ask learners to say how they worked something out. They could tell their method to the person sitting next to them. Or they could tell their method to the class.

Validate mental calculations. Children will not see the point in having to write down a method of arithmetic for a calculation they can do in their head. Nor should they have to: mental calculation is a perfectly good way of working out the answers to number questions, so encourage its use.

3 Developing formal written calculation methods

This course mainly focuses on teaching children in the middle school years, so they have probably met all of the four rules (addition, subtraction, multiplication and division) and been taught formal methods of calculation. This does not guarantee that learners have learned these methods effectively! Nor is there any guarantee that if a learner can operate a method and get lots of right answers that they understand how the method works.

As a teacher, therefore, you are striving to teach effective methods, procedural understanding of how those methods work, and conceptual understanding of the mathematics which underpins the methods.

3.1 Formal written methods of addition

Of the four basic arithmetic operations, the addition of whole numbers and associated written methods of addition tend to be reasonably well grasped. It is the other three operations where learners have difficulties.

Activity 7 Reflecting on two methods for adding three-digit numbers

Allow 10 minutes

Watch Video 3 on formal written methods and then reflect on the difficulties a learner might experience when using either of these methods.

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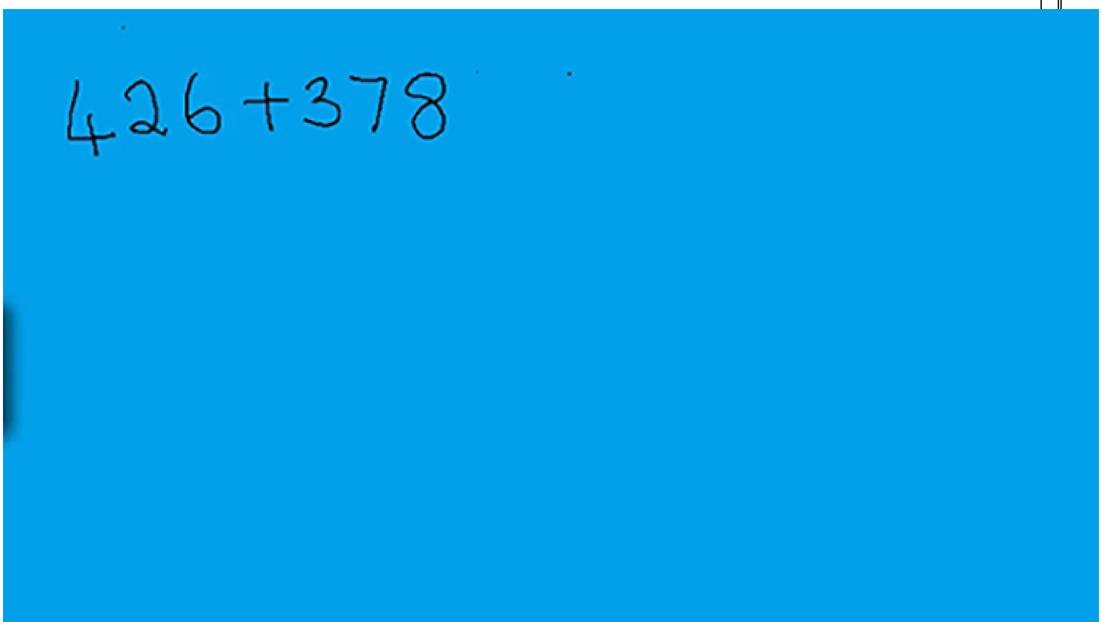
Video 3 Demonstrating formal written methods for addition



Discussion

If learners have developed mental calculation skills then these will be built on when moving to written addition methods. If learners have not developed good mental calculation skills, using written methods will be much more difficult.

If you think back to the early years, remember that learners operate addition by counting out the sets of objects and then combining the sets. If they have not progressed from this stage to being able to add up abstract numbers mentally then adding large numbers using a written method will be cumbersome.

A learner who struggles to add 6 + 8 and who uses their fingers or makes marks on paper (and I have seen both these activities used by secondary school pupils) will struggle to add 60 + 80. If they only operate addition by counting sets of objects then they are unlikely to have developed a robust concept of place value and therefore no appreciation that 6 tens + 8 tens gives us 14 tens. This is why it is so important that learners build a firm foundation of understanding in the early years. It also implies that expecting learners to move on to the next stage (e.g. of doing formal written addition before they can mentally add abstract numbers) is not appropriate. As a teacher, you may need to 'go back to basics' with learners who are in this position.

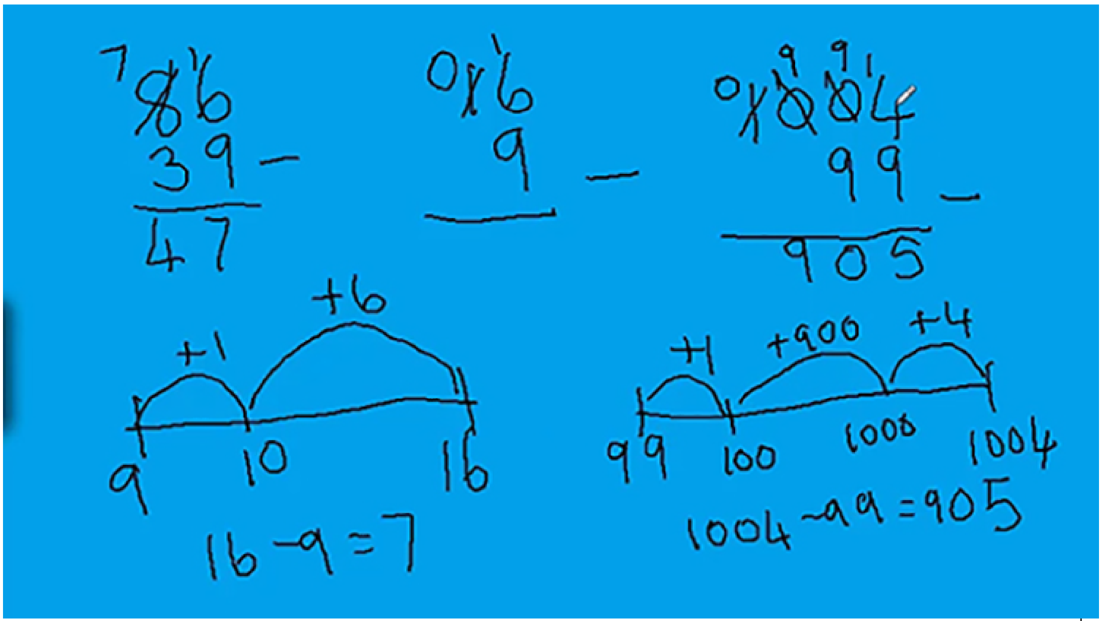
3.2 Formal written methods for subtraction

Subtraction is experienced as being more difficult than addition. Subtraction is the inverse of addition and knowing this can help learners to use complementary addition in informal methods of subtraction.

Video 4 demonstrates the decomposition method (or column method) of subtraction and the empty number line method which uses complementary addition. This latter method is a visible example instantiation of the method which many people use mentally.

Video content is not available in this format.

Video 4 Choosing the best method for subtraction



Activity 8 Choosing the most appropriate method of subtraction

Allow 8 minutes

Consider the following subtractions and decide whether it is more appropriate to use the decomposition method or the empty number line method.

a.

b.

c. The length of a journey which starts at 9.55 a.m. and ends at 11.05 a.m.

Discussion

If the numbers in the subtraction are close together then it makes more sense to 'count up' from the smaller to the larger, the method known as known as complementary addition. This is especially useful if the numbers bridge over a 10s or 100s sort of number or over the hour in a time calculation.

3.3 Multiplication and division

The next video demonstrates two methods for long multiplication and discusses the pros and cons of each method.

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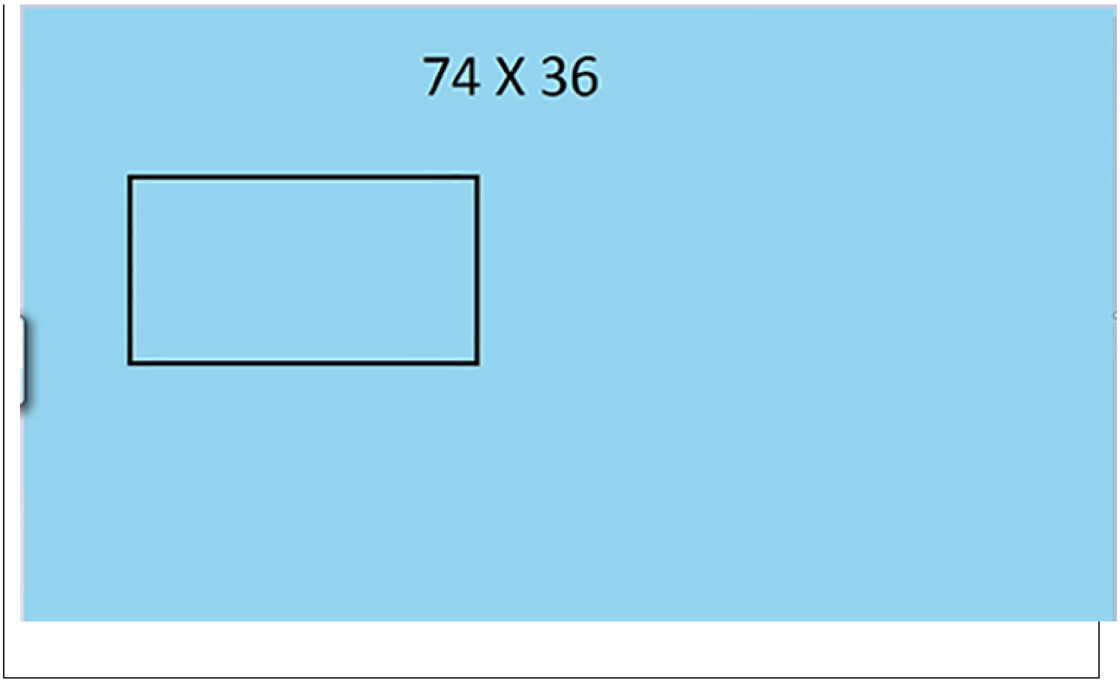
Video 5 Two methods for long multiplication



Division

Division is the arithmetic operation where learners experience the most difficulty. Division is the inverse of multiplication which means that it is possible to approach division by complementary multiplying. For example, to answer the question 78 ÷ 6 you can ask 'How many 6s make 78?'.

One way to answer the question in this way is to use the 'chunking' method where you add 'chunks' of 6 until you reach 78. It saves having to add 6 at a time, which could be laborious. For example:

(you know that 78 is more than 6 lots of 10, so start there)

(you have 18 left to make the 60 up to 78, that's 3 lots of 6)

So

Or you could start with 78 and keep subtracting 6 until you have nothing (or a number smaller than 6) left: For example:

78

78 ÷ 6 calculating by subtracting sixes

Answer

Whether working up to 78 or down from 78, the method is basically finding how many 6s there are in 78. This method clearly becomes harder with larger numbers and requires knowledge of multiplication tables and good mental arithmetic. It is, however, an intuitive method of dividing.

The traditional method of division is known among teachers as either the bus stop method (because the working looks like a bus stop),or the ‘gazinta’ method (because you ask how many times 6 ‘goes into’ each number).

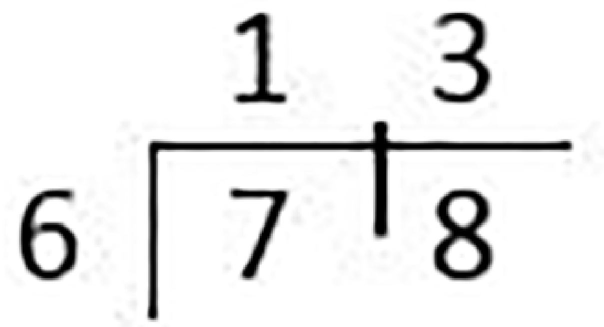


Figure 5 One formal method of short division

Figure 5 shows that 6 goes into 7 once with 1 left over. So, put the 1 (for the once) above the 7 and the 1 left over next to the 8. Now read it as 6 goes into 18, which is 3. Put the 3 over the 8. This is an easy method to operate if you can remember the procedure. It is not so easy to understand why it works. Yet many learners prefer it to having to use the ‘chunking’ method.

The bus stop method is efficient to use with larger numbers as well, as long as your multiplication tables are good, and you can do some mental calculation. When dividing by two-digit numbers, I prefer to use and to teach a version of this short division method.

Activity 9 Solving a division problem

Allow 10 minutes

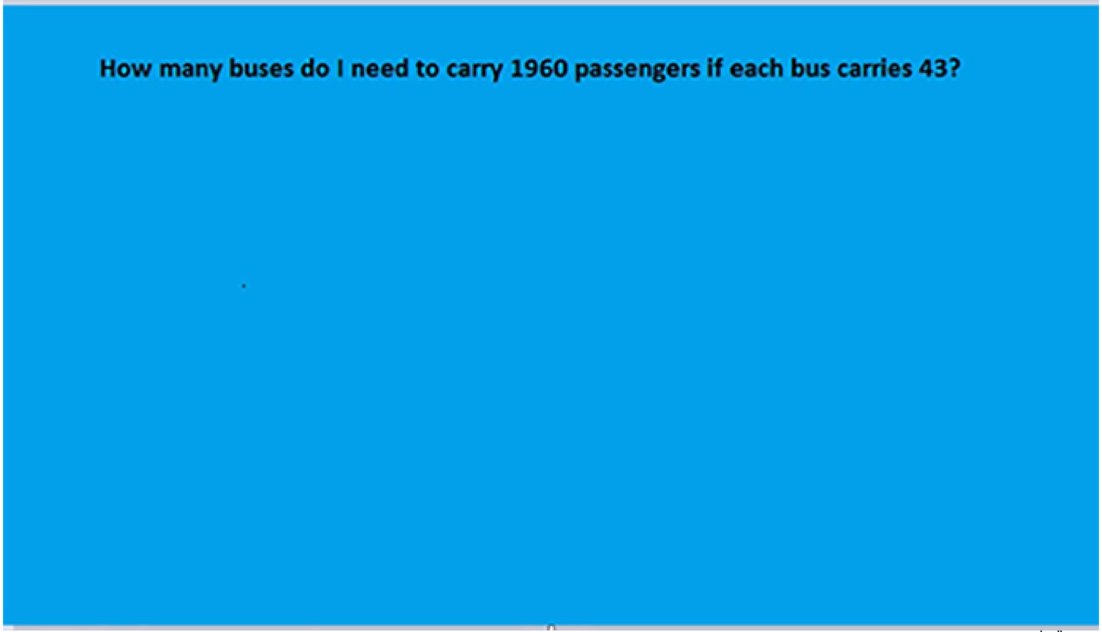
Suppose you have 1960 pupils and teachers in a school to ferry to an event in buses which each carry 43 passengers. How many buses will be needed?

So, you are trying to find out how many times 43 goes into 1960 and you need an extra bus for the remainders.

Spend ten minutes ‘having a go’, using whichever method you prefer. Then watch the video.

Video content is not available in this format.

Video 6 Two ways to solve a division problem


How many buses do I need to carry 1960 passengers if each bus carries 43?

4 The concept of place value and decimals

Place value underpins our number system, so understanding it is important in being able to calculate with numbers. Even learners in secondary education can be unsure of the place value system, so it is important to provide activities which support their conceptual understanding of it.

4.1 The decimal point and calculations with decimals

The decimal point is a symbol which is placed between the whole part of the number and the fraction part of the number. It signifies that the digit to its left is a units digit and the digit to its right is a tenths digit. All the rest of the digits fall into place to the left and right. Nothing more, nothing less. The decimal point is not some sort of bridge to jump over or obstacle to be overcome but many learners treat is as if it is.

Here is an example of a common mistake.

Since 5 + 5 makes 10, the learner has written 10 in the tenths column, presumably because they do not think that carrying over the decimal point is allowed. They see the decimal point as splitting the numbers into two parts which need to be dealt with separately.

However, a knowledge of place value means that the 1.5 is seen as 1 unit + 5 tenths, which when doubled is 2 units + 10 tenths (convert to 1 unit) which is 3 units.

One way of helping learners to appreciate this is to practise decimal times tables. Here are the first ten multiples of 0.3:

0.3, 0.6, 0.9, 1.2, 1.5, 1.8, 2.1, 2.4, 2.7, 3.0

It is tempting to say 0.12 after 0.9 but think about what it means in terms of the value of each digit. Bridging over the whole numbers is always the crunch point for the decimal times tables. They provide a useful exercise for helping learners see that numbers are carried into the next column on the left in just the same way as in whole numbers. They also look the same (as in having the same digits) as their related multiplication tables but including the decimal point.

You can think of all numbers as being decimals if you think of whole numbers as having an invisible decimal point after the units digit. You usually only make the decimal point visible if there are digits after it.

Activity 10 Using decimal times tables

Allow 3 minutes

Write out the first ten members of the 1.5 times table. You will know that you have them correct if the last one is 15.0.

4.2 Becoming familiar with decimals

Adding and subtracting decimals can be done using the same methods as adding and subtracting whole numbers as long as the digits are lined up under the correct column headings, keeping the decimal points underneath each other.

Learners can struggle to handle decimals, so it is helpful to use real-life instances with which the learners are familiar.

Numbers with one decimal place are evident on a classroom ruler set out in centimetres (Figure 6).

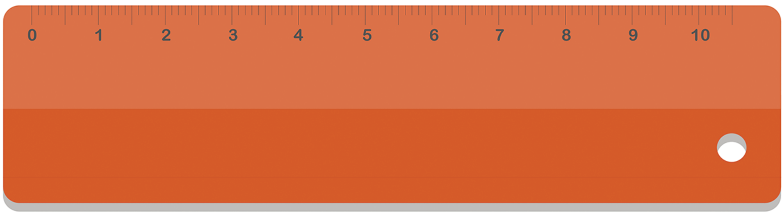


Figure 6 A ruler showing centimetres and tenths of centimetres

There are 10 gradations between each whole number (of centimetres), so each interval is 1 tenth or 0.1. Rulers are effective number lines which learners have easy access to.

A useful activity to use with learners who struggle with decimals is to ask them to draw a straight line across a page. Next, they should put a mark near the beginning and label it A. Ask them to put another mark 4.2 centimetres along the line from A and to label it B. Next ask them to put another mark 2.9 centimetres along the line from B and label it C. Now ask them to measure from A to C. Then ask how they can check their answer (i.e. add up 4.2 and 2.9). This activity (and you can use more than 2 measurements) is useful in encouraging the accurate use of a ruler and measuring and working with decimals.

Numbers with two decimal places will be familiar to your learners through the use of money. Having said this, today many people use contactless card payments and may not even think about their transactions in terms of exchanges of money. Yet, I often used to say to my learners 'think of two decimal place numbers as money' and it helped them to feel they could cope with the calculations.

Now watch Video 7 for more advice on decimal number lines.

Video content is not available in this format.

Video 7 Decimal number lines with 1, 2 and 3 decimal places

Since we use a number system in base ten, special things happen when you multiply or divide by powers of 10. A place value table is useful for explaining what happens. Now watch Video 8 for more information on using a place value diagram.

Video content is not available in this format.
Video 8 Multiplying and dividing by powers of 10 using a place value diagram

thousands	hundreds	tens	units		tenths	hundredths	thousandths
	1	2	3	.	4	5	
	1	2	3	.	4	5	

5 This week's quiz

Check what you've learned this week by doing the end-of-week quiz.

Week 1 practice quiz

Open the quiz in a new tab or window and come back here when you've finished.

6 Summary of Week 1

Now that you have completed Week 1 you have learned how early counting leads to early addition, which then leads to subtraction, multiplication and division. Careful development of these skills helps children to achieve a sense of how numbers work, providing a firm foundation for future mathematics learning. You have explored varied methods, both mental and written methods, for adding, subtracting, multiplying and dividing. Finally you have looked at place value and decimal numbers. You should now be able to:

- briefly describe how number sense develops in young children
- demonstrate a variety of mental, informal and formal written methods of basic arithmetic
- understand how place value underpins our number system in base ten, for both whole numbers and decimals.

Next week you will learn about the conceptual basis of fractions and percentages, the pitfalls learners often meet when studying fractions in particular and effective approaches to the teaching of fractions and percentages.

You can now go to Week 2.

Week 2: Teaching and learning: percentages and fractions

1 Equivalence as a concept applied to fractions, decimals and percentages

In mathematics the term 'equivalence' is used to mean 'has the same value' or 'can be substituted for'.

Fractions, decimals and percentages each have two main uses:

- Fractions, decimals and percentages are numbers, for example, , 3.2, 78% are all numbers and have a position on a number line
- Fractions, decimals and percentages can be used to denote a proportion of a quantity, for example we can find of £6, 3.2 of 5 kilograms or 78% of 250 people.

Confident learners may be able to move between the number meaning and the proportion meaning of fractions, decimals and percentages but many learners find this confusing.

Consider the following well-known equivalences:

If you think of these as numbers then their equivalence means that they all occupy the same position on a number line. Their equivalence in this case means 'has the same value'.

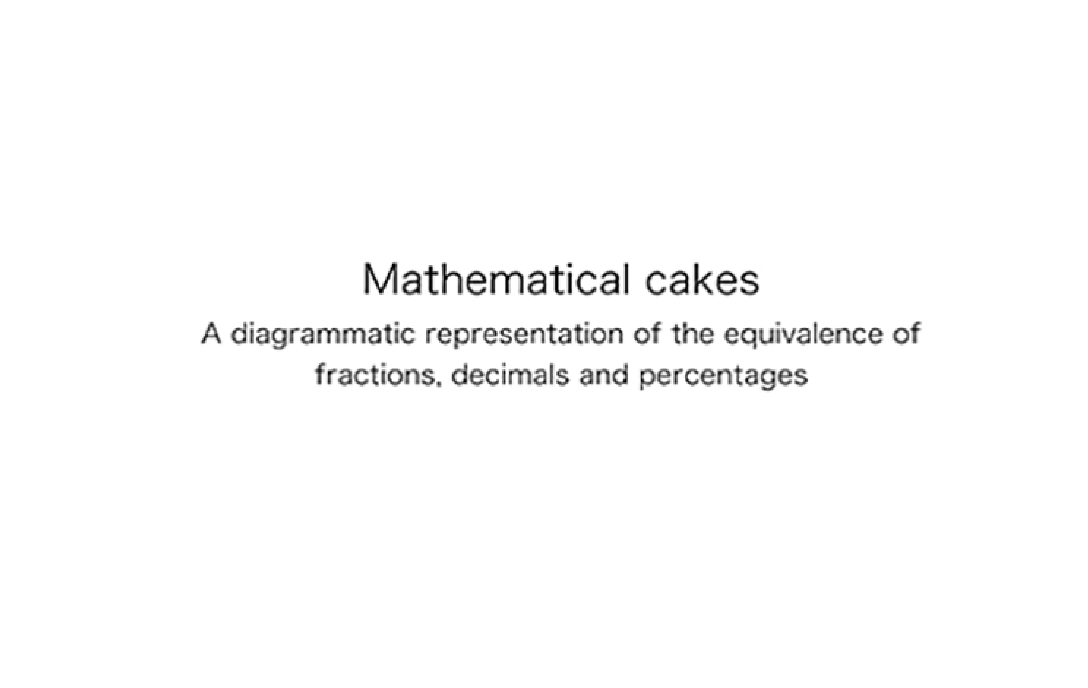
If you think of these as proportions then their equivalence means that finding of a given quantity is the same as finding 0.5 of that same given quantity or 50% of that same given quantity. Their equivalence in this case means 'can be substituted for'.

1.1 Diagrammatic representations of equivalence

The first video shows a mathematical cake which is shaped like a rectangle and has been cut into ten slices. Learners can draw their own mathematical cake on paper. It can have a varied length but it is useful to ensure that the width is 10 cm, which means each slice is 1 cm wide. Diagrams can be very useful aids to teaching concepts in mathematics.

Video content is not available in this format.

Video 1 Mathematical cake: a diagrammatic representation of the equivalence of fractions, decimals and percentages



Activity 1 Finding equivalences to one-quarter and three-quarters

Allow 5 minutes

Look at the mathematical cake diagram which was shown in the video. Use it to find equivalences to and .

Discussion

The mathematical cake diagram can be used to demonstrate the equivalences of percentages, decimals and fractions in tenths.

One-quarter can be found by finding the mid-way point between 0 and one-half. Three quarters would be found mid-way between one-half and one-whole of the cake.

One-quarter is 25% and 0.25.

Three-quarters is 75% and 0.75.

The fraction wall

Figure 1 shows a fraction wall divided into strips of the same length which can be split into equal parts. The fraction wall can be used to identify equivalent fractions.

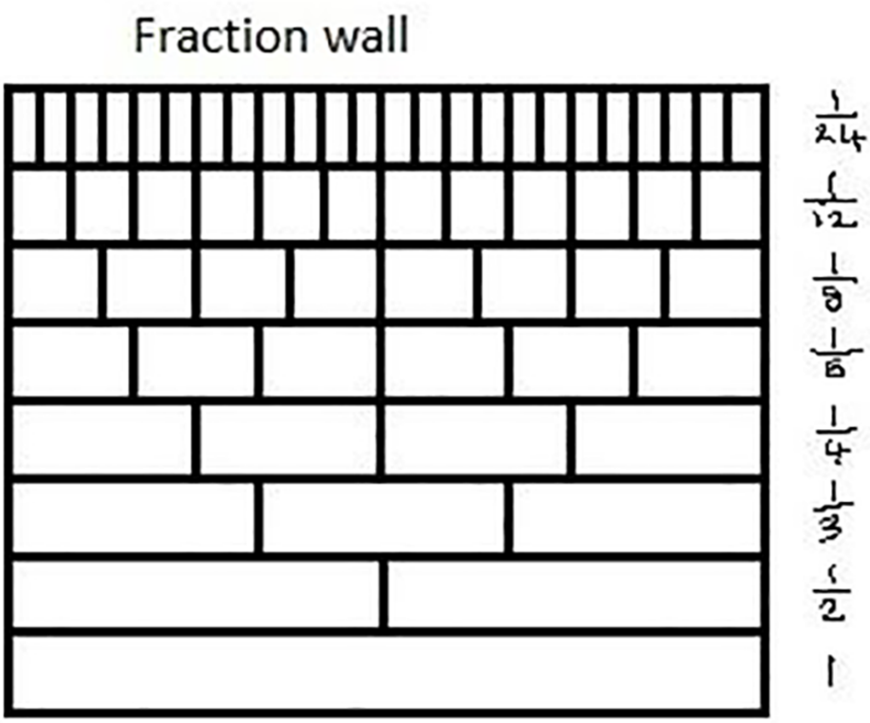


Figure 1 Fraction wall

Activity 2 Finding equivalent fractions

Allow 5 minutes

Use the fraction wall to find as many fractions as you can that are equivalent to .

Use the fraction wall to find as many fractions as you can that are equivalent to .

1.2 Challenges to understanding the equivalence between fractions

You have been introduced to two diagrams which can help people learn about the equivalence of fractions, decimals and percentages through visualisation.

Activity 3 Thinking of other useful diagrammatic representations

Allow 5 minutes

Can you think of any other diagrams which you have used, or seen in use, which help learners to understand fractions, decimals and percentages?

The concept of equivalence is important in mathematics. Can you think of other areas of mathematics where equivalence is used?

Discussion

Equivalence is an important concept in mathematics. Algebra is a particular area of mathematics where equivalence is important. Equivalent algebraic expressions are different ways of denoting the same relationship.

2 Approaches to teaching fractions

In this section you will meet again the ideas of equivalence in fractions. You will also be introduced to some different approaches to teaching the calculation of fractions, including the use of diagrams.

Activity 4 Where are fractions used in mathematics?

Allow 5 minutes

The topic of fractions is a substantial one in the school curriculum.

List as many of the different learning activities as you can which relate to the topic of fractions.

Discussion

Did you think of any of the following?

- finding fractions of amounts
- finding equivalent fractions
- ordering a list of fractions from smallest to greatest or vice versa
- adding and subtracting fractions
- multiplying and dividing fractions
- and using fractions within other topics:
- solving problems using proportion
- algebraic equations which involve fractions
- calculating probabilities

2.1 What are fractions?

Fractions are numbers made of two integers (positive and negative whole numbers), one acting as the numerator and one acting as the denominator.

(Fractions typically refer to quantities which are smaller than 1. In this case the numerator is smaller than the denominator. Fractions with a numerator larger than the denominator have a value greater than 1. They can also be written as mixed numbers; the whole number part and the fraction part).

Consider the fraction illustrated in the diagram below which shows of a rectangle shaded. The rectangle has been divided into 4 equal parts and 3 of these parts have been shaded.

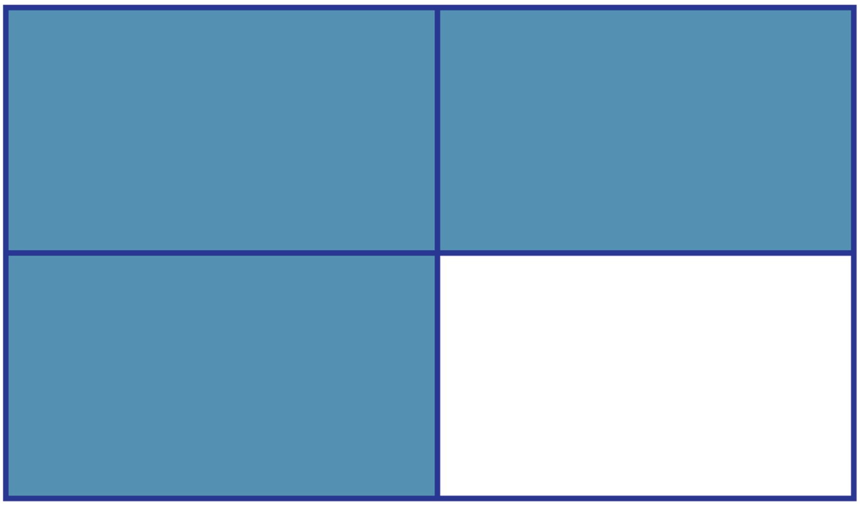


Figure 2 A rectangle shaded to show three-quarters

Three- quarters of any quantity can be found using the same process. is found (by dividing by 4 to calculate four equal parts), and then multiplied by 3 to give . This is referred to as a part-whole relationship (Figure 2).

However the number can also be taken to mean that 3 has been divided into 4 equal parts and in this case the value of the fraction is equal to the numerator divided by the denominator. This interpretation of the fraction is used when converting fractions to their equivalent decimals. If you input into a calculator you will get 0.75, which is the decimal equivalent to . This takes the meaning of a quotient (the result of dividing one number by another).

Essentially a fraction such as can have two meanings (the whole-part relation and the quotient) and this can cause difficulty in understanding for some students. One way to address this difficulty is to tell students, for example, that means 3 lots of of something, and it can also mean . This is not the only such case in mathematics, where symbolic notation can take on different meanings and which causes difficulties in students' understanding.

2.2 Ordering fractions

The understanding of equivalence among fractions can also cause a challenge because numbers which effectively have different labels can mean the same quantity of proportion (depending on how they are being used). As an example and are equivalent fractions. They can both be shown to be equivalent to the simplest form of by 'cancelling down' i.e. dividing the numerator and denominator in each fraction by the same common factor.

Since the value of the fraction is determined by the relation between the numerator and the denominator it can be difficult to compare the sizes of the two fractions. The usual method when comparing the size of two fractions is to find equivalences which have the same denominator.

Activity 5 Which fraction is larger?

Allow 10 minutes

Find the larger of these two fractions: or .

Discussion

and which is larger?

Both the denominators 3 and 8 are factors of 24 (the least common multiple or LCM) so you can find equivalent fractions whose denominators are 24

= multiplying numerator and denominator by 8

= multiplying numerator and denominator by 3

So is the larger fraction

2.3 Adding and subtracting fractions

Learners are often told that the denominator in the fraction denotes what kind of a fraction it is (halves, thirds, etc.) and that it is only possible to add or subtract the same types of fraction, i.e. the denominator has to be the same number. This is where the difficulty lies as learners struggle to find out which number should be the common denominator and what needs to be done to change each fraction so that they do have the same denominator. Several rules and procedures exist to help learners to add and subtract fractions but, since the conceptual basis is poorly understood, learners typically do not remember the rules and procedures. This often leads to fractions being taught again and again with little success.

It is worth looking at other methods which build on and develop learners' conceptual understanding of fractions. The next section gives an approach to teaching the adding of fractions which is based on learners' prior knowledge of finding fractions of a quantity.

Activity 6 Shading a grid

Allow 10 minutes

Try the following activity.

Draw a 6 by 4 grid. Next shade the grid according to this schedule (with no shading overlapping):

Shade in red

Shade in blue

Shade in green

Shade in yellow

Shade in grey

The whole rectangles should have been shaded once the activity is completed.

Discussion

Did your grid look like this, or at least have the same number of parts per colour?

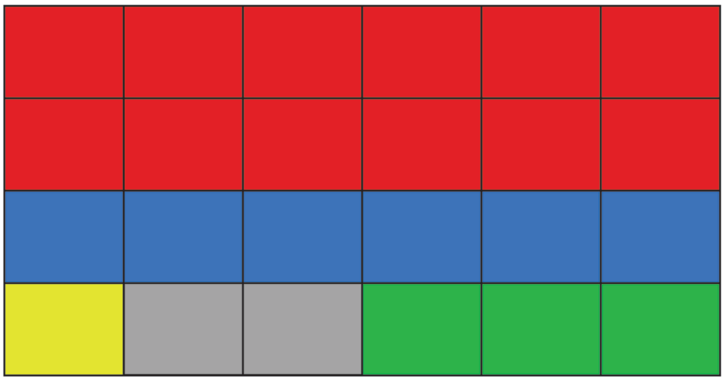


Figure 3 shading fractions of a rectangle

Activity 7 Use the grid to add fractions

Allow 5 minutes

I hope it is clear that each box in the rectangle is since the grid has been split into 24 equal parts. This should help you (and your students) to calculate the following additions of fractions based on the shaded grid.

Discussion

Did you get these answers?

With some experience of adding fractions using this grid shading methods students should begin to see that they are looking for a common denominator which would have been the number of parts in the grid. Subtraction can be carried out in a similar way by thinking about shading parts on the grid and then rubbing some of them out.

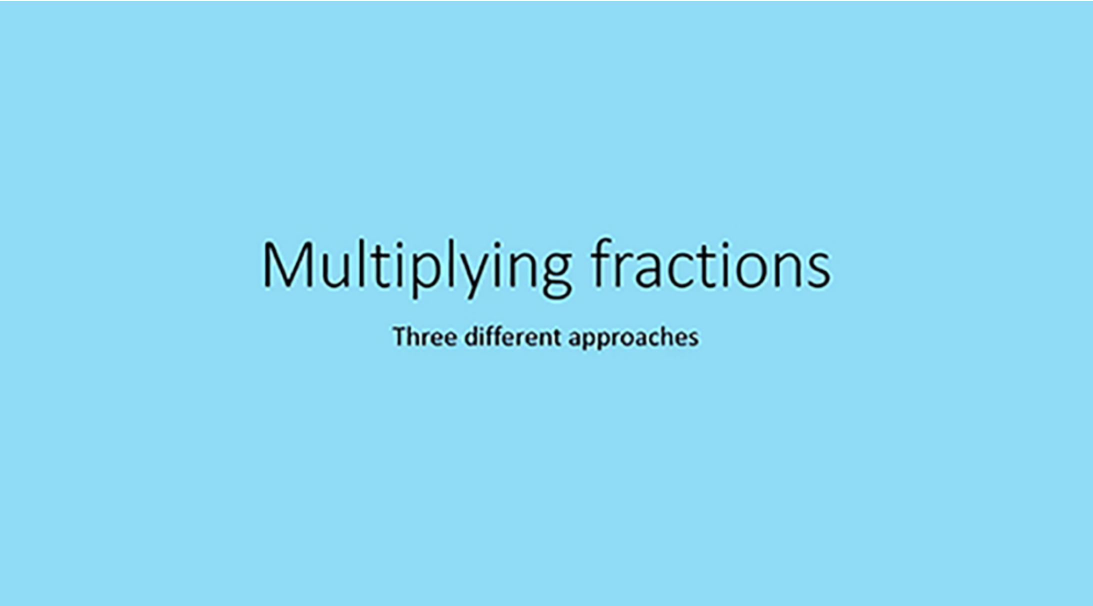
2.4 Multiplying fractions

The procedure for multiplying fractions is straightforward but the conceptual understanding is not so easy to develop.

Watch video 2 on multiplying fractions which demonstrates three different approaches.

Video content is not available in this format.

Video 2 Multiplying fractions using three different approaches



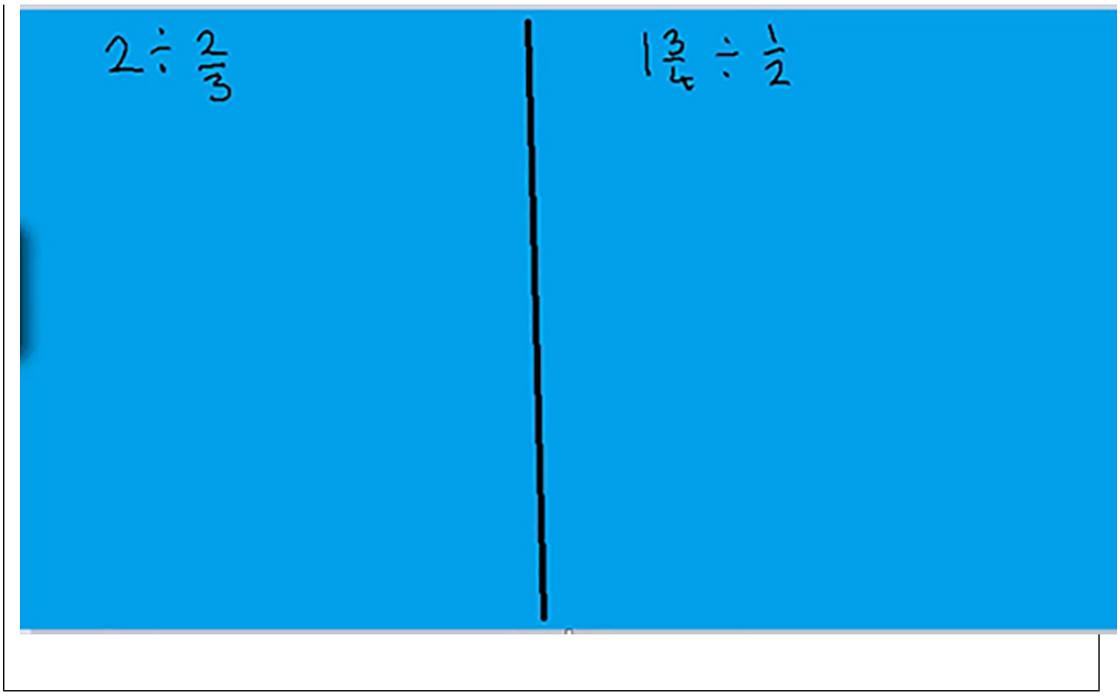
2.5 Dividing fractions

The standard procedure for dividing by a fraction is one of those where learners can get incredibly confused. ‘Turn it upside-down and multiply’ is the common mantra but which fraction (if there are two) and why does it work?

Division is the most difficult of the arithmetic functions, even for whole numbers. It can help to go back to remembering what division means. The grouping method of division is most useful when dealing with fraction division. Watch Video 3 for some useful approaches to teaching division with fractions.

Video content is not available in this format.

Video 3 Dividing with fractions



The ‘turn it upside-down and multiply’ method works because turning a fraction upside-down produces its multiplicative inverse. Fractions are unlike the integers in that they have a multiplicative inverse which can be used in fraction division because multiplying is also the inverse of dividing.

In the next section you will meet the mathematical cakes model again as a way to help learners to find percentages of amounts.

3 Approaches to teaching percentages

Percentages are typically easier to handle than fractions as they are effectively fractions with a denominator of 100 (hence 'per cent' which means 'out of one hundred'). This makes it very easy to compare the size of percentages. But remember that, for example, 1% of a large amount could well be bigger than 10% of a small amount.

3.1 Calculating percentages of amounts

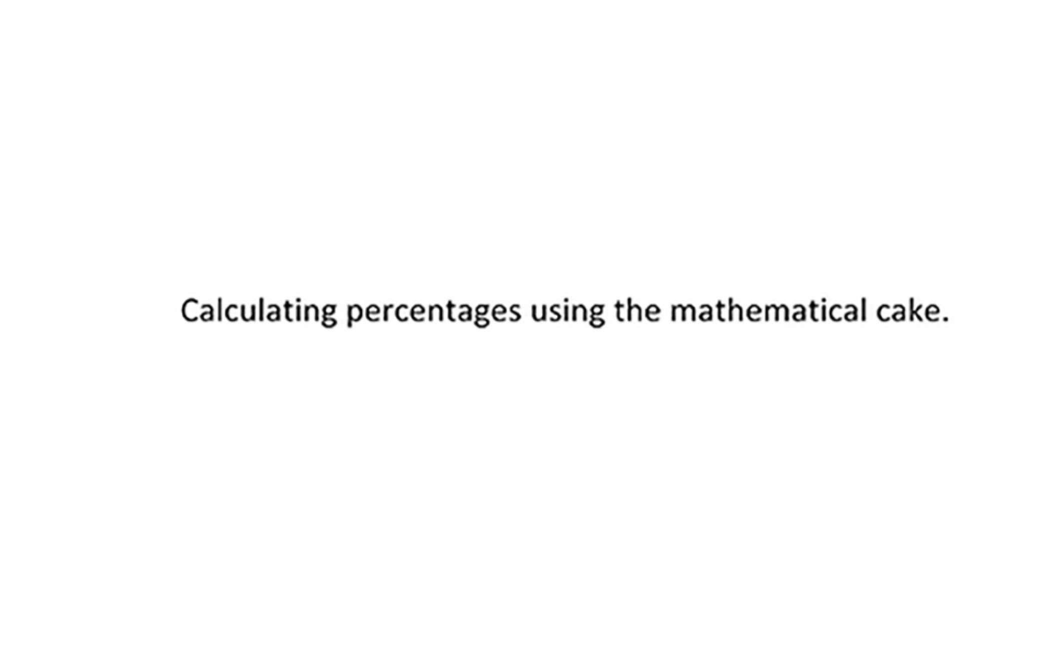
When calculating a percentage of an amount, many learners reach for the method of putting the percentage over one hundred and multiplying by the amount over one. As an example, 35% of 60 is calculated below using this method.

This method works if you are good at arithmetic and can remember the procedure! However, it is more intuitive to use some common knowledge: that 10% is equivalent to one-tenth, 1% is equivalent to one-hundredth, 50% is equivalent to one-half, and 25% is equivalent to one-quarter. Other percentages can be worked out from these.

The mathematical cake, which you met in Section 1 on equivalence between fractions, decimals and percentages, also proves useful in helping younger learners to understand the concept of percentages of amounts. It can be used to find multiples of 10% of amounts by imagining the cake cut into ten slices as before. Watch Video 4 to find out about a useful approach to working out percentages of an amount.

Video content is not available in this format.

Video 4 Using the mathematical cake diagram to find percentages of an amount



Answer

Once learners have been introduced to the mathematical cake diagram, they will certainly be familiar with finding 10% of an amount and multiples of 10% of an amount. It is also intuitive to find 25% and 50%.

For example, 35% of 60 can be found by doing the following:

-
- (multiplying 10% of 60 by 3)
- (finding half of 10% of 60)
- (adding the results of finding 30% and 5%)

Activity 9 Find percentages of 60

Allow 10 minutes

A useful exercise to do with your learners is to ask them what percentages of 60 they can find, encouraging them to modify the methods they have previously used for finding simple percentages of amounts. Try this one yourself (Figure 4).

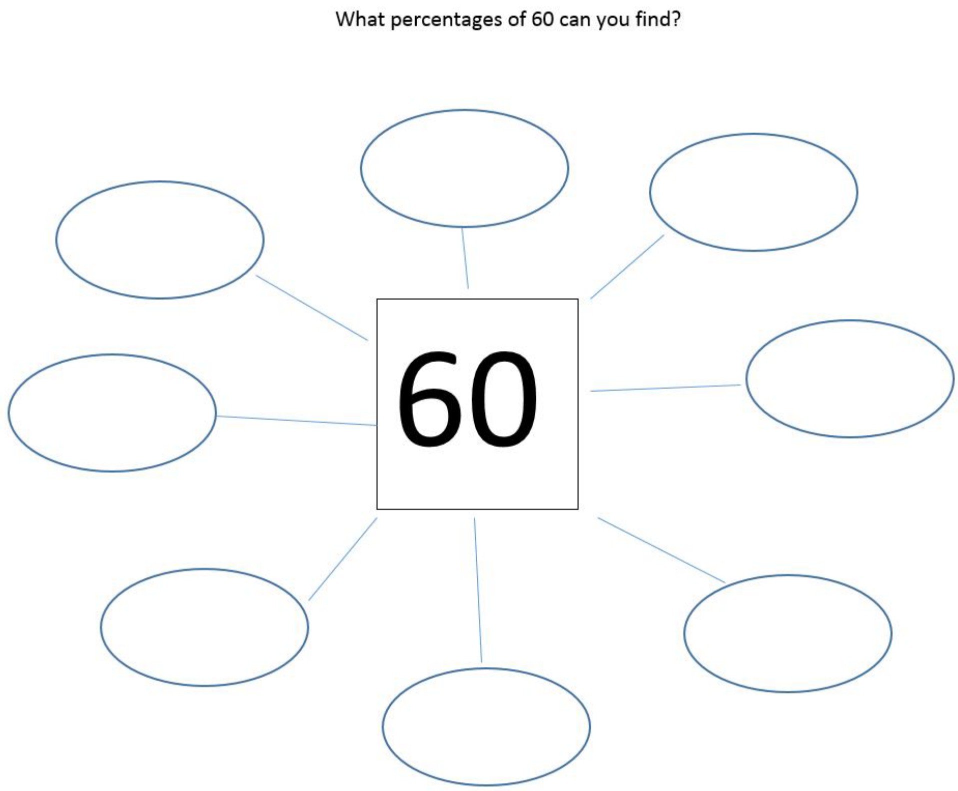


Figure 4 Finding percentages of 60

Discussion

There are many possible questions and answers that you could have posed in the diagram.

Examples could be 50% of 60, 30% of 60, or 25% of 60.

Extend learners' understanding by posing questions such as: 15% of 60, how could you find that? What about 76% of 60? Or 120% of 60?

Ask learners to pose 'What percentage of 60?' questions themselves.

So far, if your learners have been using the methods shown above, they will have found percentages of amounts by making reference to the equivalent fractions.

Decimal equivalents can also be used when calculating percentages of amounts and this is a more helpful form when using a calculator. Referring back to the mathematical cake diagram in Section 1.1, remember that etc.

To convert from percentages to decimals, the percentage has been divided by 100. This follows from an earlier observation about fractions, which stated an alternative meaning of the numerator and denominator, i.e. numerator divided by denominator. In other words:

For example, if calculating 76% of 60, the learner keys into the calculator:

However, it becomes easy to convert 76% straight away to 0.76 and then multiply by 60.

It is worth raising a caution here about using the percentage button on a calculator. Since the percentage button functions in different ways on different makes of calculator , it is best not to use it. Instead, encourage learners to convert the percentage to a decimal and then multiply by the amount.

3.2 Percentage change: using multiplying factors

Consider the following problem.

A customer goes into a restaurant and pays for a meal. Because they had the 'early bird' menu, their bill is discounted by 25%. There is also a 10% service charge added to their bill. What difference does it make to the customer's bill if the waiter deducts the 25% and then adds the 10% service charge, or alternatively, adds the service charge and then makes the deduction?

Activity 10 Does the order matter?

Allow 15 minutes

Work out the solution to the above problem in both ways and see what the difference is. Don't forget that the second percentage change is enacted on the result of the first percentage change, i.e. having deducted 25% from the bill, the 10% service charge is worked out on the resulting discounted bill.

You could choose to calculate the solution of the problem using different amounts for the bill. Or you could prove it once and for all using a general case, which requires you to use algebra, by using a letter to represent the amount of the bill.

Discussion

One form of the solution uses algebra to find the general case:

Let the cost of the meal be *c*.

For 25% deduction followed by 10% increase:

For 10% increase followed by 25% deduction:

Were you surprised at the result? It does not matter which way round the final bill is calculated because a 25% deduction followed by adding on 10% works out the same as adding 10% first followed by deducting 25%.

To see why this is, consider the use of multiplying factors for each of the percentage changes.

A 10% increase means that, having started with 100% of the amount and adding an extra 10%, you end up with 110% of the amount.

A 25% decrease means that, having started with 100% of the amount and subtracting 25%, you end up with 75% of the amount.

Using decimals for the multiplying factors (although you can also use fractions as the multiplying factors), an elegant solution to the problem can be found as follows:

Subtracting 25% then adding 10% gives c

Adding 10% then subtracting 25% gives c

Both multiplications come to $0.825c$ because it does not matter which way round you multiply (this is known as **commutativity**).

In short, a percentage change can be calculated by using the appropriate multiplying factor, which is calculated by considering what has happened to the original 100%.

The use of multiplying factors for percentage change does make the calculations much simpler. However, it may be a difficult concept for younger learners to grasp. If this is the case, it is better to allow learners to work the long way round.

In mathematics it is always best to work with the conceptual understanding that learners have demonstrated and to build on this carefully. Learners can do, understand and succeed when they have grasped the conceptual underpinning for the mathematics they are doing. This was summarised by Richard Skemp (1976)as relational understanding. Otherwise, if there is no understanding, learners will resort to trying to learn rules which are easily forgotten. (Skemp referred to the learning of rules and procedures without a conceptual underpinning as instrumental understanding.)

3.3 Finding the original amount when given the result of a percentage change

It is necessary to understand what percentage change is before this type of problem can be understood.

Finding the original amount from the result of percentage change requires some understanding of what happens when algebra is manipulated. However, first consider two easier examples where intuitive thinking can be sufficient.

Activity 11 25% more sunscreen

Allow 10 minutes

A ‘special offer’ bottle of sunscreen promises 25% more than usual and holds 500 ml. How much does the bottle of sunscreen normally hold?

Discussion

Here is one way to arrive at the solution:

25% more means one-quarter more than the usual amount in the bottle of sunscreen. This means that the special offer holds:

You need to find out how much 100% or 1 whole of 500 ml.

This gives 100 ml. Then multiply by 4 to get the whole of the amount in the normal size bottle, i.e. 400 ml.

You can see that the solution was found by dividing by 5 and multiplying by 4. Remembering that a fraction can also denote numerator divided by denominator, this is equivalent to multiplying by the fraction

This means that one is the inverted (upside-down) version of the other and, if they are multiplied together, you get 1.

It also works by using the equivalent decimals 0.8 and 1.25, which can be checked on a calculator.

Activity 12 A coat is reduced in a sale

Allow 10 minutes

A coat is reduced in a sale by 10% and the sale price is £81. What was the original price of the coat?

Have a go at this problem yourself.

Discussion

The sale price is 90% of the original price and you need to find 100% of the original price.

90% of the original price = £81

10% of the original price = £9 (dividing by 9)

100% of the original price is £90 (multiplying by 10)

3.4 Using multiplying factors to find an original amount given the result of a percentage change

Now consider the same problem using algebra.

The sale price of a coat is £81 and this is the result of a 10% discount. Let *p* be the original price of the coat.

10% discount leaves 90% or 0.9 of the price

You know that 0.9 *p* = 81

Divide both sides by 0.9 to get *p*

So, the original cost of the coat is £90.

This method can be used with much harder numbers and it is a simpler and more elegant way of reaching the solution. However, learners need to understand the method and why it works. This is known as **procedural understanding**. If learners do not have this procedural understanding then the method becomes a rule to be learned and there are too many such rules in mathematics if your approach is to learn rules rather than develop conceptual understanding.

4 This week's quiz

Check what you've learned this week by doing the end-of-week quiz.

Week 2 practice quiz

Open the quiz in a new tab or window and come back here when you've finished

5 Summary of Week 2

After completing Week 2 you have met some mathematical diagrams which can be used to represent fractions, percentages and decimals. You will have seen how such diagrams can be effective in helping to:

- demonstrate equivalence in fractions, decimals and percentages
- show how to calculate fractions and percentages of quantities
- support methods for addition, subtraction, multiplication and division of fractions.

An important message from Week 2 is that it is important to build on learners' understanding of how mathematics works; conceptual understanding, and of how mathematical methods work; procedural understanding.

Next week you will build on the learning from week two when you learn about proportional reasoning, one of the important ideas from mathematics.

You can now go to Week 3.

Week 3: Developing understanding of proportion and ratio

1 Developing proportional reasoning and an appreciation of multiplicative structure

Proportional reasoning involves thinking about relationships and making comparisons between two or more quantities using multiplicative reasoning. It is a complex way of thinking which requires learners to think about numbers and values in relative, rather than absolute, terms.

Proportional reasoning is a hugely important part of school mathematics as it underpins many other topics. For instance, it is instrumental in working with fractions, decimals and percentages, as well as similarity and enlargements. Proportional reasoning also assists learners when working on other topics throughout the mathematics curriculum, from simple probability to problems involving trigonometry.

In addition to being relevant to several topics in the mathematics curriculum, proportional reasoning also has many practical applications. It is used to calculate best buys, to work with scale drawings and maps, to adjust recipes or create particular concentrations of mixtures and solutions and to perform currency conversions.

1.1 Additive versus multiplicative thinking

For many young learners, comparisons are described in additive terms and groups are compared used subtractive language.

This form of reasoning is based on **additive thinking**.

For example, in both images in Figures 2 and 3 there are some green and pink cubes. The ratio of green cubes to pink cubes, expressed as G : R, remains the same in each picture.

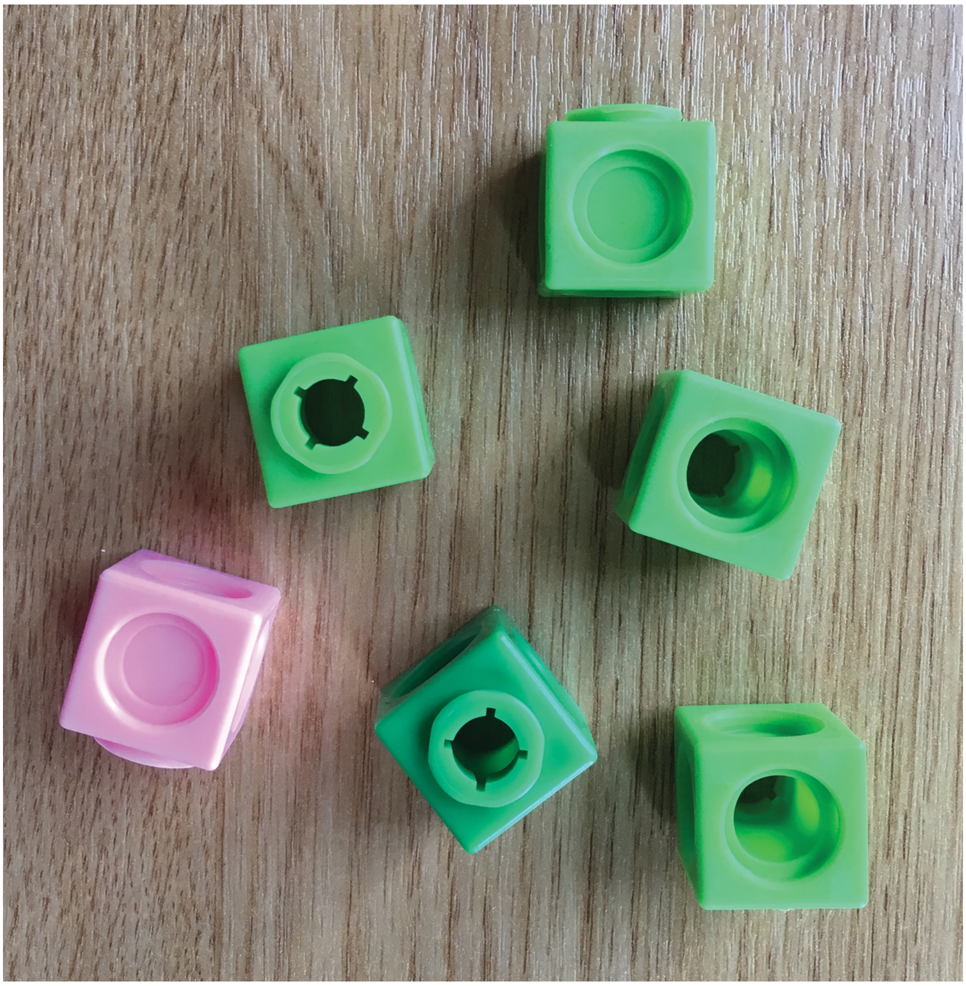


Figure 2 Green and pink cubes

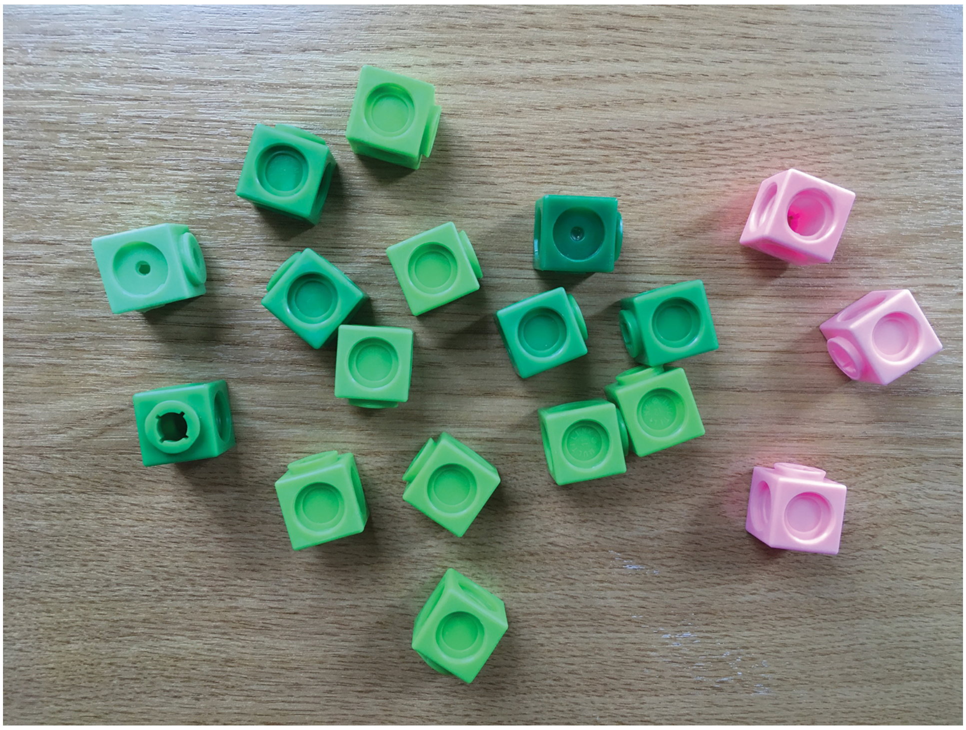


Figure 3 More green and pink cubes

If a learner is using additive thinking, they will describe the change from 5 to 15 green cubes as an addition of 10, and the change from 1 to 3 pink cubes as an addition of 2.

When a learner uses **multiplicative thinking**, they will describe the same changes as multiplying by 3.

Being able to describe proportional situations using multiplicative language is an indicator of proportional reasoning.

Activity 1 Growing dogs

Allow 5 minutes

Look at Figure 4 and then consider the following.

Ronnie, the miniature dachshund, weighed 2 kg as a puppy.

Now Ronnie weighs 5 kg.

Ralph, the French bulldog, weighed 9 kg as a puppy.

Now Ralph weighs 13 kg.

Which dog has grown more?

Figure 4 Ronnie and Ralph as puppies and adults

Activity 2 Reflecting

Allow 5 minutes

Consider the problem above. How do you think learners may respond to this question?

Discussion

If learners think in absolute terms, or using additive thinking, they may suggest that Ralph has grown more because his weight has increased by 4 kg whereas Ronnie's weight has increased by only 3 kg.

If learners think in relative terms, or using multiplicative thinking, they may suggest that Ronnie has grown more because his weight has more than doubled and Ralph's weight has less than doubled.

Activity 3 Speed on a cycling track

Allow 10 minutes

Two cyclists, Ava and Isaac, are cycling equally fast around the cycling track. (Figure 5)

Ava began cycling before Isaac arrived at the track and had completed 9 laps when Isaac had completed 3.

When Ava has completed 15 laps, how many laps will Isaac have completed?

Figure 5 Ava and Isaac cycling

Activity 4 Reflecting

Allow 5 minutes

How did you approach the task in Activity 3?

Discussion

It is easy to fall into the trap here of taking a proportional approach, resulting in Isaac completing 5 laps, when actually this task requires additive reasoning.

The key words here are ‘cycling equally fast’. Ava and Isaac are travelling at the same speed, so it is not a proportional relationship. Ava had a head start, so completed an extra 6 laps, meaning that when Ava has completed 15 laps, Isaac will have completed 9 laps.

1.2 When to use multiplicative thinking

It is important for learners to distinguish between situations which require additive reasoning and those which require multiplicative, or proportional, reasoning.

The following example offers some possible discussion points for learners.

Activity 5 Questioning learners

Allow 10 minutes

As you read the problem below, think of some questions you could ask learners to promote analysis of each situation in additive and multiplicative terms. Make a note of the questions you could ask in the box below the task.

Using boxes of eggs to create questions for the classroom

Each of the cartons in Figure 6 contains some white eggs and some brown eggs. Which has more brown eggs?

Carton A

Carton B

Figure 6 Brown and white eggs

Provide your answer...

Discussion

Possible questions to ask learners

- How many brown eggs are there in each box?
- How many brown eggs are there in comparison to white eggs in each box?
- What is the relationship between brown and white eggs in each box?
- What is the fraction of brown eggs in each box?
- What is the ratio of white to brown eggs in each box?

Asking learners questions such as ‘how many brown eggs are in each box?’ encourages learners to reason additively.

Discussions about relationships lead learners to think proportionally.

Depending on the age and experience of the learners, discussions can move on to comparing fractions, ratios or percentages as ways to represent the proportion of brown eggs in each box.

1.3 Developing learners’ proportional reasoning

Teaching basic algorithms for solving problems involving ratio and proportion will not develop learners’ proportional reasoning on their own.

Learners need their **conceptual understanding** of proportion to be flexible so that they can apply it to the many varied situations in which proportional reasoning is required.

Working on repetitive exercises can support learners in remembering methods for solving particular types of ratio and proportion questions. But it does not develop their proportional reasoning or prepare them for unfamiliar problems. Instead, learners need to be exposed to a variety of different problems and approaches.

You will consider a range of problems for learners involving ratio and proportion in Section 2 of this week.

1.4 Using concrete resources

It is important to give learners of all ages a variety of concrete experiences connected with proportional reasoning to encourage them to compare, make conjectures, find relationships and generalise their learning.

Young learners may start with weighing and measuring objects in the classroom, and discussing comparisons between amounts.

Activity 6 Sharing money

When asked to share out £24 between two cousins in the ratio of 3:1, learners can use counters to represent the pounds (Figure 7). How can you use counters to demonstrate the solution to this problem? Write responses in the box below.

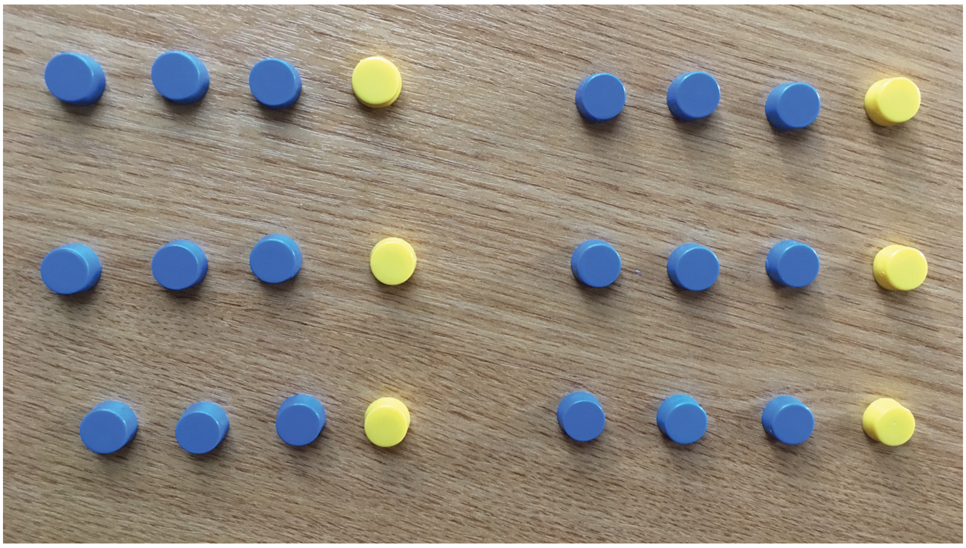


Figure 7 Using counters to share out in a ratio

Provide your answer...

Discussion

Grouping counters in groups of 1 and 3 is one way to work on this problem.

By grouping counters in this way, learners share out all the counters in the ratio of 1:3.

They can then look at how many of each colour there are altogether. Noticing that there are 6 groups of 3 and 6 groups of 1 would result in finding a total of 18 blue counters and 6 yellow counters, representing the £18 and £6 the cousins will each get.

Another way to approach this problem using counters is to create boxes (on paper or on mini-whiteboards) to represent the ratio. In this case, since the ratio is 3:1, you will need 3 boxes for the first cousin and 1 box for the second cousin.

The counters, which represent pounds, are then shared out equally between the 4 boxes. The first cousin is given 3 boxes, so receiving £18, and the second cousin is given 1 box, so receiving £6 (Figure 8).

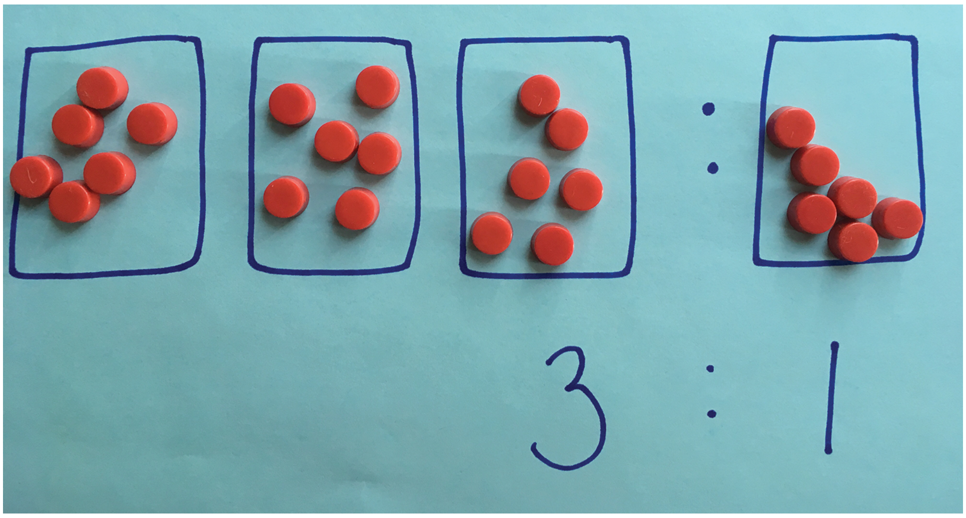


Figure 8 Using ratio boxes with counters

This approach can also be used with pens on mini-whiteboards. Learners need to create the boxes to represent the ratio and then put a mark in every box until they get to the total, in this case 24 (Figure 9).

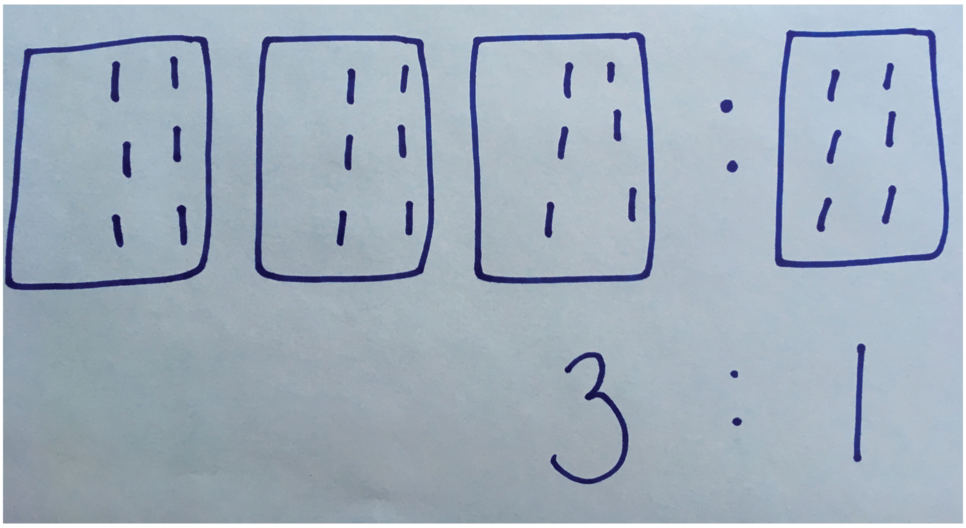


Figure 9 Using ratio boxes on mini-whiteboards

This imagery can then support learners in moving towards a more abstract concept of sharing amounts into a ratio. In other words, it can help them to understand that in a ratio of 3:1 there are 4 equal parts in total which the £24 needs to be divided equally between.

Note: When moving on to ratio problems involving fractions and decimals, using manipulative resources can become problematic. It is sensible to move onto more abstract methods with your learners before introducing these kinds of problems.

Teaching idea: Cooking with ratios

Why not incorporate some creative cooking into a maths lesson or some maths into a cooking lesson?

Figure 10 Rice crispy cakes

To make 12 rice crispy cakes you need:

- 60 g unsalted butter
- 3 tablespoons golden syrup
- 1 x 100 g bar of milk or dark chocolate
- 90 g Rice Krispies©.

Before making the delicious treats, learners could first work out how much of each ingredient they will need to make, say, 6, 24, 36 or more challenging amounts such as 30 or 50.

Decorating biscuits with icing is an even simpler recipe which could be used with younger learners.

1.5 Moving to abstract thinking and using pictorial representations

Proportional reasoning does not always involve physical objects, so learners need to be able to abstract their thinking about ratio and proportion.

When moving from concrete resources to more abstract thinking, it can be helpful to encourage learners to draw diagrams to support their thinking.

Activity 7 An example of a proportionality problem

Allow 5 minutes

If it takes Sakina 30 minutes to read 45 words, how many words could she read in 60 minutes?

Discussion

Minutes can be compared to minutes. Here 60 minutes is twice 30 minutes.

This multiplicative relationship can then be applied to the relationship between pages. This is known as a **scalar** comparison.

A pictorial representation of this approach is shown below.

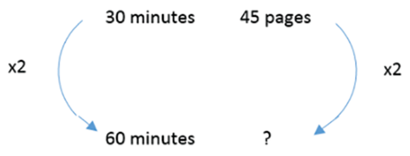


Figure 11 A scalar comparison between minutes and pages

Alternatively, pages can be compared to minutes. This is a **proportional** comparison.

Here, 45 is 1 times 30, and this relationship can then be related to the 60 minutes.

A pictorial representation of this approach is shown below.

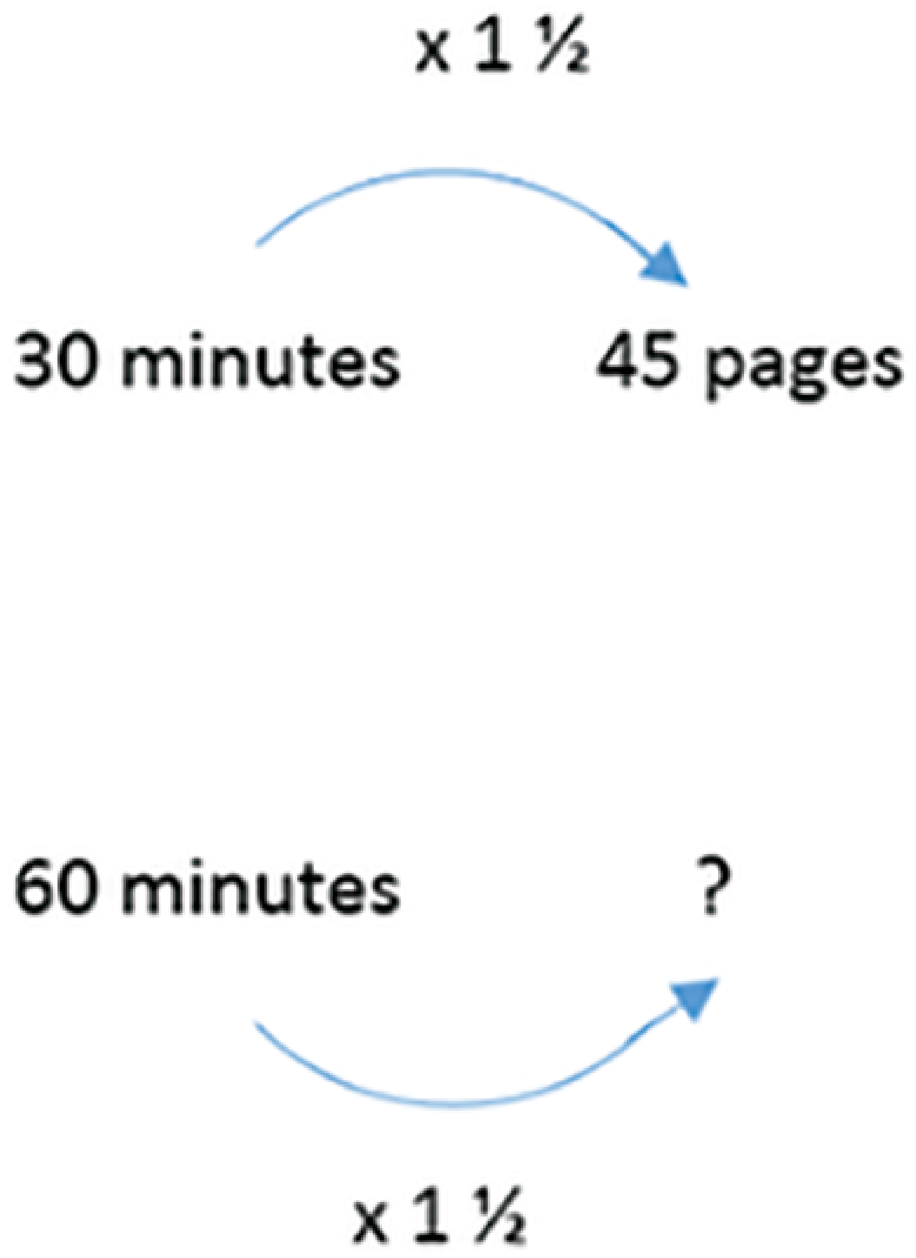


Figure 12 A proportional comparison between minutes and pages

Note that both approaches give the same result.

In the next section you will look at problem solving with proportional reasoning and discuss misconceptions that often arise.

2 Problem solving using proportional reasoning, including common misconceptions

Learners in the middle years often struggle with proportion problems. Research consistently highlights learners' difficulties with proportion and proportion-related tasks and application.

There are many common misconceptions that arise within the topic of ratio and proportion and it is important to be aware of these in order to plan to address these in your teaching.

2.1 Problem solving using proportional reasoning

Proportional reasoning questions cover a huge variety of real-life applications, from best buys to map scales.

Maps and scales

Maps and scale drawings use proportionality. In order for a map to be useful, it needs a scale which represents how much smaller it is than the place it represents. This means that journey distances can be accurately calculated.

Activity 8 Example problem: model boat

Allow 5 minutes

A scale model of a sailboat is made to a scale of 1:20 (Figure 13).

a. If the model boat is 17 cm wide, how wide is the actual boat?
b. If the boat has a sail of height 5 m, how high is the sail on the model?

Figure 13 A scale model boat

Discussion

a. The actual boat is 20 times larger, so it will be 340 cm or 3.4 m wide.
b. The model boat is 20 times smaller, so the sail will be 0.25 m or 25 cm high

Teaching idea: School or home map

Learners can create scale maps of their school or home by taking measurements using trundle wheels, measuring tapes and metre rulers. Then, using an appropriate scale, they can produce accurate scale drawings or maps.

Alternatively, learners can calculate how far away different countries are by measuring distances on scale maps in a school atlas (Figure 14).

Figure 14 Maps and an atlas

Best buys

Having proportional reasoning helps to ensure you get the best deals when shopping, particularly when it comes to offers and multi-buys. Often deals are misleading and larger 'value' packs can work out to be more expensive. It is important to be able to compare prices properly in order to make smart choices.

Activity 9 Example problem: washing-up liquid

Allow 5 minutes

Two shops, Fast-Save and Super-Mart, sell the same brand of washing-up liquid in the same size bottles (Figure 15).

At which shop is the washing-up liquid the best value for money?

Fast-Save	Super-Mart
Washing up liquid	Washing up liquid
5 for £2.85	6 for £3.90

Figure 15 Best buys

Discussion

You need to be able to compare the two prices, so you need to find the price at each shop for a common number of bottles.

One approach to this problem is to find a common multiple of 5 and 6. In this case, 30 is a sensible choice as it is the lowest common multiple of 5 and 6.

The price for 30 bottles at Fast-Save is £17.10 (6 £2.85).

The price for 30 bottles at Super-Mart is £19.50 (5 £3.90).

An alternative approach would be to find the unit price for 1 bottle of washing-up liquid at each shop. This would involve dividing the cost by 5 and 6, respectively.

Exchange rates

A common use for proportionality is currency exchange rates. When you change your money at the Post Office or travel agent 'commission free', they will use a different exchange rate when they sell you foreign currency to when they buy it, thus making a profit.

Activity 10 Example problem: Japanese yen

Allow 5 minutes

Zoë is going on holiday from England to Japan.

Before leaving, she changes 1500 pounds sterling (GBP) into Japanese yen (JPY).

The exchange rate is 1 GBP = 177.415 JPY.

Calculate the amount of foreign currency Zoë takes on holiday.

Discussion

This question is quite straightforward since you are given the unit rate for £1 in yen. All that is required here is to multiply the number of pounds by the rate, giving 266 122.50 yen.

If the rate was given in another form, for instance: 20 GBP = 3548.30 JPY, another step would be needed to calculate the unit rate.

Mixtures and solutions

Proportionality is used when producing mixtures and solutions. For example, to make concrete you need **1 part** cement, **3 parts** sand, and **3 parts** aggregate.

This is a ratio of 1:3:3, or it can be said that the proportion of cement is 1/7.

Activity 11 Example problem: orange drink

Allow 10 minutes

- A bottle contains 750 ml of concentrated orange squash.
- It is enough to make fifteen 250 ml glasses of diluted orange drink.
- How much water is needed to make 10 litres of this drink?

Discussion

This question tackles proportion in a real-life context. It is a multistep problem as, first, the total amount of diluted drink needs to be calculated before then comparing ratios.

This problem comes from the Nrich website. Suggested approaches and solutions can be found via the link in the 'Further reading' section.

Teaching idea: A giant's hand investigation

One way to engage learners in working on ratio and proportion is to leave a 'giant's handprint' on the classroom wall for them to find (Figure 16).

The learners are then asked to investigate the question 'How big must the giant be?'

Note: Week 8 of this course looks at how you might conduct statistical investigations with learners. You might find it useful to read that section before carrying out this investigation in your classroom.

Figure 16 A giant's handprint

Activity 12 Reflecting

Allow 10 minutes

a. What do the learners need to know?
b. What do they need to find out?
c. How does this problem relate to proportion?

Discussion

The learners will need to measure the 'giant's hand' and use proportion to calculate the size of the giant.

The idea behind this activity is that learners will investigate the proportions found in the human body. One way to do this is to collect data from the whole class, with each learner measuring their hand size (length, width or both) and height. This is a hands-on, practical activity which will help learners to get a sense of what proportionality means.

Whole-class discussions looking at the class data set may draw out some conclusions about the proportional relationship between hand size and height (or other measures: foot size, arm span, etc.). These relationships can then be related to the giant's hand measurements to calculate the height (and foot size, arm span, etc.) of the giant. Different learners may come to slightly different conclusions, so encourage them to explain their answers. This will help to develop their proportional reasoning.

Note: in exploring the proportions in the human body, you might like to discuss the **golden ratio** with your learners. This idea was briefly discussed in the Introduction to this week.

Some teaching ideas based around investigating the golden ratio can be found on the Nrich website.

2.2 Common misconceptions with proportional reasoning problems

In this section, you are offered several problem-solving tasks relating to proportional reasoning. As you read through each problem, consider how you would approach each task yourself and think about the issues, and possible misconceptions, that might arise for learners as they work on each problem.

Proportional problem 1: Soup

Here is a recipe for vegetable soup (Figure 17).

Vegetable soup (serves 2)

- 200 g chopped mixed vegetables (e.g. onion and carrot)
- 300 g potato
- 700 ml vegetable stock
- 1 tablespoon olive oil.

Zoë makes the soup using 600 g of mixed vegetables.

What weight of potato should she use?

Figure 17 Vegetable soup

Activity 13 Reflecting

Allow 10 minutes

Look at this learner's incorrect response below.

- Zoe needs 200 g of chopped vegetables in the original recipe.
- Now she needs 600 g of chopped vegetables.
- 200 g + 400 g = 600 g
- She needed to add 400 g of chopped veg, so she must need to add 400 g of potato.
- So, it must be 300 g + 400 g = 700 g of potatoes.

How can you explain the thinking behind this approach?

Discussion

This is a common approach to this type of problem, known as the *constant difference strategy*. The learner seems to be thinking in absolute terms. They have identified that the difference between the amount of veg and potato in the original recipe is 400 g, so have kept this difference constant by adding 400 g to the new amount of veg, rather than thinking about the relationship between the ingredients. This is an example of *additive reasoning* in place of *multiplicative reasoning*, which was discussed in Section 1 of week 3.

It is challenging for learners to identify when they should be thinking proportionally rather than additionally. Using diagrammatic representations of ratios and having discussions related to the context of the problem can support learners in moving from additive to multiplicative reasoning.

Proportional problem 2: Sharing the workload

It takes 3 workers 8 days to complete the foundations of a new building (Figure 22).

How long would it take if there was an extra worker?

Assume that all the workers will work at the same rate.

Figure 18 Building a brick wall

Activity 14 Reflecting

Allow 10 minutes

Look at the learner's incorrect response below.

How could you explain to a learner that this approach is incorrect?

- There are 8 days of work completed by 3 workers.
- So, 8 ÷ 3 = 2 2/3 days of work per person.
- Now there are 4 workers, so:
- 4 2 2/3 = 10 2/3 days.

Discussion

This requires some logical thinking, relating the answer back to the original problem.

If it takes 3 workers 8 days to complete the job, would it take 4 workers more time to complete the same job?

No, it would take them less time, so this answer cannot be correct.

If learners have been working on 'sharing in ratio' problems, they may go into automatic mode and try to use the same approach with these types of problems, causing them to make mistakes. When working on problems with ratio and proportion, the context of the problem is important.

It will take 4 workers 6 days to complete the same work.

Proportional problem 3: Paint

A particular shade of green is made by mixing 3 litres of blue paint with 5 litres of yellow paint (Figure 23).

How much yellow paint is needed to make this shade of green if you only have 2 litres of blue paint?

Figure 19 Tins of paint

Activity 15 Reflecting

Allow 5 minutes

Look at the learner's incorrect response below.

How can you explain the thinking behind this approach?

- There are 3 litres of blue and 5 litres of yellow paint.
- So, there is a total of 8 litres of paint needed.
- There are 2 litres of blue paint.
- So, to make the 8 litres, 6 litres of yellow paint are needed.

Discussion

This strategy is known as the constant sum strategy. This can be easily tackled using examples for which this strategy is impossible. For instance, if the question had asked 'How many litres of yellow paint is needed for 12 litres of blue paint?', the total sum could not have been consistent.

In this particular example, discussions about proportionality and the need for consistency in order to preserve the particular shade of green, may support learners.

Proportional problem 4: Printing press

A printing press takes exactly 12 minutes to print 14 dictionaries.

How many dictionaries can it print in 30 minutes?

Activity 16 Reflecting

Allow 5 minutes

Look at the learner's incorrect response below.

How can you explain the thinking behind this approach?

- The printing press can print 28 dictionaries.

Discussion

This learner appears to have multiplied by 2, despite 12 not being half of 30. This is a common mistake made by learners, who have previously found that doubling has been an effective strategy. This is known as the *magic doubling method,* when learners use a doubling strategy but it is inappropriate to do so.

Similarly, learners often make mistakes by *using multiplication but not by the correct factor*.

It is important to offer learners a wide range of varied examples and to have discussions about strategies, identifying that some approaches will work for particular problems but not for others.

As mentioned in the first part of this week, teaching learners methods for solving ratio problems is not enough to develop their proportional reasoning. Learners need to be able to make decisions about when to use different strategies.

Proportional problem 5: Bookshop

Martha buys 4 books for £11. In this bookshop, all books are the same price.

Billy buys 9 books from the same shop. How much do they cost in total?

Activity 17 Reflecting

Allow 5 minutes

Look at the learner's incorrect response below.

How can you explain the thinking behind this approach?

Books	Price
4	£11
8	£22
9	£23

Discussion

This learner has used an *incomplete application of the building-up method*. They have attempted to use a building-up method, by doubling the original amount to get the price for 8 books and then they have made a mistake in the final step.

In the left-hand column, the learner doubled the number of books and then added a quarter of the original number of books. On the right-hand side, this has been translated to 'double the price and add 1'.

As the teacher, unpicking this mistake means you can see that there is some understanding to build upon.

Bookshop

Books	Price
4	£11
8	£22
1	£2.75
9	£24.75

Adding the extra step into the working could help to avoid the mistake made above.

2.3 Teaching to address common misconceptions

In a topic such as this, there are many common misconceptions. It is important not to avoid these issues but instead to build opportunities to discuss these misunderstandings as part of your teaching. Through identifying mistakes and discussing why answers are incorrect, learners will develop both their proportional reasoning and their problem-solving skills.

3 Connecting ratio and proportional reasoning

According to the Oxford English Dictionary (OED) Online, **ratio** is defined as 'the quantitative relation between two amounts showing the number of times one value contains or is contained within the other'.

For example, *the ratio of male to female professors is 4 to 1.*

In contrast, **proportion** is described by the OED as *a part, share, or number considered in comparative relation to a whole*.

For example, *the proportion of greenhouse gases in the atmosphere is rising.*

In other words, a **ratio** is used to describe a comparison between two (or more) objects, amounts or values, and a **proportion** is used to describe one object, amount or value, using a comparison to a whole.

For example if the **ratio** of boys to girls in the class is 2 to 3, or 2:3, the **proportion** of boys is 2/5.

Since the proportion of boys in the class is expressed as a fraction of a whole, it is not necessary to know the proportion of girls. If there are 30 children in the class and 2/5 of them are boys, it can be inferred that 3/5, or 18, of them are girls.

In contrast, when expressed as a ratio, the '2' is meaningless without the comparison to the '3'.

Proportional thinking and reasoning is knowing the multiplicative relationship between the base ratio and the proportional situation to which it is applied.

Understanding ratio and proportion is more than merely being able to perform appropriate calculations, being able to apply rules and formulae or manipulating numbers and symbols in proportion equations.

3.1 Using simple ratios to make comparisons

First, look at Figures 20–22 and then do the activity based on them.

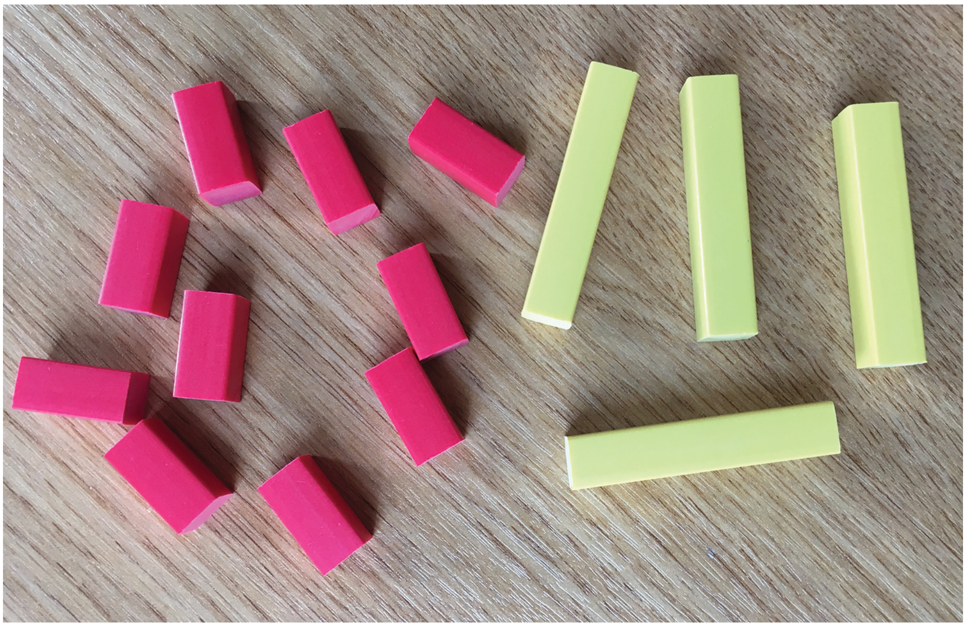


Figure 20 Using Cuisenaire rods to represent ratios

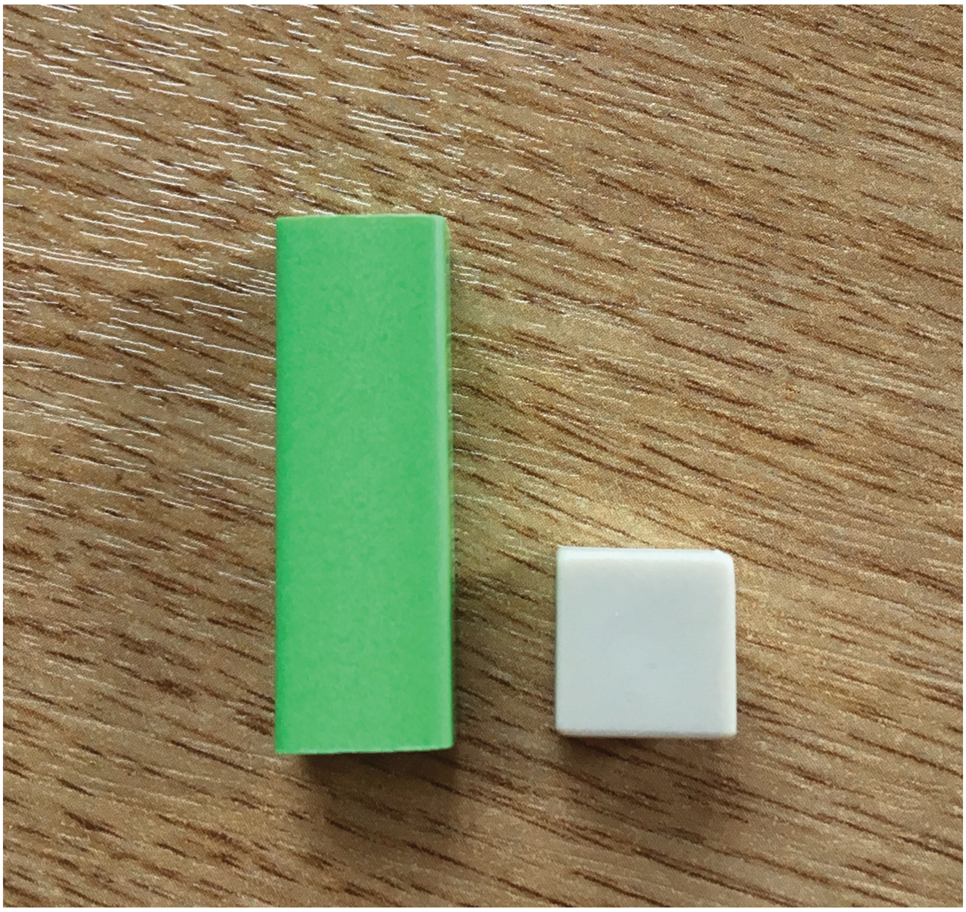


Figure 21 Using Cuisenaire rods to compare the ratio of lengths

Figure 22 Using multilink cubes to represent ratios

Activity 18 Writing ratios

Allow 5 minutes

Write the ratios for each of the images shown above in the simplest form.

a. What is the ratio of red rods to yellow rods?
b. What is the ratio of the lengths of the green and white rods?
c. What is the ratio of orange to white cubes?

Discussion

1. 8:12 which simplifies to 2:3.
2. 10:4 which simplifies to 5:2.
3. 1:3.

Using manipulatives

When first introducing ratio and proportion, it is important to have a range of concrete resources and manipulative materials available for learners to use. For example, multilink cubes, counters, coins and Cuisenaire rods. (Figure 23)

Figure 23 Concrete, manipulative resources

Cuisenaire rods are a great resource for comparing ratios as the rods can all be related in size. For example, the red rod is twice as long as the white rod, so if the white rod is assigned the value of 3 for a particular question, the red rod will represent 6.

Rods are a great tool for exploring multiplicative relationships.

Note: learners need to be given some time to familiarise themselves with the rods and the relationships between the different rods before they can be useful in the classroom.

If you do not have access to a set of Cuisenaire rods, there is a virtual version of them available on the Nrich website, although this does not provide the same concrete resource that the physical rods do.

Figure 24 Edible resources

Depending on the age of the learners and the teaching setting, using edible resources for work on ratio and proportion can be a concrete resource while providing a meaningful real-life context for learners (Figure 24). The ability to cut up certain foods, such as cakes or pizzas, into halves, quarters and so on, means that fractional ratios can be explored in a practical, hands-on way.

3.2 Connecting ratio and proportion

Throughout this week you have looked at various applications of proportional reasoning. Finally, you will look at how notation for ratio and proportion are linked.

Recapping what was discussed at the start of this section, a ratio represents a relationship between two quantities. A proportion represents one quantity in relation to the whole.

For example, a tropical drink uses 150 ml of orange juice and 100 ml of mango juice.

The ratio of orange to mango, in the simplest form, is 3:2, as for every 3 parts of orange juice, 2 parts of mango juice is needed.

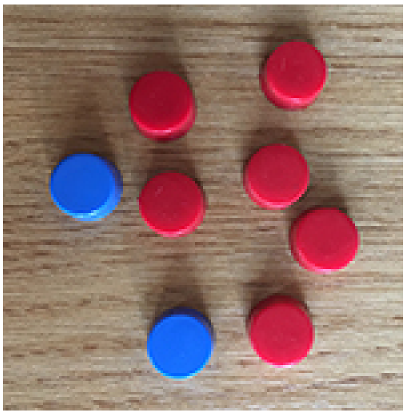
Expressed as a fraction, the proportion of orange juice in the drink is

as there are 5 parts in total and 3 of these are orange juice.

Activity 19 Matching ratio and proportion

Allow 10 minutes

For each of the sets of counters displayed, write the ratio and proportion in the box below.

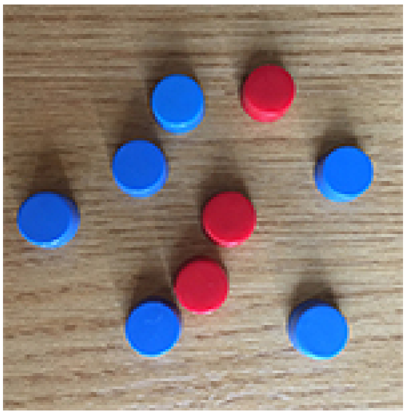


Provide your answer...

Answer

Ratio of red to blue is 3:1

The proportion of red is

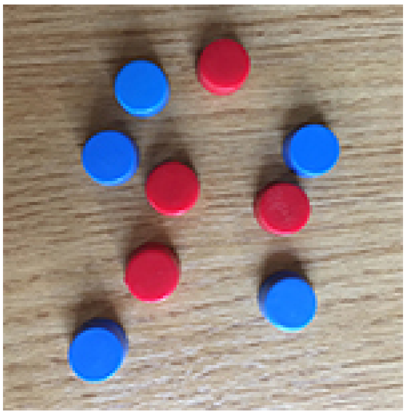


Provide your answer...

Answer

Ratio of red to blue is 1:2

The proportion of red is

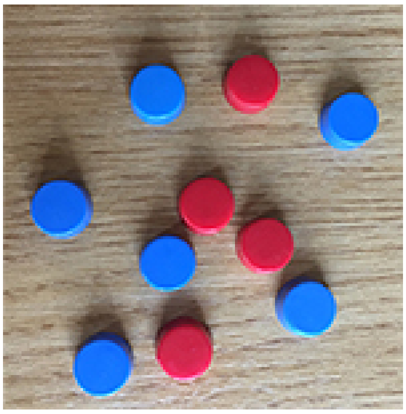


Provide your answer...

Answer

Ratio of red to blue is 4:5

The proportion of red is

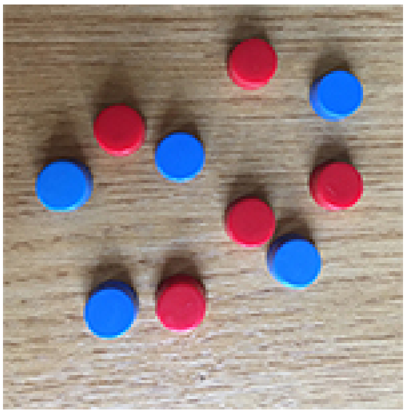


Provide your answer...

Answer

Ratio of red to blue is 2:3

The proportion of red is

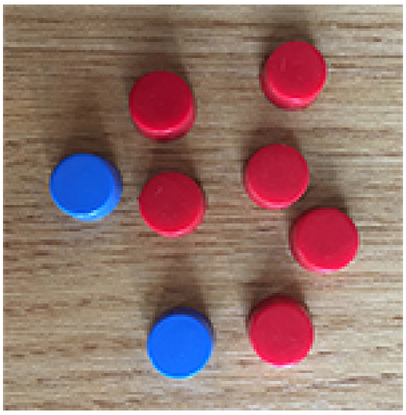


Provide your answer...

Answer

Ratio of red to blue is 1:1

The proportion of red is



Provide your answer...

Answer

Ratio of red to blue is 1:3

The proportion of red is

3.3 Using ratio tables

One way to support learners in developing their own mental strategies for solving proportion problems is through the use of ratio tables.

Ratio tables are a way to **symbolise** the problem and can support learners in finding strategies for solution. They encourage approaches such as halving, doubling, and multiplying by 10.

Activity 20 Reflecting

Allow 10 minutes

Two learners have used ratio tables to work on the problem shown below. They have both taken slightly different approaches.

Have a look at their workings below. Can you identify each of their strategies?

Problem: Seedling plants come in boxes of 35 plants. How many plants would be in 16 boxes?

Seedling plants – Sophie's strategy

Boxes	1	2	4	8	16
Plants	35	70	140	280	560

Seedling plants – Alejandro's strategy

Boxes	1	10	2	6	16
Plants	35	350	70	210	560

Discussion

Sophie's strategy

- *Repeated doubling to get to 16 boxes.*

This works because 16 is a power of 2. If the question was asking for 15 boxes, she could use the same strategy but then subtract one lot of 35.

Alejandro's strategy

- *Multiply by 10.*
- *Separately also multiply original amount by 2, then multiply by 3 to get 6 lots.*
- *Add 10 lots and 6 lots to get 16.*

Alejandro has used a **building-up strategy**. He did not just use a scalar multiplier. Instead, he had to find different parts and sum these. This can be known as the **addition and scaling method**.

In the next example, Alejandro was given the unit amount (that 1 box contained 35 seedlings). This meant that he only needed to use multiplication.

In the second example below, the unit cost is not given, so some division, as well as multiplication, is required.

Activity 21 Proportion problem

Allow 10 minutes

Think about how you might approach this problem before reading the learner's response below.

Problem: Mangoes are 2 for £3. How many could you buy for £7.50?

Mangoes – Yasmin's strategy

Mangoes	2	4	1	5
Price	£3	£6	£1.50	£7.50

Discussion

This problem is slightly different from the previous examples because it is not as clear how to get from £3 to £7.50 as it was to get from 1 to 16. Because £7.50 is not a multiple of 3, it cannot be made using either Sophie's or Alejandro's strategy.

Yasmin's strategy:

- *Double to get £6.*
- *Need £1.50 more so adding another £3 will be too much.*
- *Half the original amount to get £1.50.*
- *Add £1.50 to £6 (1 lot to 4 lots) to get £7.50.*

3.4 Continuing to develop learners' proportional reasoning

This week discussed how learners might approach different problems relating to proportion and ratio. You have considered how misconceptions could be addressed and the types of resources that might help learners to develop their understanding.

Ratio and proportion underpin many key mathematical topics and have applications in STEM subjects as well as real-life contexts. It is important that learners are given ample opportunity to practise, problem solve and discuss proportional reasoning.

4 This week's quiz

Check what you've learned this week by doing the end-of-week quiz.

Week 3 practice quiz

Open the quiz in a new tab or window and come back here when you've finished

5 Summary of Week 3

In Week 3 the focus has been on proportional reasoning and its link to multiplicative relationships. You have seen how the use of concrete objects can support early understanding of proportional reasoning, progressing to pictorial representations and, later, abstract mathematical notation as learners' thinking becomes more sophisticated.

You have worked on a range of proportional reasoning problems, looked at examples of common misconceptions and considered how these could be addressed.

You should now be able to:

- define what is meant by proportional reasoning
- recognise when a learner is using additive reasoning or when they are using multiplicative reasoning
- use effective approaches for teaching proportional representations
- have an awareness of the common misconceptions which learners meet when working through proportional reasoning problems.

Next week you will study the first of two weeks on algebra, looking in particular about how the foundation of algebraic understanding is introduced and developed.

You can now go to Week 4.

Week 4: Introducing algebra

1 Mathematical sentences and algebraic expressions

It is tempting to define algebra as the strand of mathematics that uses letters for numbers. However, the roots of algebraic thinking lie in paying attention to quantities and their structural relationships. One definition that respects this broader view holds that algebraic thinking can be can characterised by:

- Indeterminacy: the problem involves unknown quantities.
- Denotation: the unknown quantities are represented, for example by words or symbols.
- Analyticity: the unknown quantities are operated on as if they were known numbers (Radford, 2014).

In the next section you will be asked to have a go at an algebraic task. At the end of the section you will consider how it meets these three requirements.

1.1 Writing algebraic expressions

In this section you will write some algebraic expressions which arise from a simple problem.

Activity 1 Triangle perimeters

Allow 10 minutes

The shape in Figure 1 is made of two congruent triangles. Each triangle has sides labelled 3 cm, *a* cm and *b* cm. Find an expression, in terms of *a* and *b*, for the perimeter of the combined shape. You do not need to simplify it.

Find another expression. And then find another.

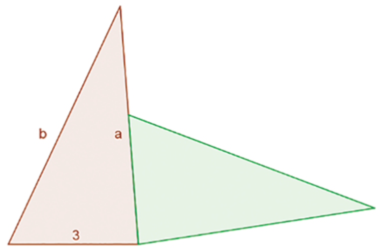


Figure 1 Triangle perimeters

The purpose of this task is to work with unknown quantities – the side lengths *a* and *b* and the perimeter – which can be represented in different ways: as (labelled) parts of the diagram, in words and in symbolic expressions. A good way to start is by reminding yourself what perimeter means, and by labelling the side lengths of the green triangle (which one is 3 cm long and which is *a* cm?).

There are different methods for calculating a perimeter, and each of these could give rise to a different symbolic expression. This is the reason for asking you for ‘another expression. And then another’. People working on this task in a group are likely to find a range of symbolic expressions but if they simplify these, they should all come to the same result.

For many learners, handling the congruent triangles will help them perceive how the combined shape is made, which sides are the same length and how to express the ‘leftover bit’ as a difference or a subtraction. We have avoided using right-angled triangles (so as not to remind you of Pythagoras’s Theorem) but this is not a distraction for younger learners. They can cut a rectangle in half diagonally, label the shortest side 3 units, and rearrange the pieces.

Discussion

Teachers have found that a key feature of encouraging early algebraic thinking is **productive lingering**. This is a phrase to describe asking learners to compare and explain their answers when they are still in the form of expressions that show the thinking processes and the mathematical operations in a calculation, and before they are tidied into a final answer.

Learners can express their method in words, or as a calculation using numbers, or, eventually, as an algebraic expression. These word- or number-based expressions can be called **mathematical sentences**, and they are the foundations of algebraic thinking.

In Japanese classrooms, teachers have a special word for the expression that leads to a final answer. It is called the ‘shiki’. Teachers ask their learners to work out what is the appropriate shiki to use for a problem, and ask them to say the shiki as well as the final answer.

Activity 2 Reflecting

Allow 10 minutes

How many expressions did you find for the perimeter?

In the table below, match the algebraic expressions with the corresponding explanation in words.

Match the algebraic expressions 1–5 with the 'mathematical sentences' A–E.

1	P = (a + b + 3) – 3 + (a + b + 3) – 3	A	On the green triangle it's only a and b because the 3 is missing. On that one it's 3 and b and a but then you've got to take 3 away again.
2	(a + b) + 3 + b+ a – 3	B	The perimeter is 3 add side b then it's the little bit down there, then it's add side b and side a. That bit is what's left over when you put the 3 next to a.
3	P = 3 + b + (a – 3) + b + a	C	To make the perimeter – for each whole triangle it's a plus b plus 3, but it's missing 3 from each one.
4	(a + b + 3) + b + (a – 3)	D	Count all the sides, it's a plus b plus 3, doubled, and then you miss out 3 twice.
5	2 (a + b + 3) – 2 x 3	E	That side is the same as the missing side so you have all of one triangle. For the other triangle you have side b and just a bit down side a, a less 3.

Discussion

All of the expressions are equivalent to 2a + 2b but they represent different thinking processes.

You should have matched 1C, 2A, 3B, 4E, 5D.

The next section examines the idea that you can express mathematical reasoning in words or in algebraic expressions. The mathematical reasoning that you are doing in this task is giving instructions for calculations without actually doing the calculation.

1.2 Expressions and terms

An algebraic expression is a calculation, written in standard arithmetic notation, where one or more numbers has been replaced with a symbol. If an expression involves addition or subtraction then the parts that are added or subtracted are called 'terms'. The video below shows the parts of an equation which are algebraic expressions and then shows how the expressions are made up of algebraic terms.

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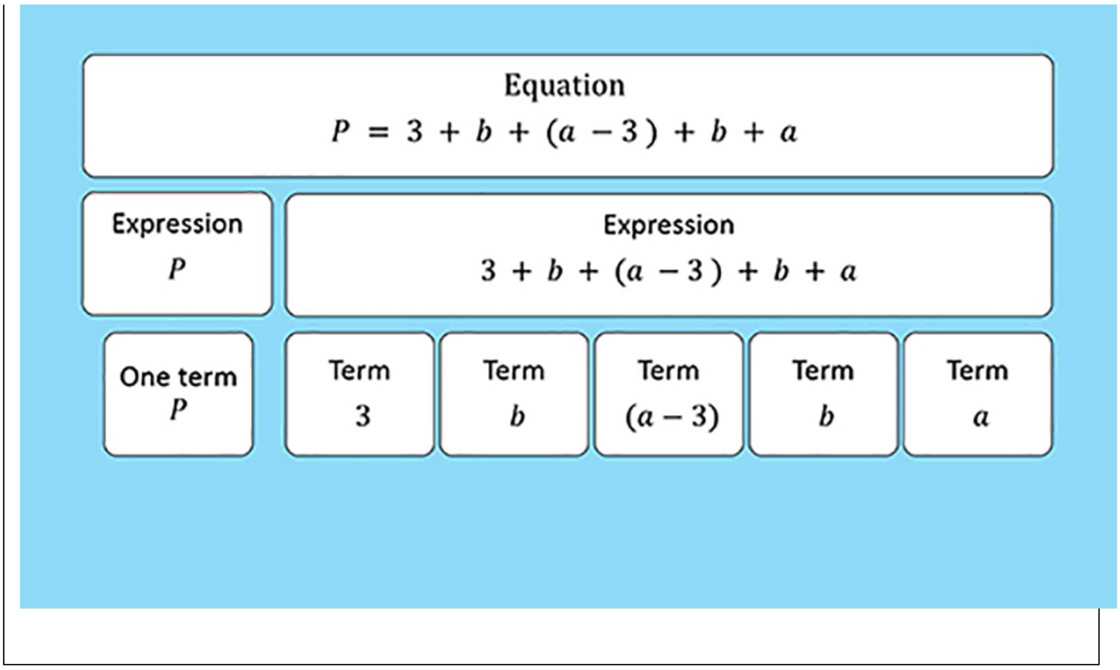
Video 1 Expressions and terms



Differences between talking in mathematical sentences and writing algebraic expressions

It is useful for teachers to compare spoken explanations and algebraic expressions.

The explanations in Activity 2 above are written in informal language, not in whole sentences, but as learners might say them.

Learners typically explain ideas in several short sentences, each with one operation. The sentences may follow the order of their thinking (for example, the first instruction will involve ‘adding’ since the aim is a perimeter). This is unlikely to match the order of the terms in the conventional expression.

When we read, we read from left to right. When we talk, our sentences usually start with the main subject. However, in numerical or algebraic expressions, the first term is not the most important one, nor is it always the unknown one, as in the examples above. The order in which you do operations is not necessarily the same order as you read and write the symbols.

Note how learners typically use ‘pointing’ words such as ‘this’, ‘that’ or ‘there’ to indicate a quantity that they want to refer to, but cannot name it without losing the thread of their thinking. In the classroom, they may also point to the diagram or one of the sides while they are talking. Gestures can be an important way of signalling their awareness of an unknown quantity. In an expression, this quantity might appear in brackets, or as a more complex term.

These are steps towards algebraic thinking. Learners have paid attention to an unknown quantity and are communicating about it.

How would you as a teacher build on the learner’s talk, and rephrase each explanation to make a more precise mathematical sentence? Think about using words such as ‘difference’ and ‘less than’.

(We say *more* precise here since it is not always easy to express mathematical relationships concisely and precisely in words – that is why algebra was invented.)

Activity 3 Reflecting

Allow 5 minutes

Look back in the diagram in Activity 2, how would the task be different if a and b were replaced by specific numbers? How could you work with learners to keep the focus on mathematical sentences?

Discussion

When given specific numbers, learners tend to work towards closed answers consisting of a single term. For example, if the triangles were labelled with sides 3, 5 (for a) and 7 (for b), they would add up as they went around the perimeter and get the answer 24. You would need to ask learners to write down mathematical sentences to keep the focus on the instructions, not the answer, such as writing down $10 + (5 – 3) + 5 + 7 = 24$.

When there are specific numbers, learners find it easier to identify a number that they can use for the ‘leftover bit’.

For a general side a, the difference is written $a – 3$. It cannot be simplified further but you can write it instead of a number and give the instruction to add it to the other sides. Learners often like to use brackets to show that this is an expression that is being treated as a single object.

One strategy for helping those who are stuck on this part is to say ‘Suppose a was 5, how long is that bit?’ There is then a danger that they spot that the difference is 2 using only their knowledge of the specific numbers 5 and 3. You can help them pay attention to the general structure by recapping that the difference is 2 because $5 – 3$ is 2, so the important operation is subtraction.

Activity 4 Varying the problem

Allow 10 minutes

What happens to the perimeter if you combine the two congruent triangles into the new shapes in Figure 2?

Use the mathematical extensions for greater depth:

What other shapes and perimeters can you find with these triangles?

Could you use these triangles to make a shape with perimeter $2a + 2b + 3$?

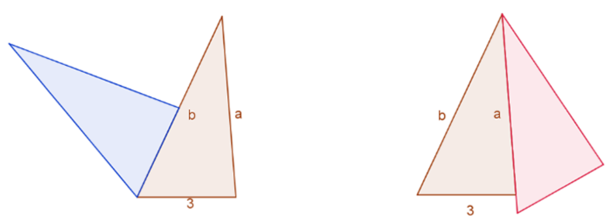


Figure 2 Four triangles

Discussion

If you join the two triangles together with the whole side of one triangle set against the same length side of the other triangle, you can arrange the triangles into many shapes, including a rectangle, a parallelogram and a kite. Whatever the shape, the perimeter will always simplify to an expression that is double the sum of two of the sides.

If you change the rule slightly, then you can find some other perimeters. For example, they could meet just at one point, or the sides could overlap by only half their length.

Activity 5 Is this task algebraic?

Allow 5 minutes

The beginning of this section defined algebraic thinking as involving indeterminacy, denotation and analyticity. In other words, it involves thinking about indeterminate quantities that are represented in words or symbols and then treated analytically.

Do you consider the triangle task algebraic?

Discussion

Although the task is set in a geometric context and needs an idea of perimeter, it is algebraic.

It involves *indeterminacy*, as learners are working with the unknown quantities a, b and p, and these are treated *analytically* since they are operated on as if they were known numbers. The quantities are represented on a diagram to give a concrete basis for the reasoning. Learners also represent, or *denote*, them symbolically using letters and numbers and iconically using words to identify and name the quantities (‘the difference’, ‘the little bit’) or to ‘point’ to them (‘that one’). The fact that the answer cannot be reduced to one specific number or closed answer is also characteristic of algebraic thinking.

2 Finding the general in the particular

Early algebra involves moving between the particular and the general.

When we talk about moving to the general in mathematics (also called generalising), we are concerned with finding features of a situation, thinking about how they may change and extending our reasoning to cover those changes. This has many similarities with our aims for learning – we *want* our learners to generalise beyond today's lesson.

You have seen that algebraic thinking involves working with unknown quantities, so you can think of this movement in two ways.

First, teachers and learners can choose different **representations** for the unknown quantities. Some representations are closely linked to their own particular classroom experiences, while others draw on generally accepted conventions for symbolic notation.

Second, the **indeterminate quantity** being represented can be thought of as having a specific (but unknown) value or it can be thought of as a variable, capable of taking a range of values. Mathematical statements about a specific unknown are likely to lead to finding its value; whereas statements about a variable say something more general, since they must hold for any of its possible values.

This section looks at these two movements, starting with representations.

2.1 Representations

The work of Jerome Bruner (1966) has been influential in early algebra. He identified three modes of representation for mathematical objects: the enactive, the iconic and the symbolic, which move broadly from the concrete to the abstract. Learning a new concept is supported by meeting each of these modes and translating between them.

- In the **enactive mode**, the concept is represented through learners acting on concrete, physical objects. For algebra, the concept of an unknown quantity can be represented by a bag or tin containing an unknown number of counters. The teacher can ask questions such as 'What happens if three counters are placed in a tin? What if one counter is taken out? What if there are ten tins with the same number of counters in each?' A variable can be represented as a changing length, for example learners can measure the height of a sunflower.
- In the **iconic mode**, a picture illustrates the unknown quantity (Figure 3). The picture takes its meaning from the learner's previous enactive experiences.
- In the **symbolic mode**, the concept is represented in an abstract or a conventional way. Bruner considers words (such as 'five', 'add' or 'tin') to be symbolic in the same way as conventional letters and signs are (e.g. 5, + or t).

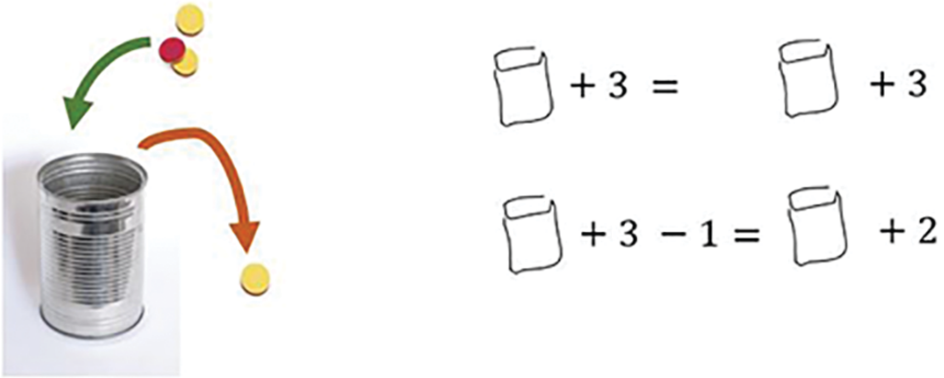


Figure 3 Representing operations on an unknown quantity iconically

Concrete–pictorial–abstract

Some teaching traditions describe a learning progression through concrete–pictorial–abstract and develop their materials accordingly. There are often intermediate stages in the final translation towards symbols/abstract representations. For example, Figure 3 shows two different pictorial/iconic representations of the tin activity. One use photographs of objects and arrows that represent the actions of adding and taking away counters. In the other, the learner has drawn their picture of a tin and combined it with familiar symbols such as +, 3 and =.

Bruner's three modes offer a useful way of planning for progression, but their order does not have to be interpreted too strictly. Teachers have found that even young children can reason algebraically about a quantity that they have not handled, when the meaning is carefully established through discussion. Equally, what *feels* concrete or abstract to a learner can change with experience. Words such as the 'price of a book' are considered as symbolic for younger learners, but they can feel iconic once their language has developed. For example, learners often choose to use symbols such as *b* (*book*) or *p* (*price*) which remind them of the words they are using for a quantity. A learner studying A-level mathematics will be able to use *x*, *y* or any other letter for the price and will eventually have enough experience of manipulating symbols with pencil and paper that this will start to feel like a concrete activity on which she can base further learning.

Making sense of symbols

Research in early algebra suggests that learners need to establish meaning in two ways. First, they need to understand how calculations and arithmetical relationships between quantities are expressed using the familiar symbols for numbers and operations. This is why teachers emphasise forming mathematical sentences in words and in number symbols as well as calculating answers. Second, learners need to make sense of working with an algebraic symbol to represent an indeterminate quantity. These are both necessary stages, which lay the foundations for manipulating symbols out of a context. If learners move too quickly to symbolic representation they risk interpreting the symbols incorrectly, introducing misconceptions that use their prior experiences from number work to deal with the troublesome letter.

Activity 6 Reflecting

Allow 10 minutes

Look at this question and the range of responses. Which would you expect to see from your learners (try it with them)? What is the misconception that underlies each answer?

When you have finished the task watch the video.

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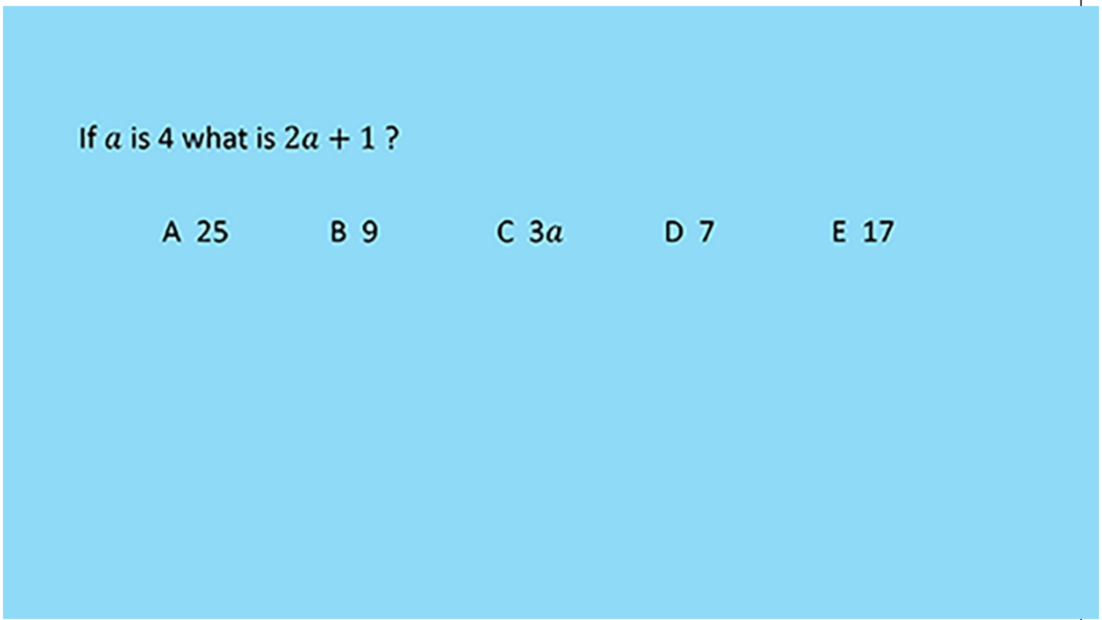
Video 2 Discussing the misconceptions behind wrong answers



2.2 Indeterminacy: moving from specific unknowns to variables

When learners are introduced to symbolic representations, they often represent a quantity that has a specific but unknown value. The learner may be asked to find this value.

Two examples of such problems are missing number problems and Think of a Number (THOAN) problems.

Activity 7 Missing number problems

Allow 10 minutes

Have a go at the following problems.

Missing number problems

$? + 3 = 13$
$? + 2 + 3 = 13$
$4 + ? + 3 = 13$
$1 + 3 + ? = 13$

$10 + 3 = ? + 2$
$10 + 3 = 7 + ?$
$10 + 3 = 6 + 1 + ?$

Think of a number problems

Think of a number, multiply it by 3 then add 2. Tell me your answer and I can work out your number.

Harry thought of a number, multiplied it by 3 then added 2. The answer was 17. What was Harry's number?

Think of a number, double it, add six, divide it in half, and subtract the number you started with. The answer is three. Why?

Figure 4 Missing number problems

Activity 8 Reflecting

Allow 5 minutes

What makes these *algebra* problems, rather than simply number problems?

What do you expect learners to learn from these problems?

Now watch the two videos that discuss these questions.

Video content is not available in this format.

Video 3 Missing number problems

Missing number problems

$? + 3 = 13$
$? + 2 + 3 = 13$
$4 + ? + 3 = 13$
$1 + 3 + ? = 13$

$10 + 3 = ? + 2$
$10 + 3 = 7 + ?$
$10 + 3 = 6 + 1 + ?$

Think of a number problems

Think of a number, multiply it by 3 then add 2. Tell me your answer and I can work out your number.

Harry thought of a number, multiplied it by 3 then added 2. The answer was 17. What was Harry's number?

Think of a number, double it, add six, divide it in half, and subtract the number you started with. The answer is three. Why?

Video content is not available in this format.

Video 4 THOAN

Think of a number problems

Think of a number, multiply it by 3 then add 2. Tell me your answer and I can work out your number.

Harry thought of a number, multiplied it by 3 then added 2. The answer was 17. What was Harry's number?

Think of a number, double it, add six, divide it in half, and subtract the number you started with. The answer is three. Why?

2.3 Comparing algebraic expressions

Learners find it easier to follow word problems that are posed in a way that matches a typical sentence structure, such as the THOAN problems above. However, even a small variation on these will start to disrupt the order. For example, it is much harder to work on a problem like this one (which comes from an examination for 11-year-olds):

> Younus takes three large letters to the Post Office. He pays with a £20 note. He gets change of £14.80. How much did each letter cost to post?

Earlier you have seen there is a progression in moving from working with a quantity that represents a specific unknown to one that can take a range of values. You will return to the ideas of variables and generality from a different perspective in Week 5, when you will look at introducing functions.

In the middle years, learners will move between these perceptions of number as they tackle different problems. Some learners strongly prefer to think of an unknown quantity as having a specific value. They experience what researchers call 'a compulsion to calculate'. You can test this preference by asking diagnostic question such as these:

Activity 9 Reflecting

Allow 5 minutes

How did you answer these questions?

Did you choose specific values for *m* and *n*? Do you see now that you did not need to?

Discussion

The answers are: 11, 51 and 17.

The first question involves such a familiar number bond that you are very likely to have thought of: *m* as 6.

The second was chosen to have the same structure but vary the number involved so that it becomes easier for you to pay attention to the comparison of 2*m* and 2*m* – 1. Some of your learners will not do this. Instead they will work out *m* = 26 and then calculate .

The third question again has a similar structure but here *m* and *n* can take a range of values, so they are not specific unknowns. Again, some learners will choose values, for example 10 and 5, and then work out 10 + 5 + 2 = 17. Some will show awareness that these are not the only possible values by trying another pair, for example 7 and 8.

Another way is to use tasks that allow learners to treat quantities as variables while providing some support by being embedded in a number context.

The strategies discussed above lead learners to a correct answer but they reveal the ‘compulsion to calculate’. These learners have difficulty in dealing with unknowns by comparing algebraic expressions, and this will prevent their progression to more complex algebra.

One way of working on this difficulty is for teachers to ‘productively linger’ with algebraic problems – asking learners to explain their thinking and listen to others. Obtaining the answer is not enough: learners should use two different methods and be able to compare them.

Activity 10 Symbol sums

Allow 10 minutes

This task involves working with several unknowns and there are several different ways to approach it. Have a go at it yourself.

Each symbol in the grid in Figure 5 has a value. The totals of the rows and columns are shown. Can you find what each symbol is worth?

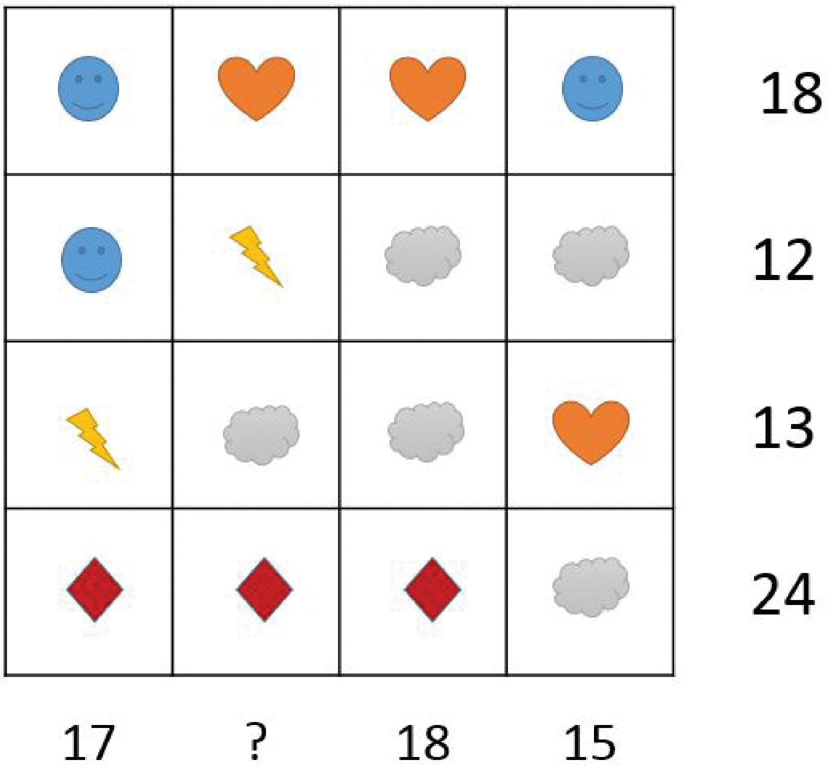


Figure 5 Symbol grid

Activity 11 Thinking points

Allow 5 minutes

- Where would you encourage learners to start?
- How long would you leave learners to work independently before you ask them to listen to other people's strategies?
- What kind of verbal explanations would you expect from learners? How do you want them to record their thinking?
- Is the task finished when the first learner works out the values? How could you 'productively linger' over this task to encourage algebraic thinking?

You can find a similar activity at https://nrich.maths.org/1053

Discussion

You could start the problem in several ways:

The first row gives you the sum of heart and smiley.

Compare it with the fourth column to find the value of a cloud.

Compare the second and third rows to find that heart is 1 more than smiley.

The fourth row shows a cloud has to be a multiple of 3: try 3 or 6 or 9.

Compare the third and fourth column to see that a diamond is 3 more than a smiley.

The third row and third column show a diamond is 5 more than a lightning.

Activity 12 Symbol sums continued

Allow 5 minutes

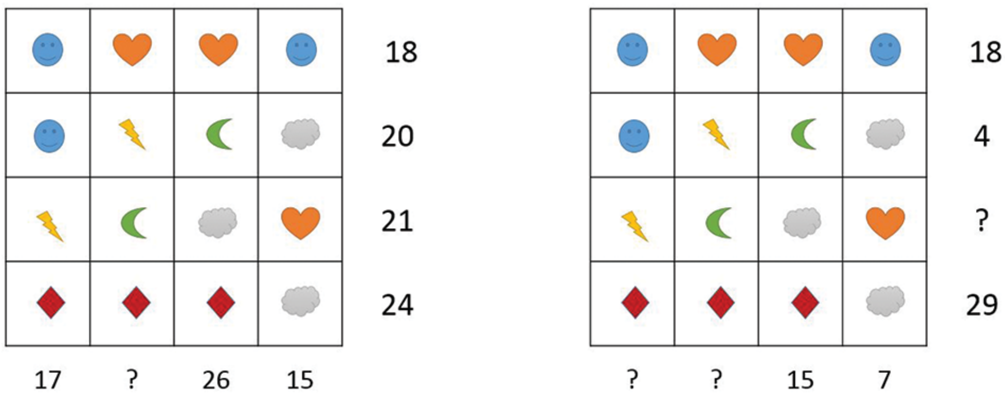


Figure 6 More symbol grids

It is a good exercise of your own mathematical knowledge to invent similar problems and check that they can be solved.

In the two grids in Figure 6, the symbols in the grid are the same but the level of mathematics is slightly harder. Can you see why?

3 The equals sign

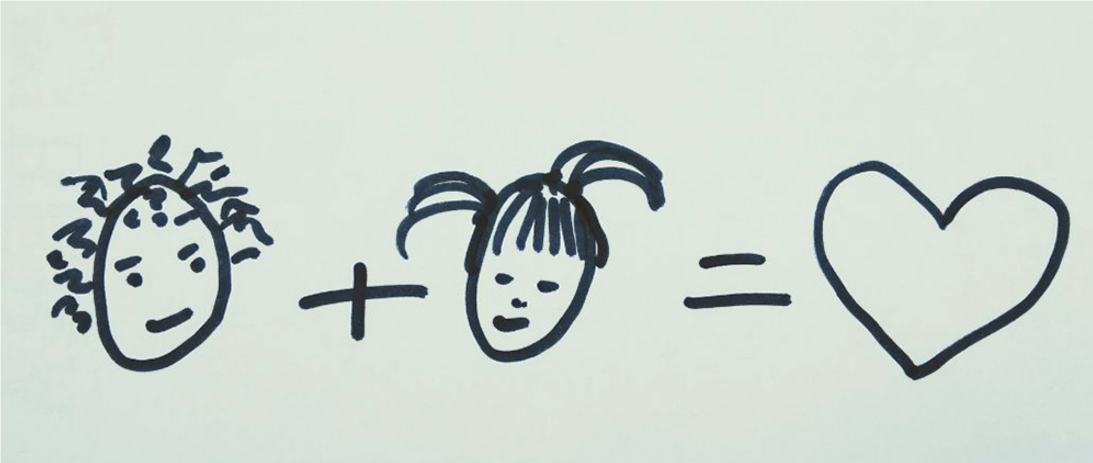


Figure 7 Friends = love

One of the strong findings of early algebra research is that learners in the middle years need to change their perception of the equals sign. This sign is recognised from an early age (Figure 7), and it is usually understood as standing for an instruction such as ‘makes’ or ‘work it out now’. This is called the **operational** understanding of the equals sign and it works well for number problems. Text books, worksheets and even national examinations use the equals sign in this way:

To make progress in algebraic thinking, learners need to develop a new **relational** understanding of the equals sign. In algebra, the equals sign only makes sense when it is used as a way of comparing two expressions and saying that they have the same value as each other.

All of the following equations can be understood as statements that express a *relationship* between two quantities:

Two other **relational** symbols are the inequality signs < and >. Tasks that require learners to decide which of these symbols to use can develop a relational understanding of the equals sign.

Activity 13 Relationships between expressions

Allow 5 minutes

Which is bigger: $3a – b$ or $2a + b$? Choose values of a and b and decide. Record your results using the signs <, > and =.

Discussion

You could write:

If we choose then and , so .

If then and so .

Or then and

Activity 14 Reflect

Allow 5 minutes

You might have noticed the different ways in which the three responses to the previous question were written.

How would you ask your learners to record their results?

Does this ensure that they have practice in selecting and writing the correct signs, in comparing two expressions?

a	b	$3a - b$	$2a + b$	Write your inequality or equation here
1	2	1	4	$3a - b < 2a + b$
10	3			
6	3			

Discussion

Using the equals or inequality signs between numbers or symbolic expressions is correct mathematics. Learners should not write =, < or > on its own.

Using a table is often helpful for keeping track of several calculations.

a	b	$3a - b$	$2a + b$	Write your inequality or equation here
1	2	1	4	$3a - b < 2a + b$
10	3	27	23	$3a - b > 2a + b$
$3a - b$	$2a + b$	15	15	$3a - b > 2a + b$

3.1 Relational understandings

Treating the equals sign as a relation is important for two aspects of algebraic thinking. The first is **maintaining equal value**. The two expressions on either side of the equals sign are equivalent; their values balance. Anything that is done to one side of the equation must be done to the other side, in order to keep that balance. This aspect is seen in the common metaphor of an equation as a balance scale (Figure 8).

Sometimes this aspect is referred to as sameness. However, looking at the balance makes it obvious that there is not the *same* expression on both sides. Instead, they have *equal value*.

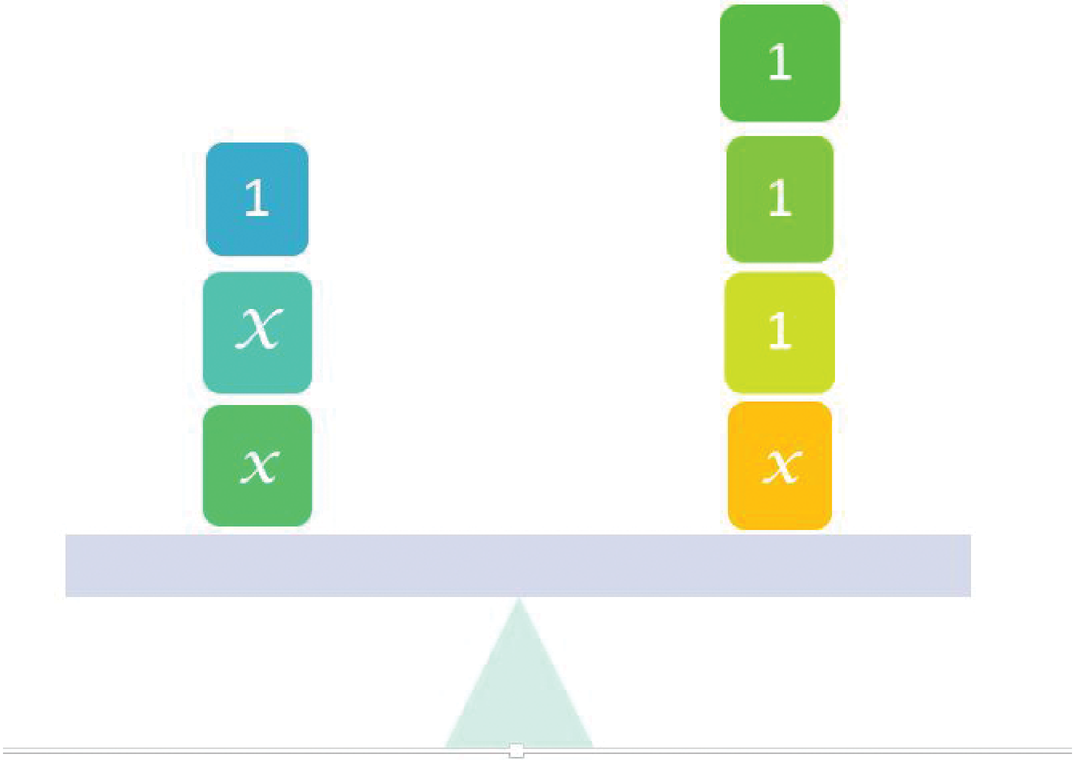


Figure 8 Balance

What is the equation represented by this balance?

What happens if you take 1 from both sides of this balance? If you take the number represented by x from both sides?

The second benefit of the relational understanding of the equals sign is that it allows **substitution**. If two expressions are equal in value then one can be substituted for the other. This is useful in simplifying expressions and solving equations.

You met substitution in the symbol grid activities. Watch the video below to see how.

The equals sign as substitution is used twice in this process: the first time it is used to replace an indeterminate quantity (expressed symbolically as a sum) with a number that has equal value. The second time the recall of number facts is used to determine a number that could be substituted for symbolic quantities, and a new equation can be written.

Video content is not available in this format.
Video 5 The equals sign – substitution

The equals sign - substitution

Substitution or ‘do the same to both sides’

It is useful for teachers to be aware of the substitution aspect of the equals sign as learners will use this in their speech. They are likely to draw on the substitution when they know the number facts involved (here it is 9 + 6 that equals 15). When they do not know the number facts, they will use the principle of ‘maintaining equal value’ and ‘do the same to both sides’ in order to work out 15 – 9 = 6. Both aspects need to be recognised and taught.

3.2 Always, sometimes, never true?

As previously discussed, algebraic expressions are like mathematical phrases or sentences. Equations are like mathematical statements. A mathematician will always ask whether or not an equation is true.

When equations contain no unknown quantities, use calculation to check if they are true:

is an equation but it is not true.

is an equation that *is* true.

When equations include unknown quantities, they may be true for only one value of the unknown:

is only true when n is 11.

If $n = 10$ then $2n – 5 = 17$ is false.

They may be true for several values:

is true when $n = 3$ and when $n = –3$.

They may be true for all values of the unknown:

In formal mathematics you can use the sign ≡ to show that these expressions are ‘identically equal’:

Sometimes an equation is given as a definition or a formula for working something out. We can assume that someone else has done the work of showing it is always true. For example:

(definition).

(formula a learner could prove).

(formula that a learner could not prove!).

Activity 15 Always, sometimes or never true

Allow 10 minutes

Decide whether these equations are true for all values of the variable, or only some values or none.

$2n + 3 = 3 + 2n$	$b + 12 = b + 16$
$4 + 2e = 6e$	$p + 12 = f + 12$
$2t - 3 = 3 - 2t$	$k + 5 < 20$
$3(m + 3) = 3m + 3$	$4(3 + c) = 12 + c$
$p^2 > 4$	$q^2 = 10q$
$x^2 > x$	$16s^2 = (4s)^2$

Figure 9 Always, sometimes, never true?

Discussion

Trying values will help you see that there are no values of b or m that make their equations true: they are *never* true. Using algebraic reasoning will help us see that e = 1, t = 1.5, c = 0, q = 10 are specific values that make their respective equations true. Also, some equations are true for a range of values: k < 15, p either greater than 2 or less than -2, and x greater than 1 or less than 0. A final one, that is more subtle because it involves two variables, is that p + 12 = f + 12 is true only when p = f. All these equations are *sometimes* true.

Finally the equations 2n+3 = 3+2n and $16s^2 = (4s)^2$ are true for all values of n and s. The first is because addition is commutative, and the second because squaring applies to the whole bracket. They are *always* true.

4 This week's quiz

It's now time to complete the Week 4 badge quiz. It's similar to previous quizzes but this time, instead of answering five questions, there will be 15.

Week 4 compulsory badge quiz

Remember, this quiz counts towards your badge. If you're not successful the first time, you can attempt the quiz again in 24 hours.

Open the quiz in a new tab or window and come back here when you've finished.

5 Summary of Week 4

In Week 4 you met the idea that algebra is more than simply using letters for numbers. Algebraic thinking involves an appreciation of structure and generality. It allows us to work with unknown quantities as if they were numbers, using words or symbols to represent those numbers. You have seen how algebraic thinking can be supported by the progression in the use of concrete, pictorial and abstract representations.

You should now be able to:

- describe algebraic expressions and algebraic terms
- support learners to move between working with numbers and working with symbols
- deal carefully with the equals sign and be aware that it is often used to mean 'makes the answer of' but that in algebra it has the meaning of equivalence (i.e. both expressions on the right and left of the equal sign have the same value).

Next week you will build on what you learned in week four to begin to think about functional reasoning and specifically how unknown quantities can vary.

You are now halfway through the course. The Open University would really appreciate your feedback and suggestions for future improvement in our optional end-of-course survey, which you will also have an opportunity to complete at the end of Week 8. Participation will be completely confidential and we will not pass on your details to others.

You can now go to Week 5.

Week 5: Introducing functions and graphs

1 Function machines

The function machine is a way of thinking about three aspects that make up a function: the inputs, the outputs and the rule that determines their connection (Figure 2). The representation as a machine connects them in one diagram (just like seeing a toaster makes you think of bread and toast).

You have already seen examples of such thinking in the 'Think of a Number' problems. The input is the number you think of, you calculate given a rule and you tell someone the answer – the output.

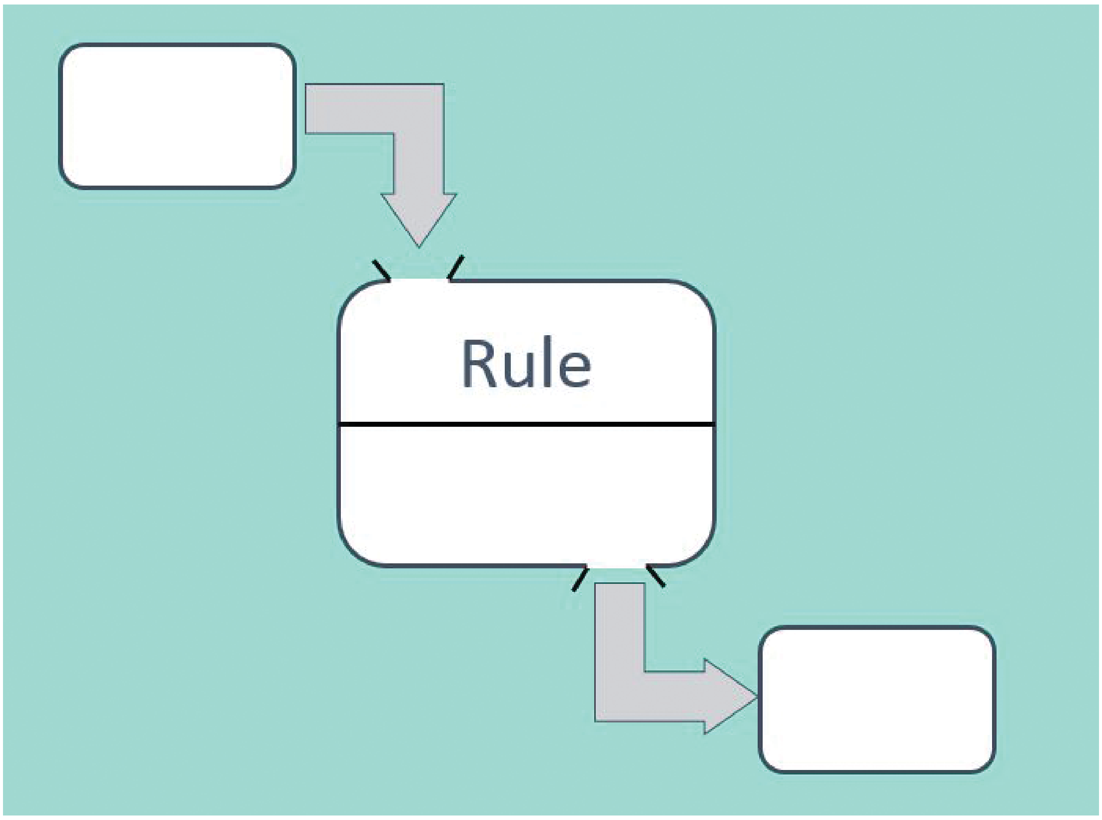


Figure 2 Function machine

The rules for functions can be as simple as 'double' or 'stay the same' and as complicated as you like. (It does not even have to be expressed in symbols.) The rule *does* have to be 'well defined', meaning that for each input number there is only one possible answer.

This condition means that two of the examples in Figure 3 are not functions:

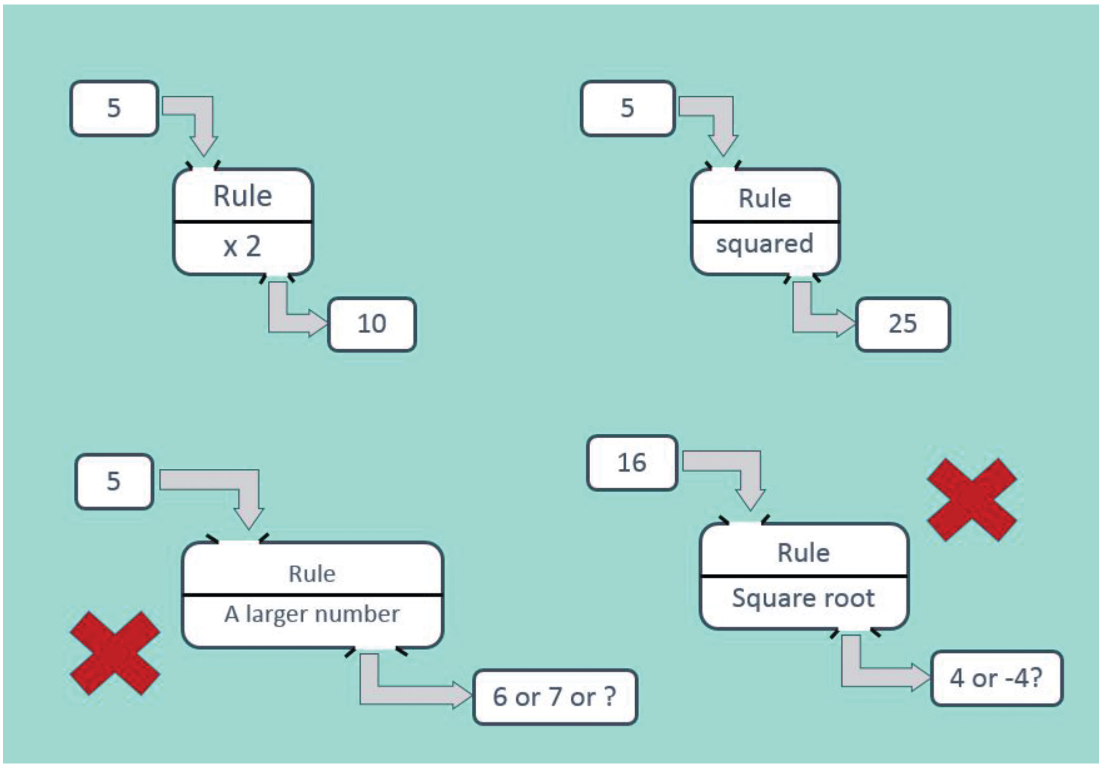


Figure 3 Examples and non-examples of functions

As you saw with Think of a Number problems, the function machine representation is powerful because learners can work with it at the level of taking a specific unknown quantity – the input – and calculating with it to get an output. In this sense, the function is a process that they can carry out.

The machine representation also lends itself to more sophisticated tasks requiring learners to go beyond simply finding outputs. Teachers can introduce the idea of the input as a quantity that can take *any* value. They can set tasks where the process of writing and comparing the rules and outputs is more important than what numbers they are. The two activities you will carry out in this section illustrate this kind of task.

The goal of working in this way is laying the foundations for learners to treat quantities as variables and to use functions fluently in later mathematics.

For example:

- the squaring function is used in finding the area of a circle
- the cubing function is used in finding volume
- the sine function is used in trigonometry (and, of course, the cosine and tangent functions).

The 'machine' analogy suits these examples since it corresponds to learners' experiences of pressing the calculator key for that function (Figure 4). Before calculator technology, all of these functions were either calculated or looked up in special booklets of tables.

Figure 4 Functions on your calculator

Activity 1 Getting used to long expressions

Allow 10 minutes

On a piece of paper, write an expression that summarises the following sequence of functions.

Think of a number, then:

- add 3
- multiply by 2
- subtract 5
- divide by 3
- add 72
- multiply by 6.

Then watch the video below from the OU archives. It introduces a teacher, and shows him building up the sequence with 14-year-old learners on a board. Ask yourself:

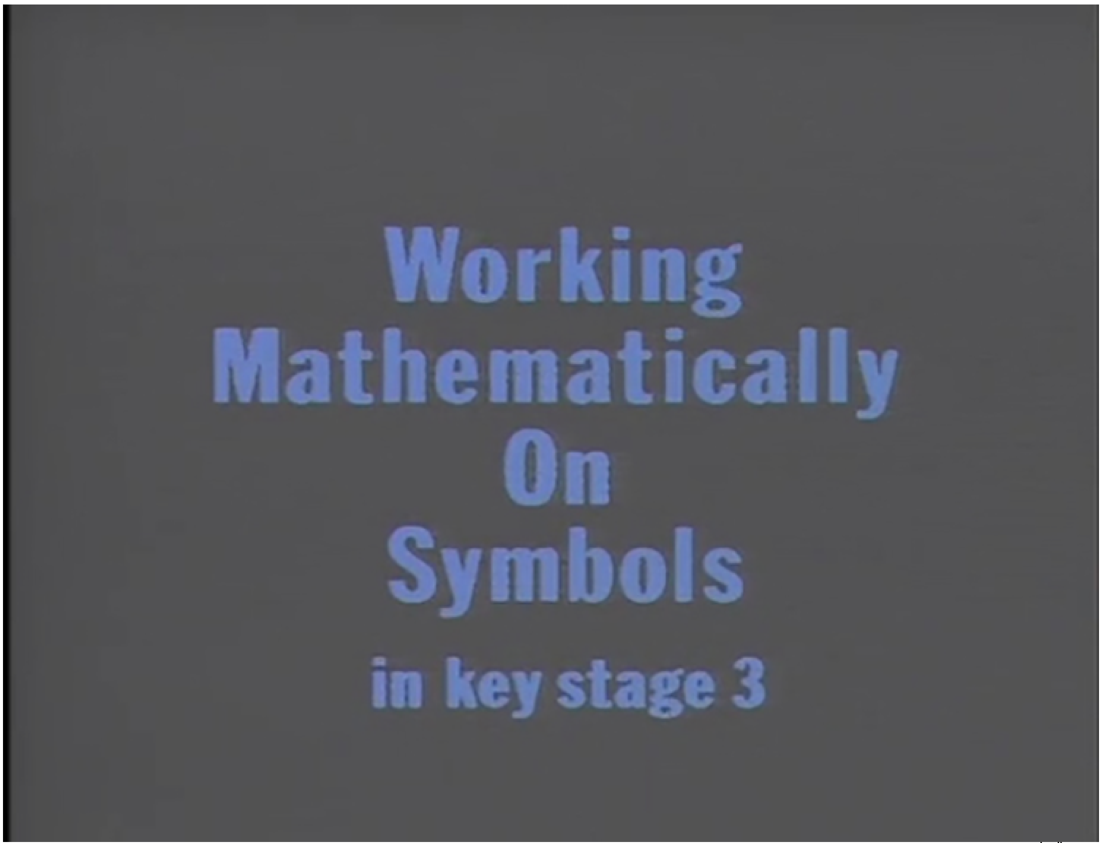
What do you think the teacher's learning objectives are?

How do learners react to calculating on an unknown quantity?

What does the teacher do to draw attention to conventional symbol use?

Video content is not available in this format.

Video 1 Working mathematically with symbols at Key Stage 3



Discussion

The emphasis of the video activity is on gaining familiarity with symbols to express the whole sequence without any intermediate results being calculated. Note how the teacher builds up the expression first by 'doing' each function, and then by 'undoing' it. 'Doing and undoing' is an important idea in algebra. It provides an organising structure to relate processes that appear different but are the reverse of each other, such as × and ÷. This allows us to reduce the complexity of calculations.

The next activity is a well-tested teaching approach devised by Don Steward, called 'both ways'. Using two functions (× 2 and + 3) increases the level of complexity, and going both ways allows learners to make predictions about the results. This task is designed to shift attention from calculating outputs towards comparing methods of calculation. The activity is written in the way you would use it with a class of learners, although you would need to offer support for some of the calculations (for example, a negative number line).

Activity 2 Both ways

Allow 10 minutes

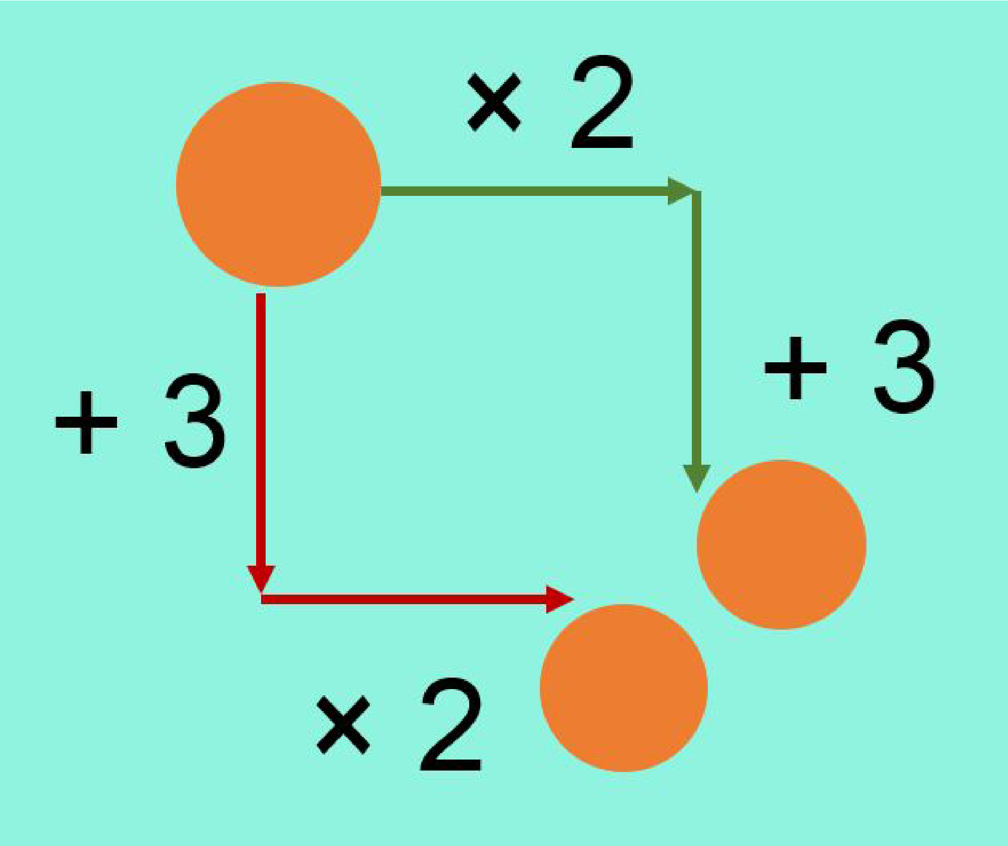


Figure 5 Both ways

You are going to choose an input number and follow the instructions on the arrows in Figure 5 – both ways. Do you think your answers will be the same?

- Try starting with these numbers: 5, 8, 3.1, –4, 100, 1000
- a million (you could write m for short)
- a billion (b for short)
- any number (n for short)
- Compare the two answers you get. Can you explain what is happening?

Discussion

You get different answers by going different ways:

13 (green way) and 16 (red way), 19 and 22, 9.2 and 12.2, –5 and –2, 203 and 206, $2m+3$ and $2m+6$, etc.

The difference between the answers is always 3, for any input number.

Using *m* for a million, etc. is a popular way of introducing letters for numbers in function machines. It matches the way we speak about such numbers in words: 2 million and 3. It is mathematically correct in that *m*, *b*, *t* all stand for numbers – one million, one billion, etc. – rather than objects such as mangos or bananas that cannot be calculated with.

Although it is a good introduction, it is not an approach to stay with. (For example, you do not want learners to write *2 million one hundred and three* as $2m1h+3$ or think that *m* always stands for a million.) This is why the sequence of examples ends with using *n* for 'any number' (the sound rhyme helps learners accept this meaning).

You could explain this result with diagrams using a rectangle for *n* and dots for each 1. The important step is that, following the red arrows, both the number and the 3 dots get multiplied by 2.

This activity shows the need to use brackets when you combine addition and multiplication in a single expression. If you are confident with algebra, you might have explained it to yourself by writing the expression for each way and expanding the brackets. However, learners in middle school are unlikely to be convinced by manipulating symbols because they do not yet feel concrete to them.

green arrows $2n+3$

red arrows $2(n+3) = 2 \times n + 2 \times 3 = 2n + 6$

$2n$ and $2n$ are the same number, but 6 is more than 3 so the difference is 3.

You can find resources related to this activity on the Median blog at https://donsteward.blogspot.com/ indexed under 'Expressions Both Ways'.

2 Sequences as functions

A sequence is a collection of numbers or objects that are arranged in order. Each of these numbers or objects is called a ‘term’, so a sequence has a first term, second term, third term, etc.

Note that this is the same word, but not the same meaning, as the ‘terms’ that you find in an expression. You do not rearrange, add or subtract the terms of a sequence. It is wise to avoid using the word in different ways within one lesson.

All of the sequences you meet in school mathematics have a rule.

Activity 3 Sequences and rules

Allow 5 minutes

What is the next term in each sequence in Figure 6?

What is the rule for each sequence? Write it in words, and as an algebraic expression if you can.

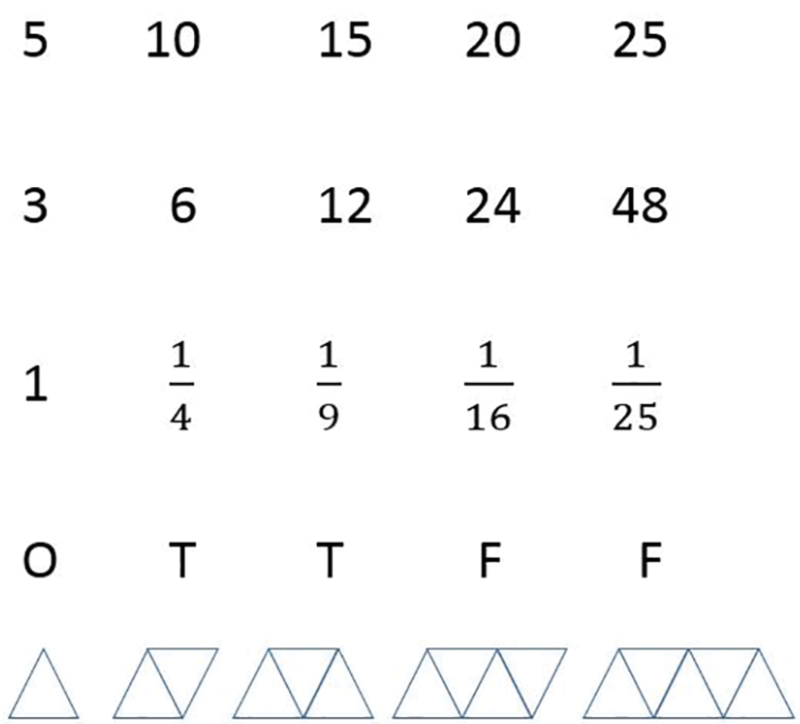


Figure 6 Sequences

Keep a note of your work on this activity as you will use it in the next section.

Discussion

For the first five sequences the sixth terms are 30, 96, 1/36 and S. The last sequence is a diagram with six triangles.

2.1 Term-to-term and position-to-term rules

For three of the sequences in Figure 6 you probably thought about how to get from one term to the next, and continued that pattern. You could have done this by adding 5, doubling, or drawing another triangle. These are all ways of describing the change between any two consecutive terms, so this is called finding a **term-to-term rule**.

It would be much more difficult to identify a term-to-term rule for the other two sequences in Figure 6. Instead, you need **position-to-term** rules. This type of rule connects the position number (1, 2, 3, etc.) to the associated term. A table is a useful representation to show the relationship between position number and the term.

Position	1	2	3	4	5	6	n
	one	two	three	four	five	six	
Term	O	T	T	F	F	S	?
(letter sequence)							
Term (fraction sequence)	1	1/4	1/9	1/16	1/25	1/36	$1/n^2$

For the letter sequence, the position-to-term rule is 'the term is the first letter of the position number as an English word'.

For the fraction sequence, 'the term is the fraction 1 divided by the position squared'. You can write this symbolically as

Term-to-term rules describe the pattern you see by moving *along* a row of the table.

Position-to-term rules describe the pattern you see when moving *down* from the position row.

The rule must describe a general pattern that holds for every term of the sequence.

Learners often prefer looking for term-to-term rules. This is for two reasons. First, they only need to look along the changes in one variable – along the terms of the sequence. Second, term-to-term rules involve simpler operations. However, a term-to-term rule is not helpful if you want to find a term that is far away from the ones that are known.

The position-to-term rule is more powerful.

Activity 4 Position-to-term rules

Allow 10 minutes

Imagine finding the 37th term in each of the five sequences in Figure 6.

Which one would be easiest?

Which one would be hardest?

Write the position-to-term rules for the first two sequences.

Discussion

Sequence	37th term	Position-to-term rule

5, 10, 15, 20, 25	185	Multiply position number by 5. n^{th} term is $5n$
3, 6, 12, 24, 48	3×2^{36} = 206 158 430 208; most calculators will not display all of these digits.	Raise 2 to the power of one less than the position number. Multiply the answer by 3. n^{th} term is $3 \times 2^{n-1}$
1, 1/4, 1/9, 1/16, 1/25, ...	$1/37^2$	n^{th} term is the fraction 1 divided by the position squared. n^{th} term is $1/n^2$
O, T, T, F, F	T	take the first letter of the position number

There is another reason to emphasise position-to-term rules. Sequences are an early example of functions. If we relate this to the function machine: the position number is the input, the term is the output and there is a rule that connects them.

When learners start to work with sequences, they work with specific values of the position numbers: 1, 2, 3, etc. These are not indeterminate quantities, so they are thinking numerically. The algebraic thinking only starts when learners consider what the rule is for a general input number, which is usually denoted by the symbol *n.* The corresponding term is called the *n*th term.

It is relatively easy to understand how the position number can change, since it follows the natural numbers. So, although, teachers may never mention the word ‘function’ when teaching sequences, by emphasising position-to-term rules they are nevertheless preparing the underlying idea of a function as a relationship between two variable quantities.

2.2 Generalising with sequences

Sequences also give learners a context in which to generalise, particularly when they are defined using diagrams or shapes. If you look at sequence resources you will find two main approaches to generalising activities. We can describe these as **looking for changes** versus **looking for structure**. Many textbooks written in the 2000s focus on the first approach, while more recent research (Kieran et al., 2016) suggests that learners also need to look for structure. The next two activities in this section exemplify each approach.

In the first approach, the intention is for learners to observe a pattern of change across a sequence of examples by asking ‘What is the same and what is different?’

Activity 5 Looking for changes

Allow 10 minutes

Each term of the sequence in Figure 7 is a house. Draw the next two terms.

What is the same and what is different about the houses?

What would the 26th house look like?

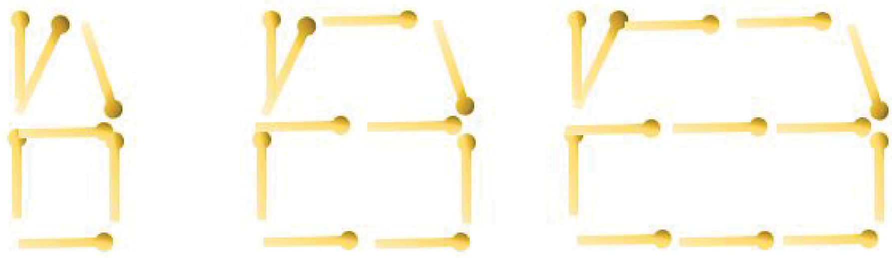


Figure 7 A sequence of houses

How many matches are needed for each of the houses? Complete the table below to make a number sequence.

Then describe a method for finding how many matches are needed for house *n*.

Position	1	2	3	4	10	26
Number of matches						

Discussion

Did you continue the pattern by looking at the changing shapes?

Or by thinking about numbers – how many matches you would need?

You will explore each of these ways of finding the n^{th} term. (This is the position-to-term rule.)

Changing shapes

Learners will come up with several different ways of looking at the way these ‘houses’ grow. As you saw in Week 4, having multiple solution methods can give rise to fruitful algebraic reasoning if there is time planned in the lesson for learners to compare their methods, correct errors and explain why they are the same.

One very common method of describing the changing shapes is to focus on the 3 horizontal rows of matches. Another method is to note that each house has the same beginning and the same end. The middle section grows each time by adding 3 matches (1 each to the roof, the floor and the middle). We will now follow this second method to show its details.

The method is described in pictures, and then words and symbols. This sequence can be used for any methods.

Pictures (see Figure 8)

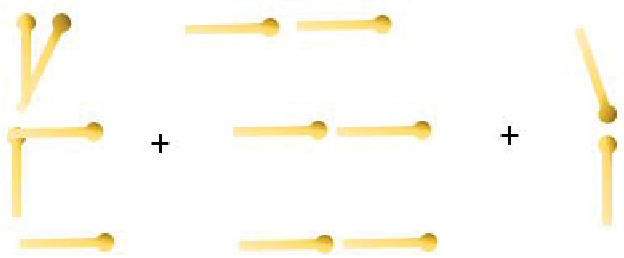


Figure 8 Growing houses

Words

The number of matches for the 10^{th} shape will be 5 (for the beginning) plus 2 (for the end) plus 3 for each of term added on after the first. You add one less group of 3 than the shape number.

Number symbols

It can be helpful for learners to see the calculations written as a sequence.

Shape 1: 5 + 2 or 7

Shape 2: 5 + 3 + 2 or 7 + 3

Shape 3: 5 + 3 + 3 + 2 or 7 + 3 + 3

Shape 4: 5 + 3 + 3 + 3 + 2 or 7 + 3 + 3 + 3

Now that we have got past 3 × 3, it is worth moving to multiplication instead of repeated addition.

Shape 5: 5 + 4 × 3 + 2 or 7 + 4 x 3

Words and algebraic symbols

Looking at the calculations shows a pattern. A method for finding the *n*th term is (starting with the part that changes) to calculate the number that is one less than *n*, multiply that by 3 and add it to 7.

In symbols this is $(n-1) \times 3 + 7$. We can simplify this by expanding the brackets:

$3n - 3 + 7$

$3n + 4$.

So, the *n*th term of the sequence is $3n + 4$.

2.3 Changing numbers

If, instead of looking at the shapes, you focused on the number of matches, you probably noticed that the terms went up the same amount each time, in threes. Sequences that have a constant difference have two names (teachers need to know these but learners do not). They are called **linear** or **arithmetic sequences**.

For linear sequences, it is possible to work only with the numbers and ignore the shapes, and some teachers prefer to work like this. They rely on learners remembering a procedure:

1. Remember that the 3 times table increases in threes.
2. Add a row in the table for the multiples of 3. This row has the rule $3n$.

3. Work out what the change is between 3*n* and the sequence you want. Adapt the rule to include this change

Let's demonstrate this procedure.

Step 1 and 2

Position	1	2	3	4	10	26
Number of matches	7	10	13	16		
$3n$	3	6	9	12	30	78

Step 3

Looking at the third and the second rows (e.g. look at 3 and 7, or 6 and 10), the difference between them is + 4, so we write the rule as

number of matches = $3n + 4$

This helps complete the table for the 10^{th} and 26^{th} terms.

Should teachers work with the shapes or the numbers? The advantage of working only with number patterns is speed. It provides a relatively quick method for finding a position-to-term rule for a given linear sequence. On the other hand, it is relatively common for learners to misapply this procedure and give the answer $3n – 4$. Another disadvantage is the reduced opportunity for thinking algebraically since learners tend to focus only on the first term of the sequence (so they are not working with indeterminate quantities). In addition, simply finding a position-to-term rule is a skill that is worth a few marks in examinations, but is not used directly to support other topics. If learners can use the numerical method without writing an extra row in the table, it is a good indication of fluent thinking with symbols. If they cannot, the teacher needs to balance the effects on their learners of spending time memorising and perfecting the procedure, or of trying to make sense of change within a context.

2.4 Looking at structure in a sequence

We now return to the second main approach, where the focus is on looking for structure in a sequence. This is similar to looking at shape, rather than number, but it uses only one term from the sequence. A *typical* term is chosen – usually the third or the fourth, since the first term is often too simple to show the structure.

Activity 6 Looking for structure

Allow 10 minutes

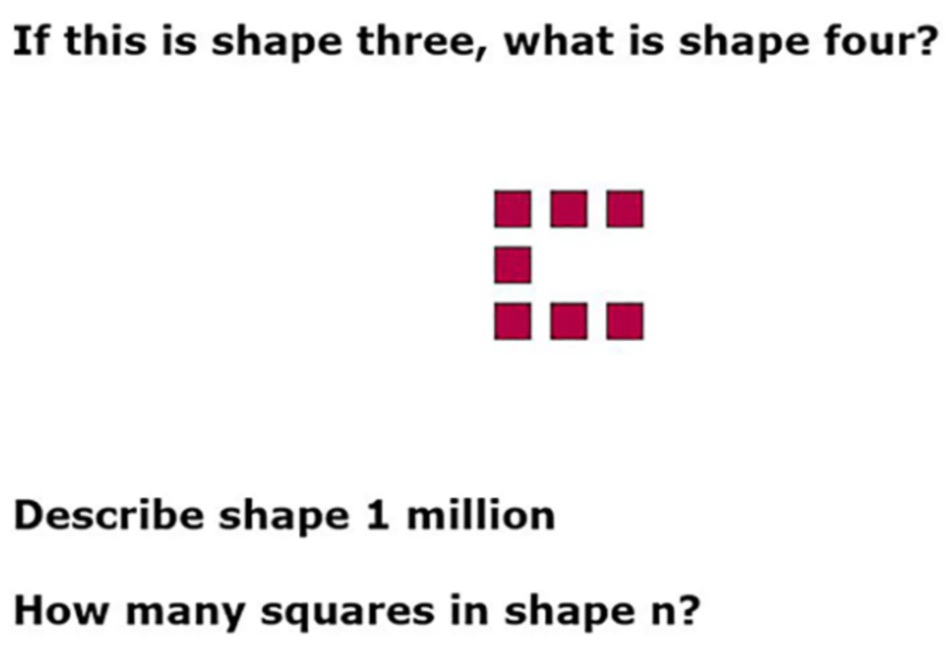


Figure 9 shape C pattern

An activity based on this approach will have two phases of exploration.

In the first short period, learners decide on their 'shape four'. They can be supported by asking 'Where do you see the "threes" in shape three?'. This phase ends with collecting several examples of 'shape fours', displaying them for comparison and asking 'If this is shape four, do we know how to make shape five?'

For this problem, you might allow the class to divide into two groups: those who want to extend only the 'arms' of the C (so that the fourth shape has 9 squares) and those who have also extended its 'spine' (so it has 10). Either method can be justified but they lead to different expressions.

The task for the second phase is for pairs to explain in their books using a labelled diagram what 'shape one million' would look like, and find how many squares would be used for any shape.

Isometric dotty paper is a good resource for creating more patterns to practise this approach (Figure 10). Ask 'If these are shape 3, what is shape 4?'

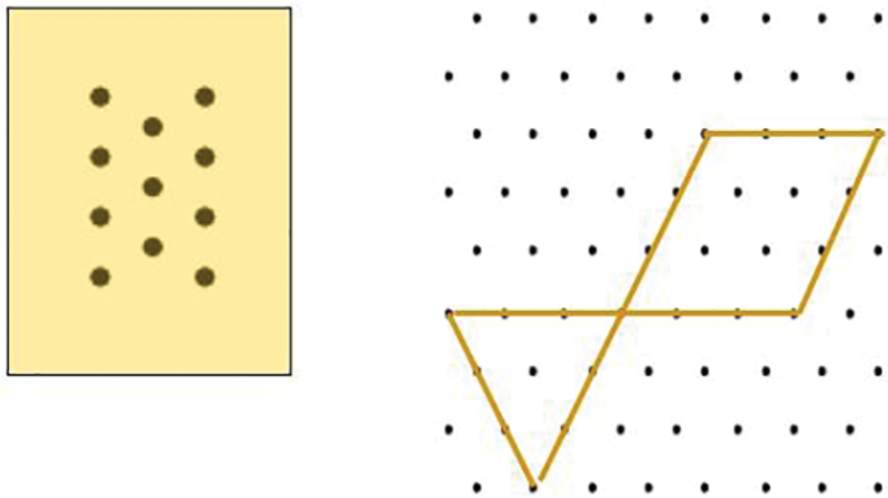


Figure 10 More shape threes

2.5 Summary

So far this week, you have met the idea of a function machine, with input and output, and the idea that sequences are functions where the input is the position number and the output is the term. In both of these representations, the function can be considered as a **process**, that is, as something that you do to the input.

In Activity 1 you became more sophisticated in your treatment of the function by *reversing* the process. In Activity 2 you *compared* two functions. In Activities 3 to 6 you also saw how a function can be considered as a *relationship between two quantities*, that is, as something to notice and describe. This relationship stems from the structural connections that link the two quantities (which gives the position-to-term rule) and it determines how they change together. This will be the focus of the next section.

3 Extending functions: variables and graphs

The idea of a function as a relationship is sometimes called **covariance**; because it describes how two quantities vary together. If your learners go on to study advanced mathematics, they will need to be able to shift fluently between the process perspective on function (associated with rules, function machines and sequences) and the relationship (covariance) perspective.

In the middle years, the goal for most learners is to have a sound understanding of a function as a process and to develop some awareness of the representations related to functions that will prepare them to notice different kinds of covariance.

In this section you will meet some of the more advanced vocabulary and concepts associated with these representations.

3.1 Variables

You have seen that an indeterminate quantity can be thought of as a specific quantity that happens to be unknown (for now), or as one of several unknown quantities. Extending this idea leads to thinking of a quantity as a **variable**, that is, as a single object that can change, or vary in some way.

Variables can be discrete or continuous.

A **discrete** variable can take only certain values, and it would be nonsensical to choose a value between them. The position number of a sequence is an example of a discrete variable. A position number can only be a whole number. It would be nonsensical to talk about the '3 and a halfth' term.

A **continuous** variable is one which can take any value. The nature of a variable may depend on how it is measured. One of the common variables that learners will meet in school is time. Time is usually a continuous variable, but we can make it discrete by taking daily, or hourly, aggregate measurements. You will return to this idea in Week 8 (Working with data).

Activity 7 Representing covariance

Allow 15 minutes

Look at the four pairs of variables below.

1. Are the variables discrete or continuous?
2. Informally, how do you visualise each pair of variables changing together?

    Time in hours / Time in minutes

    Age / Number of hours of sleep

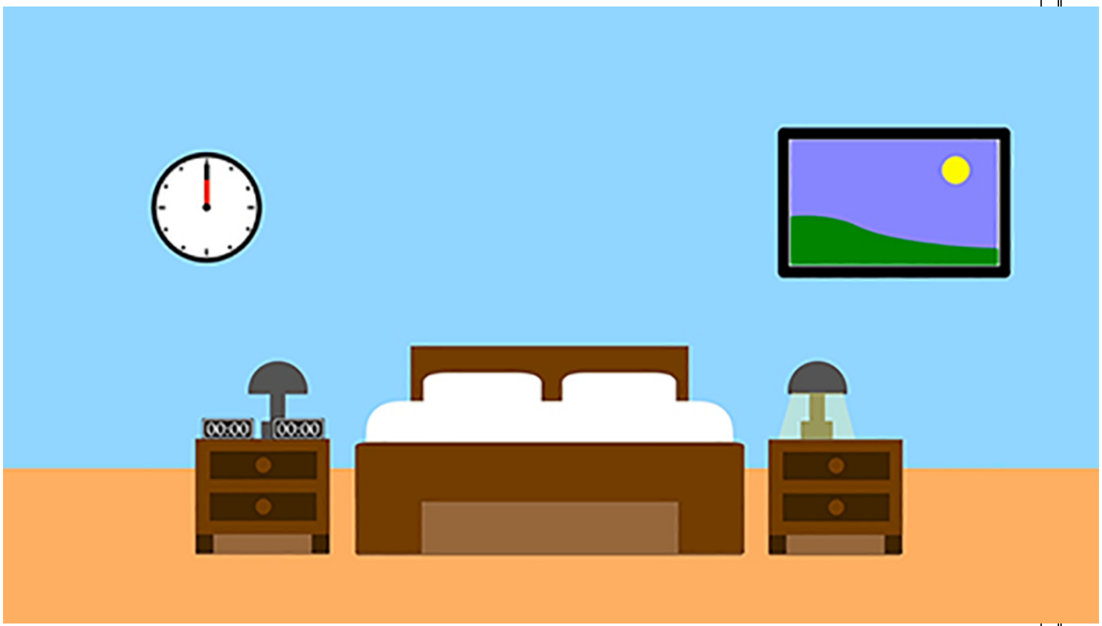
    Number of people sharing a box of 24 chocolates / How many chocolates each person gets

    Your age / Your elder sibling's age.

3. Now watch the video below which shows different mathematical representations of the first example.

Video content is not available in this format.

Video 2 Different representations of covariance



Discussion

Time in hours is usually considered as a discrete variable since we only write whole numbers of hours (except perhaps half hours)

Time in minutes could be either discrete or continuous.

An age is discrete as you are likely to measure it in years or in years and months, but not with an accuracy of days or hours.

Number of people and chocolates are discrete variables.

You might have thought of graphs, or lines growing together, or imagined groups of chocolates.

3.2 Increasing and decreasing graphs

In the Video 2, the graph of time in minutes against time in hours was described as **increasing**, since whenever the -variable increases, the *y*-variable does too.

It is also possible for graphs to be **decreasing**. The definition for a decreasing graph is: whenever the -variable increases, the *y*-variable decreases.

Most graphs are neither increasing or decreasing.

Linear graphs

In the video you saw an example of a linear (straight-line) graph (Figure 11).

If a linear graph is increasing then its gradient is positive, since when increases by 1, *y* changes by a positive number. If a linear graph is decreasing, the gradient is a negative number.

What happens when the gradient of a linear graph is 0? The *y*-value does not change when increases and so the graph is horizontal.

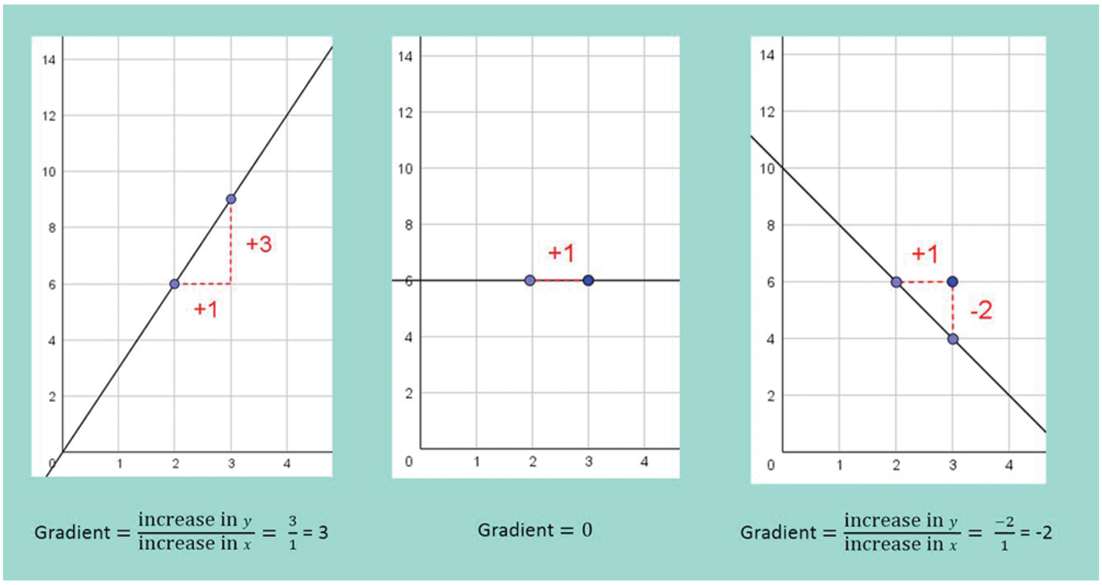


Figure 11 Gradients of linear graphs

Activity 8 Function detective

Allow 5 minutes

Recall the variables from Activity 7:

- age / number of hours of sleep
- number of people / how many chocolates each person gets
- sharing a box of 24 chocolates
- your age / your elder sibling's age.

Match each pair with one of the graphs in Figure 12 below.

Is each graph increasing, decreasing or neither?

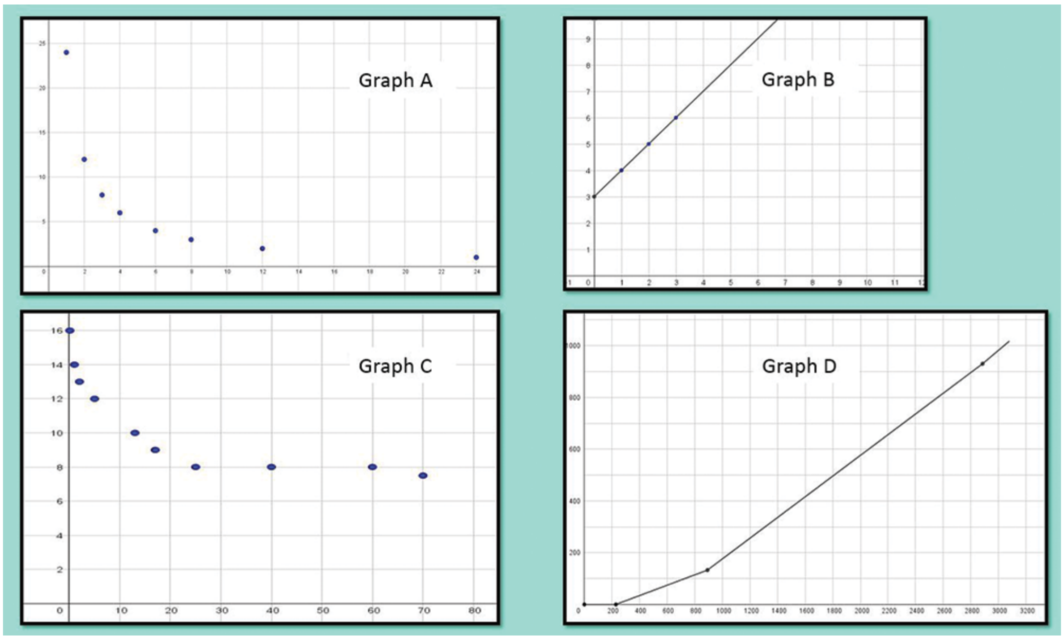


Figure 12 Four graphs

Discussion

Graph A

You should have found that graph A showed the relationship between the number of people sharing a box of 24 chocolates and how many chocolates each person gets. One way of checking this is to look for the points (1,24), (2, 12), (3, 8) that all lie on the graph. The graph is decreasing. Notice also that this graph is shown as a set of discrete points since the number of people and the number of chocolates are both discrete variables.

You could have chosen either variable for the x-axis (and the other for the y-axis) because this graph is the graph of a **reciprocal function**. The rule is that:

the number of people sharing a box of 24 chocolates x how many chocolates each person gets = 24

In symbols we could write that $xy = 24$

Or $y = 24/x$

Any function where y is a constant divided by x is a reciprocal function. A reciprocal function does not have a constant gradient since the slope changes along the graph.

Graph B

Graph B shows your age (on the x-axis) and (on the y-axis) the age of an (imaginary) sibling who was 3 years old when you are born. So it starts at (0,3). This point is called the y-intercept: it is the point where x takes the value zero. The graph is increasing. The gradient of the graph is 1 since your sibling's age increases by 1 for every year you gain. The graph shows a **linear function** since it is a straight line with a constant gradient, but it is not a **proportional relationship** (see Week 3) because it does not start at (0,0). It is drawn as a line graph since age can be considered as a continuous variable.

Graph C

This graph is neither increasing or decreasing since there are intervals (between ages 20 and 60) in which the x-value increases but the y-value stays the same. It represents age on the x-axis and recommended hours of sleep on the y-axis. It only shows some data points, not joined by a line, so the variables are being treated as discrete variables. However, in the context we are clearly meant to use the data at the given ages to infer (make predictions about) the sleep needs at other ages.

It is unlikely that there is a simple mathematical function that will describe this graph.

Graph D

This graph is not one of those shown. It is an example of a **piece-wise linear function**, since the different sections are each straight lines. It shows weekly salary on the x-axis and how much income tax you pay on the y-axis (using the 2018-19 tax bands). The curriculum in England suggests that learners should be able to read information from graphs in financial contexts. Research suggests that it is necessary to discuss the context with learners before asking them to interpret graphs. Most 11-14 year olds would not be familiar with the idea that employers take tax out of your weekly pay.

So far we have met

- the linear graph which increases regularly so we can calculate a gradient. It has equation $y = mx + c$ where m is the gradient and c is the (starting) value of y when x is zero. These graphs typically occur for costs, for travel or for conversion graphs are linear.
- the piece-wise linear graph which has straight line sections. It does not have a simple equation. These graphs occur when there are different types of cost or travel combined on one graph.
- the reciprocal graph with equation $y = k/x$ .This graph typically occurs when sharing a constant amount k.

There is one final graph that we will meet in the next section.

3.3 The exponential function

In the next two activities you will meet one last important type of function and graph: the exponential function.

Activity 9 Word of mouth

Allow 10 minutes

Figure 13 Word of mouth

Three friends share a secret. On day 2 they each tell one new person.

Then every day for a week, everyone who knows the secret tells it to one new person.

How many people know on day 2?

How many people know on day 3?

Estimate how many people will know the 'secret' on day 8.

Discussion

On day 2, 6 people know the secret.

On day 3, 12 people know.

This is a sequence you met before: 3, 6, 12, 24, 48. The term-to-term rule is 'double each term to get the next'. The position-to-term rule is $3 \times 2^{n-1}$.

By day 8, a large number of people will know. You can use the formula or a graph to find the value (Figure 14).

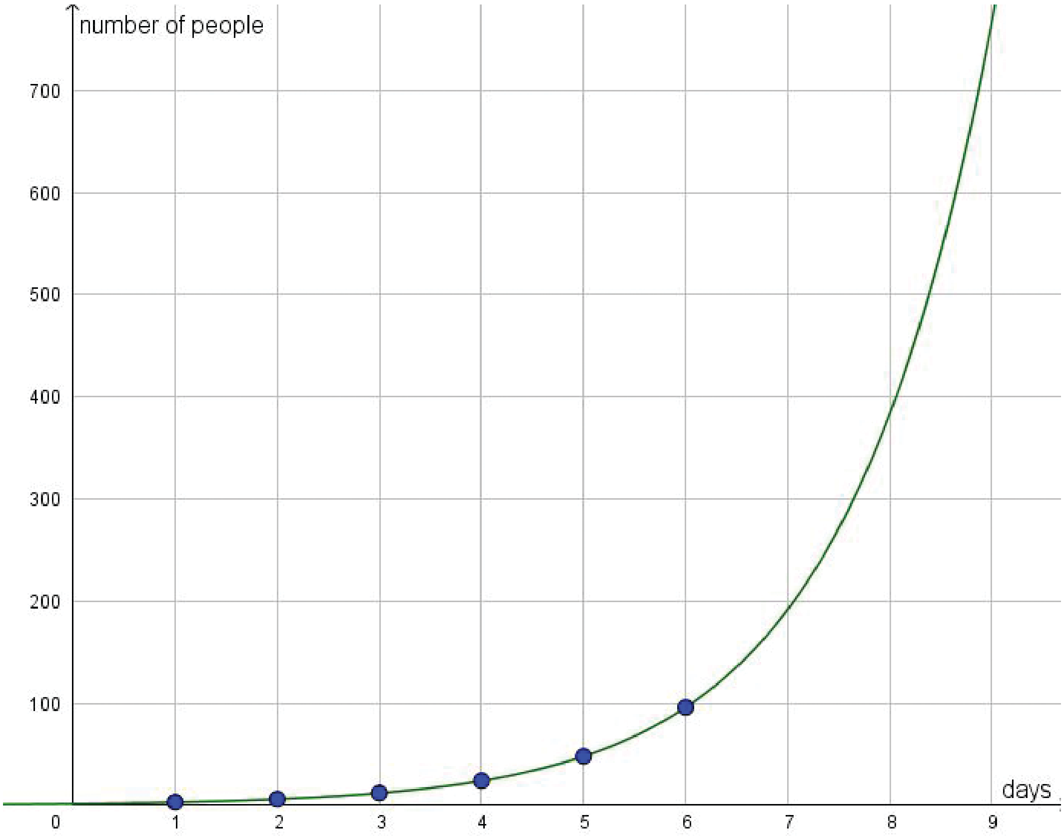


Figure 14 Sharing secrets

In this problem, the number of new people who learn the secret each day is equal to the number who know it already. This makes it an example of an **exponential function**. The characteristic feature of these functions is that their rate of change is proportional to their value. In more everyday words, this means that the higher the *y*-value, the more it grows.

It is worth comparing this with a linear function, which grows at a constant rate.

This feature lies behind the English phrase ‘growing exponentially’ which is used to emphasise rapid growth. It also makes exponential functions difficult to graph. The scale on the *y*-axis has to be large to include the points for higher values of *x*, but then the *y*-values for low values of *x* are not easy to read.

As well as exponential growth you can have exponential decay, which gives a decreasing exponential function.

Activity 10 Exploring exponentials

Allow 20 minutes

For each of these exponential functions, complete the table of values, choose a scale and draw a graph.

Which part of the function affects whether it grows or decays?

Function 1: $y = 100 \times (1.02)^x$

Function 2: $y = 100 \times (0.5)^x$

Function 3: $y = -10 \times (0.5)^x$

x	0	1	2	3	5	10
y						

Figure 15 Exponential functions

Exponential functions are used in many real-life situations.

- In biology, they describe the spread of diseases and the growth of populations.
- In engineering, they describe waves, and how movement dies away.
- In finance, they describe how money grows when it is invested at a constant rate of interest.

Although exponential functions appear on the school curriculum for the middle years, learners are expected to plot and read their graphs and not to work with their equations. This has been included only for your own knowledge asa teacher.

The role of teachers is to support learners in gaining a good mathematical understanding of graphical representation in a range of contexts. Learners need to be able to choose scales and plot *x*-values and *y*-values on a graph. They need to find the *y*-value for a given *x*-value and vice versa. They also need to interpret change on a graph. They need to know whether the graph is increasing or decreasing, and whether changes are constant or can vary. The linear, piece-wise linear, reciprocal and exponential graphs are all functions that illustrate different kinds of change and this is their value for teachers.

4 This week’s quiz

Check what you’ve learned this week by doing the end-of-week quiz.

Week 5 practice quiz

Open the quiz in a new tab or window and come back here when you’ve finished

5 Summary of Week 5

In Week 5 you have looked at functional thinking and how this is introduced using the idea of the function machine which works by operating on an input to produce an output. This led to a consideration of number sequences where the input is the position in the sequence and the output is the number (or term) at that position. Finally the idea of covariance was introduced where there are two quantities which vary together according to a relationship. The two quantities can be plotted on a graph and the shape of the graph is determined by the function itself. Although the concept of the function is a sophisticated one, the approaches you met in week five show that it is possible to carefully introduce it to learners in the middle years.

You should now be able to:

- introduce learners to the concept of functions using concrete and pictorial representations (such as the function machine) and to develop this towards more abstract representations of functions
- use other representations, with which the learners are familiar, to build on their understanding of functions
- explain to learners how two unknowns can vary together according to the rule (or function) connecting them.

Next week you will learn about the teaching and learning of geometry, including how shapes are defined according to their properties and problem solving in geometry.

You can now go to Week 6.

Week 6: Teaching and learning about geometry

1 How children think and reason about geometric shapes

The study of geometry has its foundation in the learner's experience of the environment in which they live. More than any other area of mathematics we learn about shape and space in an intuitive way through our experience. By the time children go to school, they have already learned much about shapes and have developed spatial perception in a natural way by interacting with their environment.

It would seem appropriate that any approach to learning about the geometry of shape must build on the intuitive understanding which is gleaned from our lived experience. Nevertheless, even though children come to school with an impressive understanding of shape and space pertaining to their environment, most encounter difficulties when learning about geometry.

1.1 Progression in geometric thinking

A Dutch educator called Pierre van Hiele showed how children's understanding of shape develops through increasing levels of sophistication.

At first, children learn to recognise shapes in a ***holistic*** way and to name them. This usually happens in early primary school. As they progress, children learn to ***describe*** shapes by their properties and to recognise a shape from a list of properties. These children tend to be older primary to younger secondary age, and some may never progress beyond this level.

In the third level, children or learners are able to reason more *abstractly*, to work out how one property of the shape generates other properties, and to construct a concise definition for a shape.

Later levels describe much more advanced geometrical reasoning which is appropriate for higher school and university study.

Thinking holistically

In their early school years, learners have typically been presented with shapes and they have learned their names (Figure 1). However, they will also have learned some implicit things about these shapes. One of these is orientation or 'right way up-ness'.

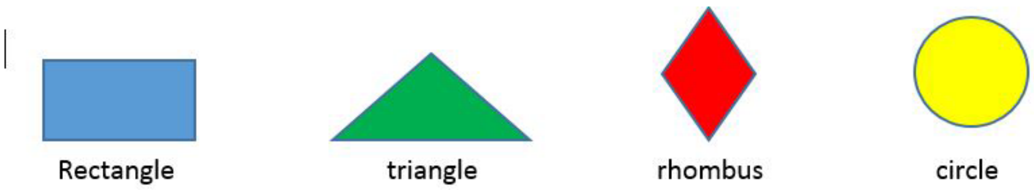


Figure 1 The common presentation of four basic shapes

It is common for children in early years schooling to be presented with shapes which are upright, in the case of squares and rectangles, and for triangles to sit on a horizontal base. So, quite often, learners decide that the orientation of the shapes is an important property and do not accept shapes in orientations which are unusual. As an example of this, consider the two squares in Figure 2 below, which are presented in different orientations.

Figure 2 A square in two different orientations

It is common for the first of these two shapes to be perceived as a square, whereas the second shape is perceived as a 'diamond' because it sits on its corner.

One way of dealing with this is to suggest that the learner turns the paper round, so that they can see the shape 'the right way up'. This is more difficult if the shape cannot be turned round, but the suggestion can be to turn the head so as to look at the shape 'the right way up'.

Moving on to description

Learners develop an intuitive feel for the look of a shape: the squareness of a square, the circular nature of a circle, and so on. If learners only see and experience 'typical' examples of each kind of shape then they may not accept shapes which are in unusual proportions or orientations. For example, the shapes in Figure 3 are all kites but only the first one is typical of the orientation and proportion shown in textbooks and on classroom posters.

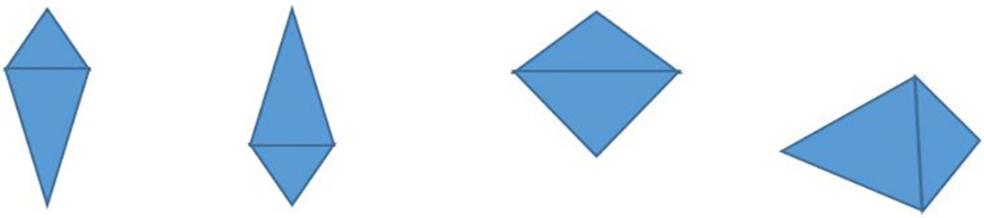


Figure 3 Four kites in different proportions and orientations

You can be sure that these shapes are indeed kites if the properties of a kite hold true for these shapes.

Activity 1 Properties of a kite

Allow 5 minutes

Write down as many properties of a kite as you can think of. Remember to use angle sizes, side lengths and symmetry.

Discussion

Here are some properties of the kite:

- Four sides, i.e. the kite is a quadrilateral.
- Two pairs of adjacent equal length sides (adjacent means the sides are next to each other).
- One pair of opposite equal angles.
- One line of symmetry.
- No rotational symmetry or rotational symmetry of order 1 (i.e. there is only one position where you can fit the kite in a 360!Warning! MS Gothic not supported° turn).

Once learners are able to describe shapes by listing their properties they will accept unusually orientated or proportioned shapes such as the kites in Figure 3. They can progress to solving simple geometrical problems using the properties of the shapes that they know.

Activity 2 Is this a kite?

Allow 10 minutes

Have a go: A shape ABCD has been drawn on the square grid below. Is this shape a kite? Explain your reasoning.

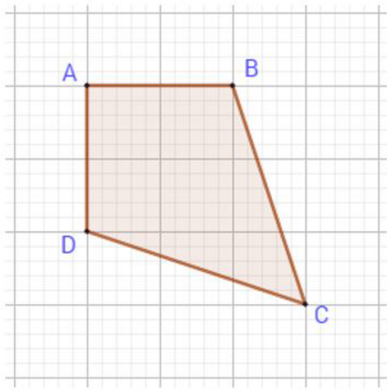


Figure 4 Is this shape a kite?

Discussion

ABCD is a kite.

Adjacent sides AB and AD are equal in length (2 units).

Adjacent sides BC and BD are equal in length (they are both the diagonals on 3 by 1 rectangles).

AC is the line of symmetry. (Points A and C are on the line. Points B and D are both one diagonal of a square away from line AC.)

Activity 3 Beginning to reason abstractly

Allow 10 minutes

In Activity 1 you listed the properties of a kite. Now work out a definition of the kite which uses only two or three of its properties. These properties are known as the *necessary and sufficient* conditions. There is more than one way to write a definition of the kite.

Discussion

Here are some definitions of the kite using necessary and sufficient conditions.

- A quadrilateral with two pairs of adjacent equal sides.
- A quadrilateral with a line of symmetry and a pair of opposite equal angles.

Did you think of any others?

1.2 Defining shapes using flexible definitions

In this section you will consider how more flexible definitions of shapes can be used to include other kinds of shapes.

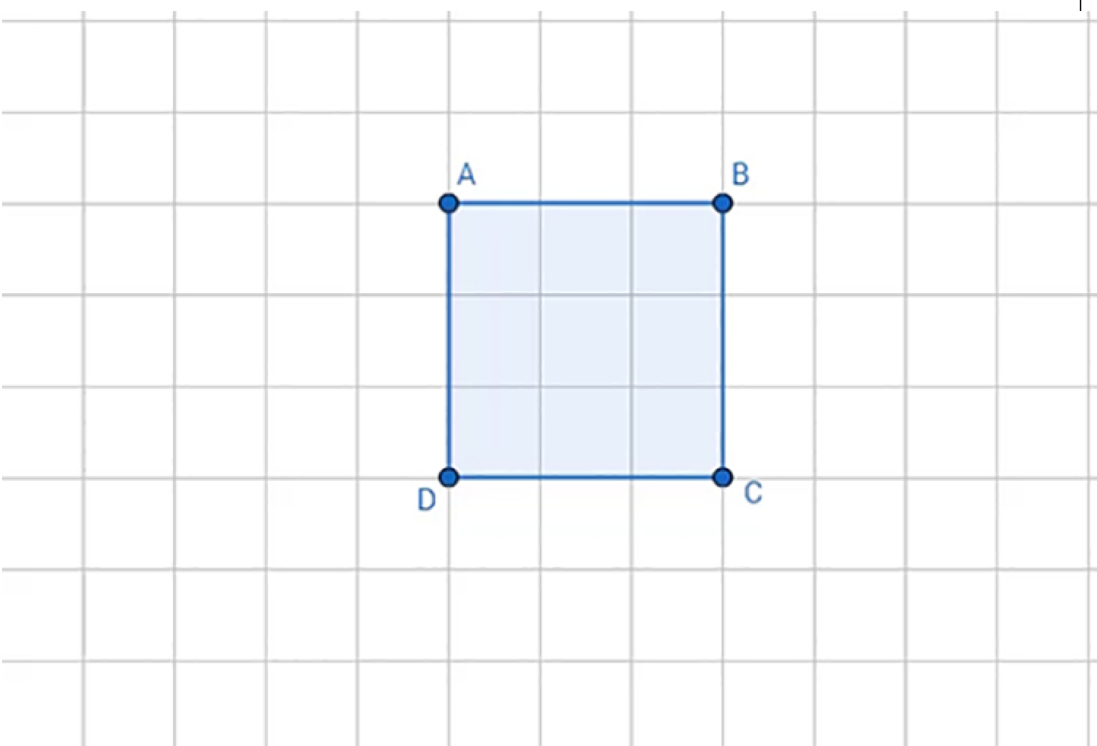
Being able to write a definition of a shape is an example of analytical thinking. As learners' reasoning develops, they are able to define shapes in different ways and to develop definitions which are more inclusive. As an example of this, consider the rectangle and the square (Figure 5).

Figure 5 A rectangle and a square

What happens if you were to take one side of the square and pull it? The following video shows this happening.

Video content is not available in this format.

A square being 'pulled' into a rectangle



You have just watched a square being pulled into a rectangle. Might a square be thought of as a special kind of rectangle? (In mathematics we call this 'a special case'.)

Activity 4 Writing definitions

Allow 5 minutes

Write down a definition of the rectangle.

Write down a definition of the square.

Activity 5 More definitions

Allow 10 minutes

Below are two definitions of a rectangle, with a subtle difference between them, and one definition of a square.

Is it possible to include the definition of a square into either of the two definitions of a rectangle?

Definition 1 of a rectangle

A rectangle has four sides and four right angles. It has two longer sides of the same length, which are opposite each other, and two shorter sides of the same length which are opposite each other.

Definition 2 of a rectangle

A rectangle has four sides and four right angles. It has two pairs of equal sides which are opposite each other.

Definition of a square

A square has four sides which are the same length and four right angles.

Discussion

The definition of a square cannot be included in definition 1 because its definition of a rectangle stipulates that there are two sizes of equal sides: the longer sides and the shorter sides.

The definition of a square can be included in definition 2 because its definition of a rectangle says that there are two pairs of equal and opposite sides. This also includes the case when these two pairs of equal sides are also equal to each other, in which case the rectangle is also a square.

The understanding that some classes of shapes can be included in other classes of shapes is called the concept of inclusion. So, for example, squares can be seen to be special rectangles. However, in order to accept the concept that squares are special rectangles, a more inclusive definition of the rectangle, such as definition 2, needs to be used.

Activity 6 Including some shapes as special cases of other shapes

Allow 5 minutes

Try to identify other inclusive relationships. For example, is there a definition of the kite which would include the rhombus? Can rectangles be thought of as special cases of the parallelogram?

1.3 The visual or physical representation of shapes

If you consider a shape, such as a square or a circle, you have a mind's-eye picture of the shape which includes the 'squareness', or the 'circleness', and this shape may be filled in (with colour) in your mind, and it may have a particular orientation. You may draw a diagram of this shape with a pencil and the outside edges are likely to have some width (or you could not see them). If you have a square or circular tile then the tile will have depth (or you could not pick it up).

The diagram and the tile are representations of the shape and as such are examples of what the shape can look like. In contrast, the mathematical shape itself is entirely abstract and is described using its geometric properties. Orientation and thick pencil lines are not included in those properties. However, teachers and learners need to work with representations of shapes as if they are mathematical objects because it is all they have to work with. It is worth being aware of the differences because learners' attention may be distracted by particular features of the physical representations of the shapes.

2 Investigating the properties of triangles and quadrilaterals

In the previous section you read about the properties of some shapes and how combinations of those properties form the definitions of those shapes. Triangles and quadrilaterals are important because they form the basis of other shapes and are generally included in the school curriculum for the middle years.

2.1 Triangles

Triangles are the most basic of the polygons (a polygon is a straight-sided shape). They have three sides, the smallest number of sides it is possible to have. Triangles also make a rigid structure, as any engineer will know.

You will know the names of the triangles which have further properties in addition to having three sides (Figure 6).

Right angled triangle		One right angle
Isosceles triangle		Two equal sides and two equal angles
Equilateral triangle		Three equal sides and three equal angles

Figure 6 Types of triangle

In addition, the isosceles right-angled triangle has a right angle, two equal sides, and two equal angles of 45!Warning! MS Gothic not supported° each.

Acute-angled triangles have three angles all smaller than a right angle.

Obtuse-angled triangles have one angle that is bigger than a right angle.

2.2 Quadrilaterals

In this section you will explore how certain quadrilaterals can be made from triangles.

Activity 7 Making shapes out of two congruent right-angled triangles

Allow 10 minutes

Draw and cut out two congruent right-angled triangles. (Congruent means they are exactly the same size and shape, same angles, etc.)

Can you make the following shapes by placing the congruent right-angled triangles together along two of their edges (i.e. no overlapping)?

- 2 different isosceles triangles
- 2 different parallelograms
- 1 rectangle
- 1 pentagon
- 1 hexagon

Discussion

Answers

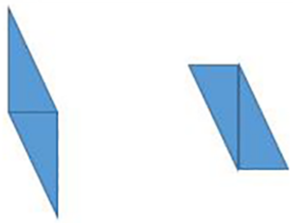
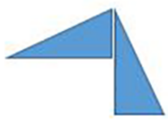
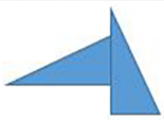
Isosceles triangles One of the right angled triangles needs to be flipped over so that each triangle is a mirror image of each other.	
Parallelograms One of the right angled triangles needs to be turned 180 degrees.	
Rectangle One of the right angled triangles needs to be turned 180 degrees	
Pentagon One right angled triangle has been given a quarter turn and the two triangles are placed together so that 2 corners coincide.	
Hexagon The right angled triangles are orientated in the same way as for the pentagon but the left hand triangle has moved down.	

Figure 7 Shapes made placing two right-angled triangles together

Activity 8 Reflecting a triangle in one of its sides

Allow 10 minutes

Draw a triangle (not a special triangle) and then draw a reflection of that triangle in one of its sides. Keep both the original triangle and the reflected triangle which should now make a quadrilateral. What kind of quadrilateral do you get?

Then do the next activity and read the discussion afterwards.

Activity 9 Reflected triangle in Geogebra

Allow 10 minutes

Click on this link !Warning! inherit not supportedhttps://www.geogebra.org/m/U6azsQb6 which takes you to a Geogebra file that contains one triangle which has been reflected in one of its sides. 'Drag' the resulting quadrilateral to investigate which shapes you can make. (If you have difficulty in opening the link, go to !Warning! inherit not supportedwww.geogebra.org and search for 'triangle reflected in one of its sides'.)

Discussion

You can drag the figure at the link above if you have clicked on the arrow tool (which is the default tool when you open a Geogebra file) and then you drag the figure on the screen. Were you able to make any of the following?

- a kite
- a rhombus
- a square
- a isosceles
- an triangle

Activity 10 Splitting quadrilaterals

Allow 5 minutes

Which quadrilaterals can be split into two congruent triangles? Complete the table in Figure 8. The third column refers to whether the triangles are reflected or rotated in the quadrilateral.

Quadrilateral	***Congruent triangles it can be split into.***	***Using reflection or rotation?***
Rectangle		
Square		
Kite		
Rhombus		
Parallelogram		
trapezium		

Figure 8 Splitting quadrilaterals into two congruent triangles

Discussion

The table can be completed as shown in Figure 9.

Quadrilateral	***Congruent triangles it can be split into.***	***Using reflection or rotation?***
Rectangle	Right angled triangles	Rotation symmetry
Square	Isosceles right angled triangles	Rotation symmetry
Kite	Scalene triangles	Reflection symmetry
Rhombus	Isosceles triangles	Rotation symmetry Reflection symmetry
Parallelogram	Scalene triangles	Rotation symmetry
trapezium	Cannot be split into two congruent triangles	

Figure 9 Solutions to splitting quadrilaterals

These tasks are a novel way to approach learning about triangles and quadrilaterals. They encourage learners to think about the shapes and their properties. Many textbook exercises consist of questions asking learners to recall shape properties, without requiring learners to think deeply about them.

The special quadrilaterals – those we have names for – are special because they have interesting properties such as symmetry, which relates to the triangles which make them up.

3 Developing problem-solving skills in geometry

So far this week, you have read about children's geometrical thinking as it develops through the early to the middle school years. This has been linked to the description of the properties of shapes. You have looked at definitions of shapes and how these can be written in an inclusive way to include some shapes as special cases of others. You have considered triangles and how these can generate special quadrilaterals by reflecting a triangle in one of its sides or rotating the triangle using a half-turn around the midpoint of one of its sides.

In this section you will think about developing learners' problem-solving skills in geometry. Many problems in geometry are entirely abstract from the point of view that there may be no practical real-world application. However, geometry can be approached in a practical way and this makes it accessible and engaging for learners. Geometry does also have real-world applications in nature, in design and architecture for example.

Three problems will be posed that you will be asked to have a go at answering. After each one you will consider how to best pose these problems for learners, including the resources needed and how to manage these in the classroom context.

3.1 Polyominoes

The activities in this section can also be used to consider reflection and rotation symmetry.

You have probably seen dominoes, which are made of two squares joined together along their complete edges (Figure 10). There is only one domino because turning it round into a different orientation does not count as making another domino. A domino has two lines of symmetry (horizontal and vertical) and rotational symmetry of order 2 (2 positions where the domino can fit within a 360º turn).

Figure 10 A domino

As you might imagine, triominoes are made of three squares placed along their complete edges. 'Tri' at the beginning of a word denotes three (like 'triangle' for a three-sided shape and 'tricycle' for a three-wheeler cycle).

There are two different triominoes (Figure 11). Remember that turning a triomino round or flipping it over does not create a different triomino.

Figure 11 The two triominoes

Activity 11 The symmetry of triominoes

Allow 5 minutes

Describe the symmetry of each of the two triominoes.

- Does each triomino have reflection symmetry and, if so, where is the line of reflection?
- Does each triomino have rotation symmetry?

Discussion

The I-shape triomino has two lines of symmetry and rotation symmetry of order 2.

The L-shape triomino has one line of symmetry and rotation symmetry of order 1 (i.e. no rotation symmetry).

Activity 12 The tetrominoes

Allow 5 minutes

Tetrominoes are made by joining four squares together along their complete edges. Try to find all of the possible tetrominoes.

How can you be sure that you have found them all?

Describe their symmetries.

Discussion

Answer

It can be useful to employ a system when trying to find all the tetrominoes.

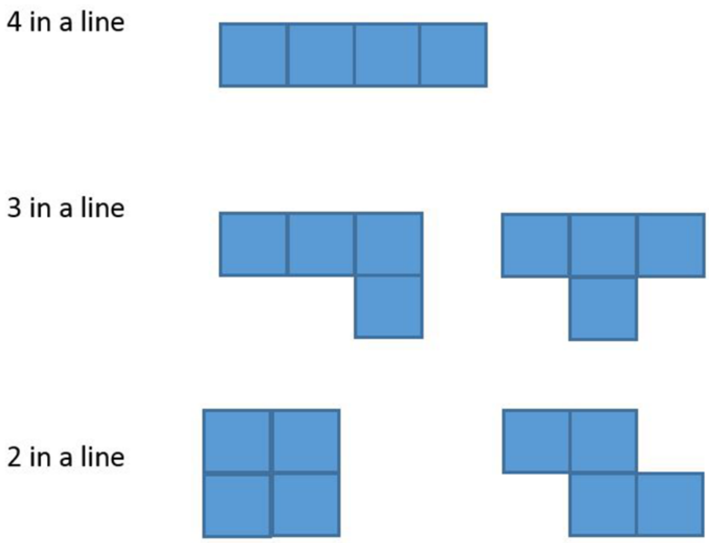


Figure 12 The tetrominoes

Activity 13 The pentominoes

Allow 10 minutes

Now try to find all 12 pentominoes (5 squares joined along their complete lengths).

Several challenges can be posed using the 12 pentominoes. It is useful to be able to give them labels, especially since many of them look like letters of the alphabet (with a little imagination). For example, one set of labels uses the letters F, I, L, N, P, T, U, V, W, X, Y and Z. See if you can match these letters to the pentominoes you made.

One challenge is to use all 12 pentominoes to make a 10 by 6 rectangle. 5 × 12, 4 × 15 and 3 × 20 rectangles are also possible. Other challenges are to make animals – a penguin, camel, dog and kangaroo are all possible.

Individual pentominoes can be used to make tessellations (or tiling patterns). For example, the P-shape and the F-shape can be tiled.

Pentominoes as an activity with learners

I remember the first time I met this activity. I was working as a Special Needs teacher in the mathematics department of a large college and was supporting the class teacher of a low attaining group of 15-year olds. She had found the activity in a publication and had printed it out. The activity engaged the learners and they were all able to make a start at it. No one individual found all of the twelve pentominoes but between the whole class all twelve were found and displayed on the board.

Activity 14 Preparing a lesson using pentominoes

Allow 20 minutes

Imagine that you are going to use the pentominoes task with the group of 15-year-olds. List the equipment you will need for the lesson.

Discussion

Think about how you will introduce the task to the learners. Can you expect a whiteboard in the room, or an interactive whiteboard? Will you draw on the board? Will you make tiles which can be moved around?

Think about the learners investigating the task for themselves and the equipment they are likely to need. Square paper is useful. It may be that learners are given square tiles to use to create the pentominoes which can then be drawn on the square paper.

Now read this:

In the lesson with the 15 year old low attaining group the class teacher began by drawing a domino and talking about it as two squares put together along their edges. She established with the class that it had to be whole lengths of the edges placed together so that

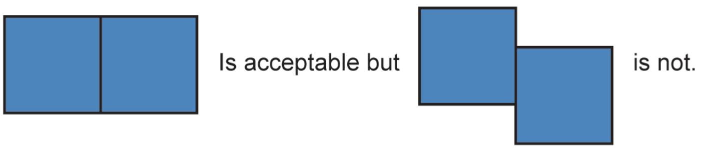


Figure 13 A lesson about polyominoes

She told the class that triominoes were made from three squares placed together along their edges, gave them some thinking time and then got learners at the board to draw two possible different triominoes.

The learners were next asked to find as many tetrominoes as they could and to draw them on squared paper. Some time was given for this and then the teacher asked for volunteers to draw the results on the board. She drew attention to the systematic way of finding all of the five tetrominoes and how this helps us to be sure that we have found all possible. It also gave her the opportunity to talk about *congruency*, which is where we have two tetrominoes of the same shape, if only we could cut them out and sit one on top of the other. A tetromino which is congruent to one of the others is not allowed as they all need to be different.

In the last part of the lesson the learners were set to task finding the twelve pentominoes, which they drew on squared paper.

The lesson was engaging for the pupils because it was simple enough for them all to make a start. They could find more of the polyominoes by drawing, using visual intuition. the task encouraged them to be logical and systematic as a way of identifying all possible shapes. The class experienced individual working but also collaborative working when they came together frequently as a group to discuss whether they had found all shapes.

3.2 Ideas for tasks using four cubes

So far, only 2D shapes have been considered but now you will think about 3D shapes or models.

Most mathematics departments have sets of interlocking cubes, or multi-link cubes as they are sometimes known (Figure 14). The bigger ones which are 2 cm along their edges are best to use as they are easy to handle.

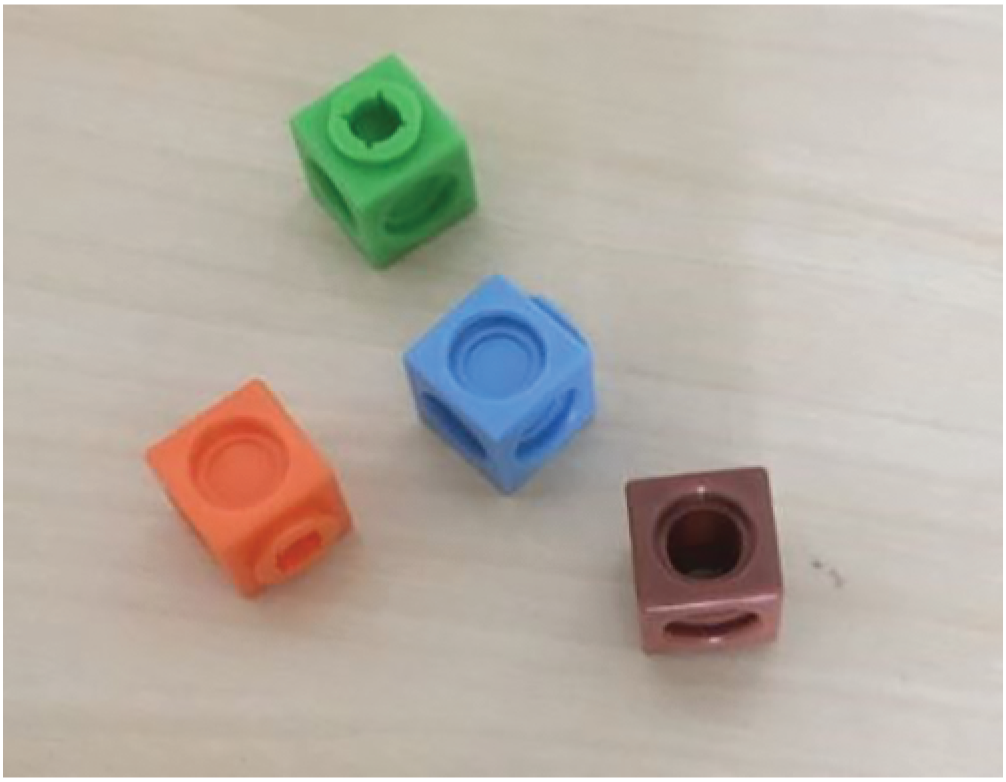


Figure 14 Four multi-link cubes

There are several tasks which you can ask learners to do with four interlocking cubes.

1. Make a four-cube model and hide it behind a book or other appropriate hiding place. Now try to describe it to another person so that they could make the same model using four cubes.
2. Find all of the different models which can be made using four interlocking cubes. How can you be certain that you have them all?
3. Which four-cube model has the largest or smallest surface area?
4. Draw the models on isometric paper.
5. Draw the plan and elevations of each model.
6. Choose one of the models, devise its net and make it up using card.

Activity 15 A task using four cubes

Allow 25 minutes

Choose one or more of the tasks above to try for yourself (having first procured four cubes). Then read the following suggestions and points regarding each suggested activity.

Make a four-cube model and describe it to another person so that they could make the same model using four cubes

Learners may find it helpful to use terms such as:

base
next to
above
in a row

Find all of the different models which can be made using four interlocking cubes

As with the pentomino activity, it helps to be systematic. In addition, shapes which are congruent to other shapes are not considered to be different. Think about cutting a 3D-shaped hole to fit the four-cube model. If another four-cube model can be fitted into the same hole then it is congruent and not different.

Figure 15 Three four-cube models

Which model has the largest or smallest surface area?

All of these four-cube models have the same volume because they have the same number of cubes. If the edge is considered to be one unit then the area of the face of a cube is one square unit and the volume of one cube is one cube unit. So, the four-cubed models each have a volume of four-cube units. However, their surface areas, which can be counted and recorded in square units, will be different.

Drawing the models on isometric paper

There is an art to drawing representations of 3D images on isometric paper. Learners may need some advice. Isometric paper is based on equilateral triangles and the orientation of the paper is important. Make sure that the paper looks like the example in Figure 16, with vertical lines of dots. If not, turn it 90º.

The first step is to draw one cube:

- Draw a rhombus shape, which will represent a square face in perspective.
- Next put three parallel lines from each of the three corners to provide the depth of the cube as shown.
- Finally complete the cube as shown.

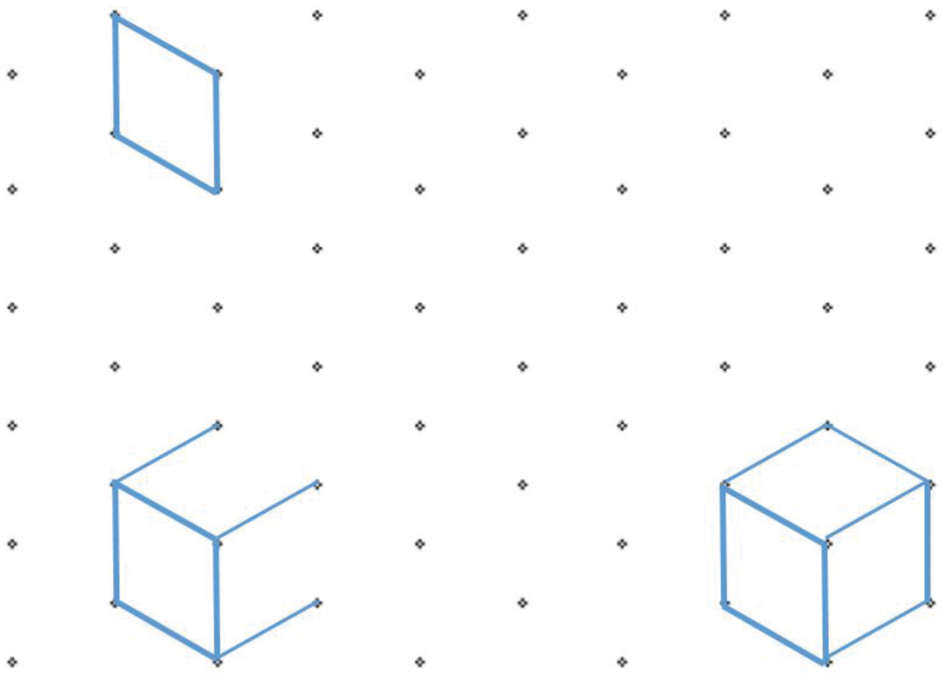


Figure 16 Drawing a cube on isometric paper

Once the learner has mastered the skill of drawing a cube on isometric paper they will be able to draw the four-cube models by extending the original cube to represent cubes added on.

Drawing the plan and elevations of each model

This work is related to the work of architects on plans and elevations for buildings. The plan is the footprint of the model, which sits on the table. The elevations are the front and the side views imagined as if they were flat.

The side and elevations of the (four-cube model) are shown in Figures 17 and 18. Squared paper is useful for this activity.

Extensions to this work could be to provide the plan and elevations for a cube model. Learners then have to make the model and sketch it in 3D on isometric paper.

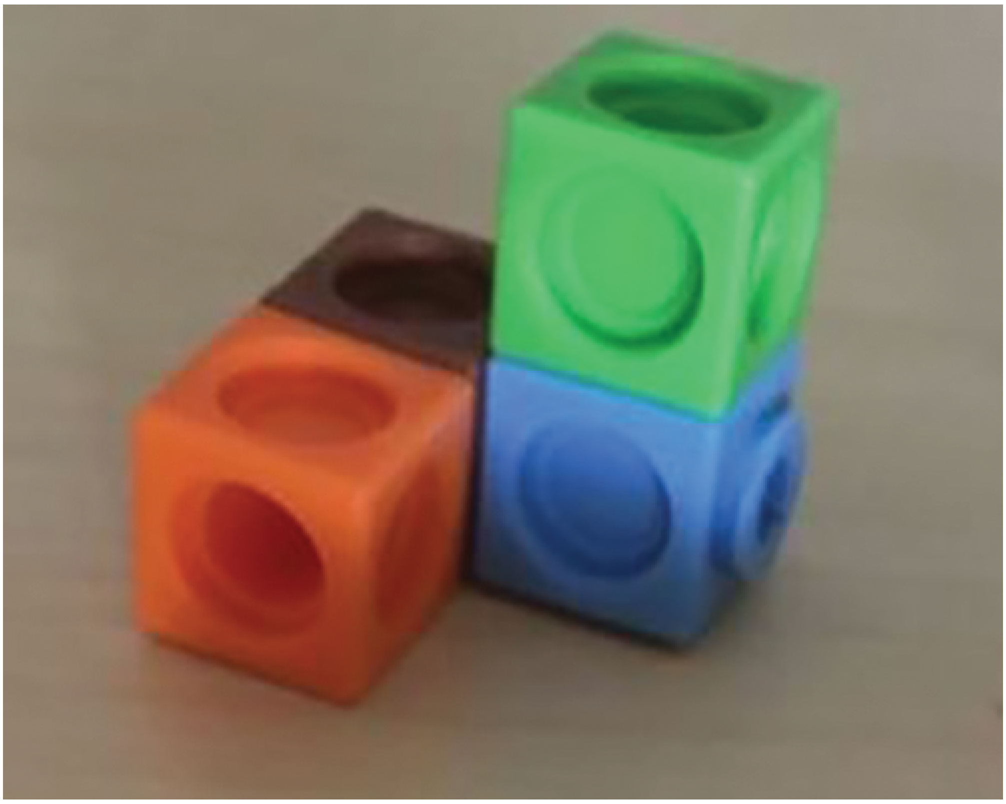


Figure 17 The four-cube model used to draw the plan and elevations

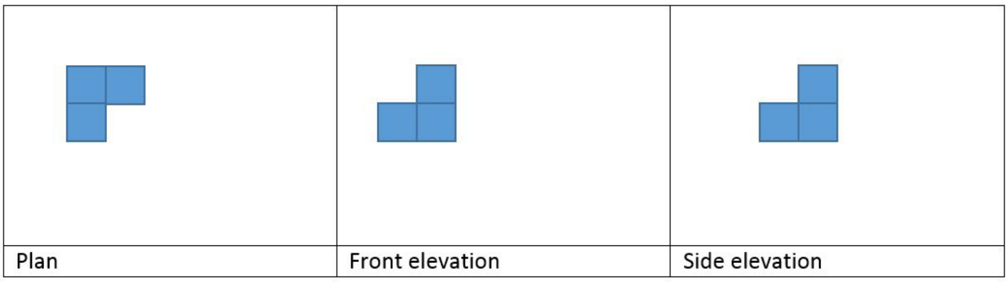


Figure 18 Plan and elevations for the four-cube model

Devise a net for a four-cube model and make it up using card

This task can be easy or hard, depending on the choice of four-cube models. It is not as difficult as it might first seem if the model is placed on to paper and the learner considers the shape of the base, what needs to be folded up to make the sides and then what needs to be attached to the sides to fold over to make the top. Trial and improvement is a useful strategy since if the net does not work it can be modified and tried again until it does work.

3.3 Enlarging shapes using repeating tiles

The activities in this section can be linked to enlargement as a transformation.

Some shapes can be enlarged when several smaller versions of themselves are laid together (Figure 19). Squares, rectangles and parallelograms all come into this category. Note how many of these shapes fit together to make an enlargement which is two or three or four times as big.

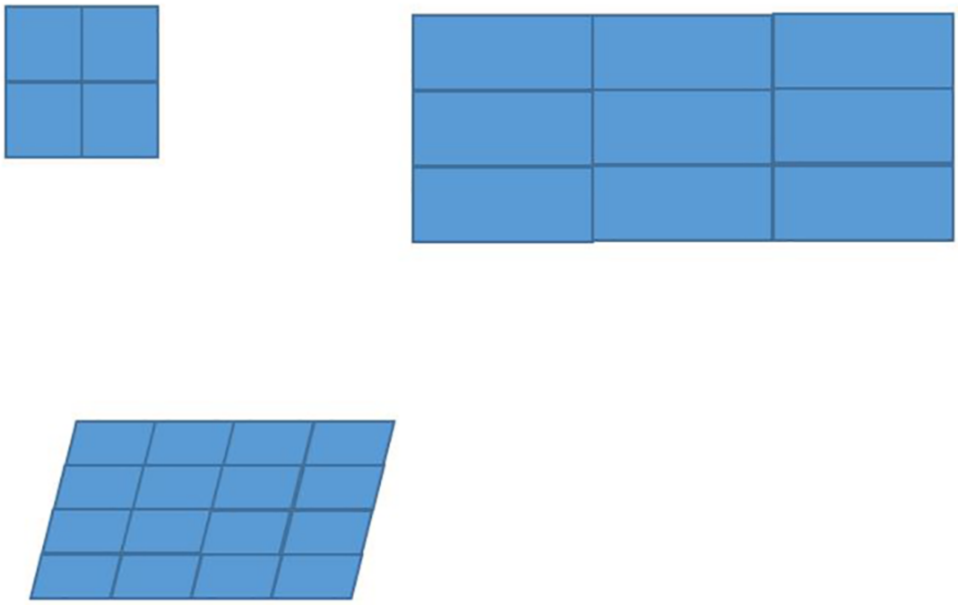


Figure 19 Repeating tiles

Triangles are also repeating tiles. Try this with different versions of a triangle. You can either make a triangle tile from card or using ICT. You may have to rotate some of the triangle tiles.

Activity 16 Using repeat tiles to make a version which is twice the size of the original

Allow 10 minutes

Explore other shapes which can be used as repeating tiles. For example, show how the shape below can be generated from four smaller versions of itself.

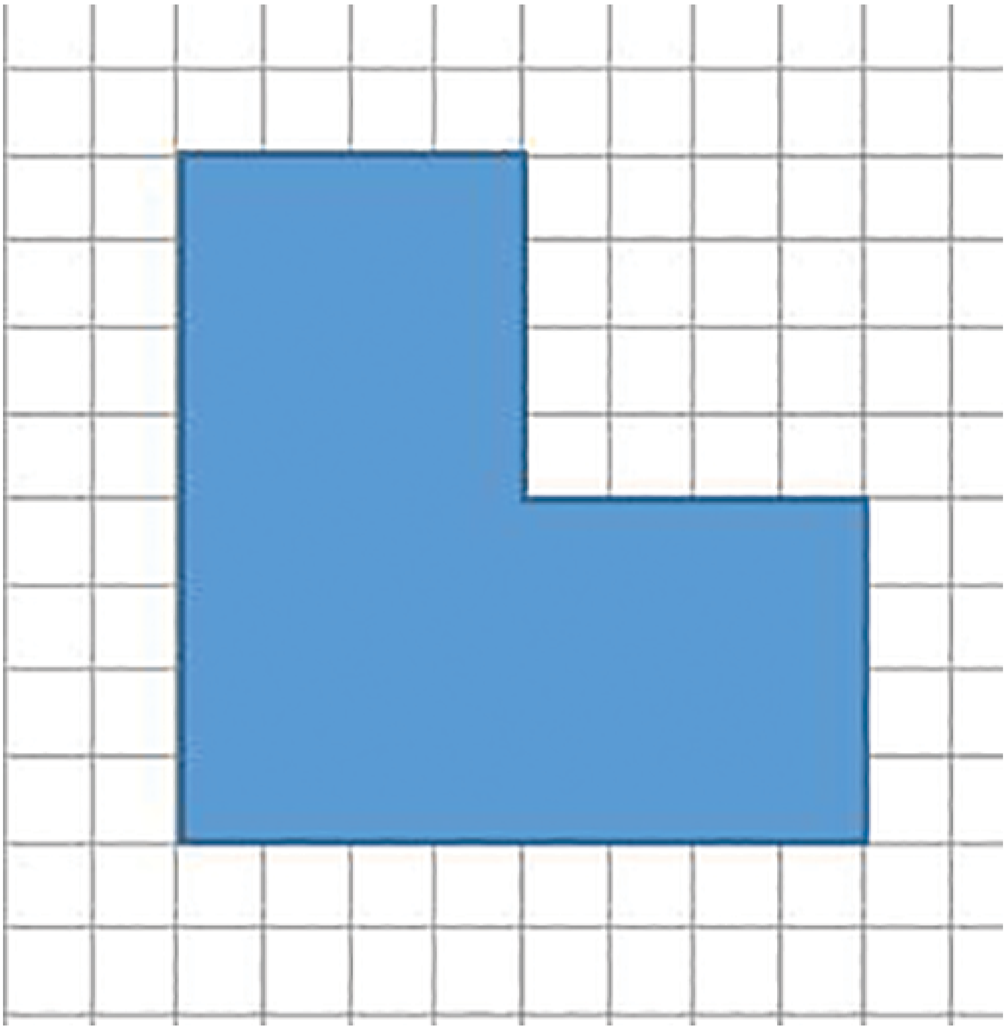


Figure 20 An L-shaped tile which can be made from four smaller L-shaped tiles

Discussion

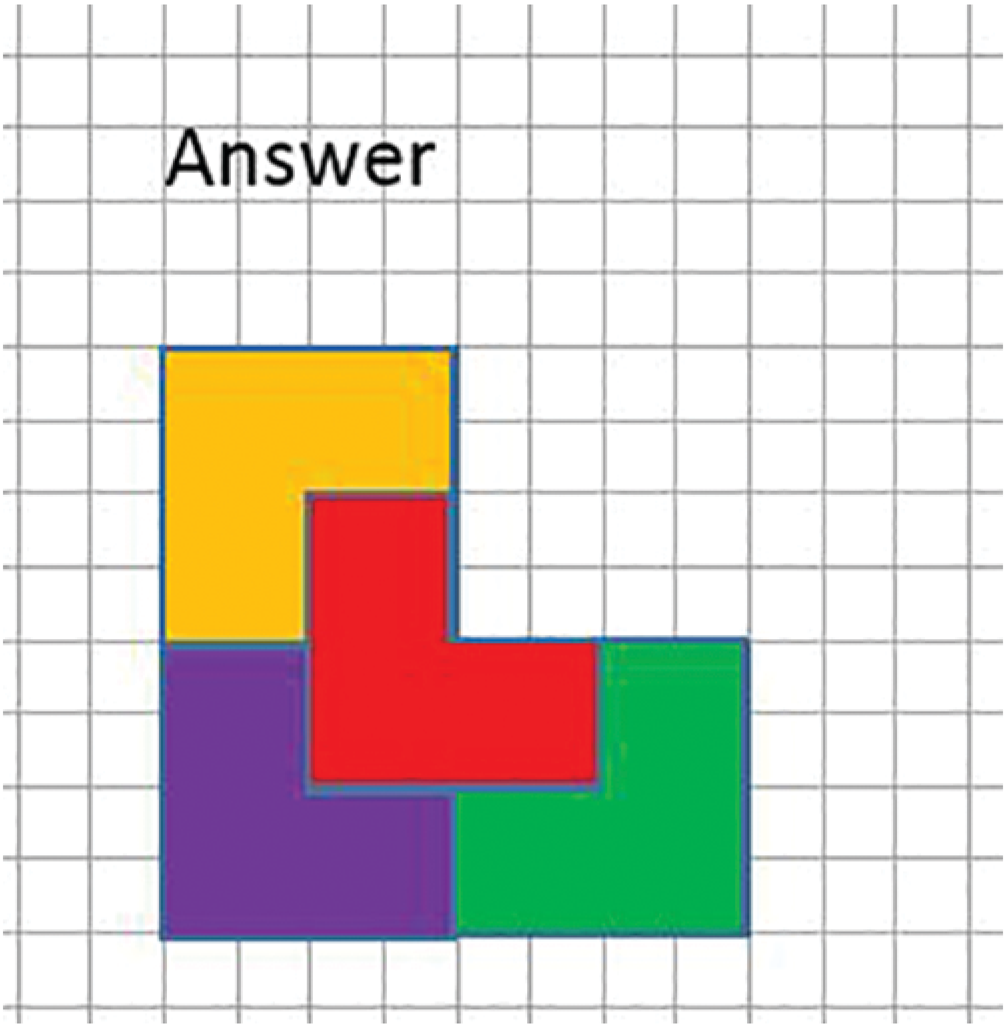


Figure 21 The solution to Activity 16

Repeating tiles as an activity with learners

This task gives the opportunity for learners to create and manipulate cardboard tiles in order to explore which shapes are repeat tiles. Alternatively, the tiles could be produced using ICT. For example, some shapes are generated by the 'Insert Shapes' facility of Microsoft Word.

Repeat tiles provide an approach to enlarging shapes because the final result of placing together several repeat tiles must be a correct larger version of the original repeat tile. When four repeat tiles are joined together then the length of each side of the enlargement is 2 times the length of the side of the original tile. In order to create a shape where the lengths of each side are 3 times the original tile, you need to use 9 tiles.

Learners can investigate the number of tiles needed to generate enlargements of 2, 3, 4 and 5 times the original tile.

This can lead to the observation that the factor of enlargement for area is the square of the factor of enlargement for lengths.

4 This week's quiz

Check what you've learned this week by doing the end-of-week quiz.

Week 6 practice quiz

Open the quiz in a new tab or window and come back here when you've finished

5 Summary of Week 6

In Week 6 you read how the study of geometry can be very intuitive, drawing on learners' experience of the lived in world and as such it is a very practical area of mathematics. On the other hand teachers and their learners work with diagrams and physical objects which are themselves representations of abstract geometrical objects. There is a tension between the specific physical representation which can be seen and touched and the abstract concept in geometry (e.g. the triangle on paper represents a whole set of all possible triangles in the abstract).

You should now be able to:

- understand that geometrical shapes have properties and that their definitions draw on these properties
- use varied activities to support the exploration of geometric shapes, both 2D and 3D
- think about how to set up the classroom to facilitate problem solving activities in geometry, including use of practical resources.

Next week you will learn about measures, how these were developed through history, and how to address the appropriate use of measures within mathematics lessons.

You can now go to Week 7

Week 7: Developing understanding about using measures

1 Measures and measuring

In this section you will consider different types of measuring scales and read about the history behind the development of measures.

1.1 Measuring scales

Quantities of things are measured using measuring tools such as a ruler or a thermometer. These measuring tools may have numerical scales and the quantity is read from the scale.

Measurement scales can be:

- ***Ordinal scales***, when things are put in order, e.g. 1st, 2nd, 3rd.
- ***Interval scales***, such as temperature, where equal differences on the scale represent equal differences in temperature. However, there may be no true zero point. This means it is not possible to say that a temperature which is 40º Fahrenheit is twice the temperature as 20º Fahrenheit.
- ***Ratio scales***, which satisfy the properties of measurement, in that equal intervals of the scale represent equal intervals of what is being measured and there is a non-arbitrary zero. Examples include length, area, volume, angle and age.

Measurements are thought of as being rational numbers (i.e. whole numbers and fractions or decimals) even though they may represent irrational numbers such as the diagonal of a square or the circumference of a circle which involve numbers such as the square root of 2 or π (pi). This links with the idea of approximation in measurement, which will be considered later.

Activity 1 Reflecting on learners' difficulties with using measures

Allow 5 minutes

Have you observed learners using measuring tools such as a ruler or tape measure, weighing scales, a thermometer or clocks?

What problems or misconceptions were evident?

Discussion

Learners may struggle with understanding that measuring (a length, a weight, a time interval, etc.) means dealing with a continuous quantity which starts with zero on the measuring scale, rather than with 1 as in counting. Research evidence indicates that when students measure lengths, they may be applying a poorly understood procedure, rather than focusing on the correspondence between the units on the ruler (seeing it erroneously as a counting device) and the length being measured (Battista, 2006).

Time measurement is given in analogue form (a round-faced clock with the numbers 1 to 12 on the face where the passage of time is analogous to the turning of the hour and minute hands) and digital form (using numbers to indicate hours and minutes passed). Time has a different base (e.g. time uses base 60 and 24) from decimal (base 10).

1.2 The historic development of measures

The development in the use of measures happened because people wanted to trade (in foodstuffs, amounts of cloth, etc.) and because communities needed to divide areas of land between family groups. Informal measurements began with reference to parts of the body, for example:

- The inch was the length from the tip of the thumb to its joint.
- The foot was the length of a man’s foot.
- The yard was the length from a man’s nose to the tip of the index finger on his outstretched arm.

I know that, if I wanted to buy some cloth from a merchant, I would have sent the tallest man in my family! Hence the need for these measures to be standardised. The first time that measures were standardised in the UK appears to have been in 1215 in the Magna Carta.

Some of the old imperial measures have strange names. For example, 22 yards is equal to one chain. My historian friend tells me that every community had a chain. This was literally a chain made of links, 22 yards long and used to measure the ground when setting out fields.

Imperial measures of length

Below is a list of imperial measures of length and how they relate to each other.

- 12 inches to 1 foot
- 3 feet to 1 yard
- 22 yards to 1 chain (real chains, see previous comment)
- 10 chains to 1 furlong (still used in horse racing)
- 8 furlongs to 1 mile
- 1760 yards to 1 mile.

Imperial measures of weight

Below is a list of imperial measures of weight and how they relate to each other.

- 16 ounces (oz) to 1 pound (lb)
- 14 pounds to 1 stone
- 8 stones to 1 hundredweight (cwt)
- 20 hundredweight to 1 ton.

Imperial measures of volume (liquid capacity)

Below is a list of imperial measures of volume and how they relate to each other.

- 20 fluid ounces (fl oz) to 1 pint
- 8 pints to 1 gallon.

If you are as old as me, you will have had to learn all of these conversions between units of measure! The United Kingdom was ‘decimalised’ in 1971 which meant that the UK changed from using money in bases of 12 and 20 (12 pence in a shilling, 20 shillings in a pound) to the current system of 100 pence in a pound. This change in the measurement units for money happened fairly quickly but metrication (the change from the imperial system to the metric system), which began officially in 1973, remains unfinished. Schools teach the use of metric units but also have to teach learners about imperial measures because there are still areas of public life where these are used. Examples include distance in miles, vehicle speeds in miles per hour (mph) and beer sold in pints.

1.3 The metric system

The metric system is based on grams (g), litres (l) and metres (m).

The prefixes always mean the same. The common prefixes are as follows.

Kilo 1000

- 1 kg = 1000 g
- 1 km = 1000 m

Centi 1/100

- 1 cm = 1/100 metre
- 100 cm = 1 metre

Milli 1/1000

- 1 mm = 1/1000 m
- 1000 mm = 1 m
- 1 millilitre = 1/1000 litre
- 1000 ml = 1 litre

Water connects all three measures!

- 1 cm × 1 cm × 1 cm = 1 cubic centimetre (cm^3) and it holds 1 ml of water which weighs 1 gram (g).
- 10 cm × 10 cm × 10 cm = 1000 cubic centimetres and it holds 1 litre of water weighing 1 kg.

1.4 Useful 'rough' metric to imperial equivalences

Below is a list of useful 'rule of thumb' equivalences between metric and imperial measures.

- 10 cm is about 4 inches
- 30 cm is about 12 inches = 1 foot
- 8 km is about 5 miles
- A litre of water is a pint and three-quarters.
- Two and a quarter pounds of jam, weighs about a kilogram.

1.5 Everyday measures

In this section you will use your everyday experience to think about appropriate use of measures.

Activity 2 Having a sense of common objects and their measures

Allow 10 minutes

It helps to have a sense of what a metre looks like, what a kilogram feels like, and so on. Take a look in the kitchen cupboard and list items which are measured in grams, kilograms, millilitres, centilitres, litres and metres.

Discussion

It can help children to consider everyday items and their measures. Here are some common measures (clearly they are approximate).

- A packet of crisps weighs 25 grams.
- An apple weighs about 100 grams.
- A bag of sugar weighs 1 kilogram.
- A family car weighs 1 tonne. (Note: imperial tons and metric tonnes are similar in size.)
- A teaspoon holds 5 millilitres.
- A bottle of wine holds 75 centilitres.
- A bottle of fizzy drink holds 2 litres.
- A ruler is 30 centimetres long.
- The room is 2½ metres high.

Activity 3 'The tailor's rule of thumb'

Allow 15 minutes

Figure 1 Gulliver in Lilliput

'The tailor's rule of thumb' was said to be used to make shirts and trousers in the past. It is quoted in the novel *Gulliver's Travels* by Jonathan Swift:

The seamstresses took my measure as I lay on the ground, one standing at my neck, and another at my mid-leg, with a strong cord extended, that each held by the end, while a third measured the length of the cord with a rule of an inch long. Then they measured my right thumb, and desired no more; for by a mathematical computation, that twice round the thumb is once round the wrist, and so on to the neck and waist, and by the help of my old shirt, which I displayed on the ground before them for a pattern, they fitted me exactly.(Swift, 1726, p73)

Work out how the Lilliputian seamstresses calculated Gulliver's wrist, neck and waist measurements.

Discussion

The mathematical computation is:

'Twice round the thumb is once round the wrist

Twice round the wrist is once round the neck

Twice round the neck is once round the waist.'

By measuring round the base of the thumb, a tailor can get a rough estimate of wrist, neck and waist measurements.

An interesting classroom activity is to get learners to see how accurate this is with string. However, caution is needed where learners have issues about body image.

1.6 Extending measures to compound measures

Compound measures are comprised of at least two other measures. Perhaps the most commonly used example is speed which is the distance travelled in one unit of time (i.e. it uses a combination of length and time measures). It is measured in miles per hour, km per hour, metres per second, etc.

When introducing compound measures to my learners I used to pose this question:

'Which is heavier: a ton of lead or a ton of feathers?'

My learners, being smart young people, would think about this question and then say 'They weigh the same because you said they weighed a ton'. I would then ask:

'What is the obvious difference between the ton of feathers and the ton of lead?'

The answer is that the ton of feathers takes up much more space than the ton of lead. This leads to a consideration of volume and weight. Lead is denser than feathers as it is more compact. We would use this to work out that density is the weight per unit volume of a substance.

One year I had a thoughtful group of learners who replied that if a ton of lead fell on you it would probably kill you whereas you would be likely to survive having a ton of feathers falling on you. Cue the class clown brilliantly miming someone climbing out of a ton of feathers!

2 Reading measures and accuracy

In this section you will look at different types of measuring scales and consider appropriate degrees of accuracy in measurement.

2.1 Measuring scales

Reading a length from a ruler or tape measure is straightforward because these measuring devices are generally marked in unit lengths and each unit length is subdivided. A metre ruler may be marked in centimetres, each of which are subdivided into tenths or 0.1 of a centimetre.

The tape measure shown in Figure 2 has inches on one side, subdivided into quarters and sixteenths, and centimetres on the other. Note that the 10s figure for centimetres is only marked for every 10 centimetres.

Figure 2 A tape measure showing inches and centimetres

Reading temperature from thermometers is usually straightforward because they are marked in degrees Celsius (ºC) and/or degrees Fahrenheit (ºF) (Figure 3). However, although each gradation may be a unit length, they may be numbered in 20s, 10s or 5s.

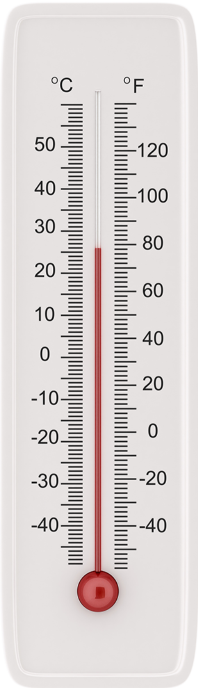


Figure 3 A thermometer showing degrees Fahrenheit and Celsius

Reading a weight from a weighing scale may be more challenging. Some scales may be marked in kg or lb but the gradations might not be in the subunits. It may be necessary to count how many intervals a unit has been divided into and enact a division to find out what each subdivision represents. See the picture of the scales in Figure 4.

Figure 4 An old-fashioned weighing machine

Reading the amount of fuel left in the tank of a car is a case in point. Fuel gauges rarely tell you how much fuel is in the tank. Instead, they tell you what fraction of a full tank is left (Figure 5).

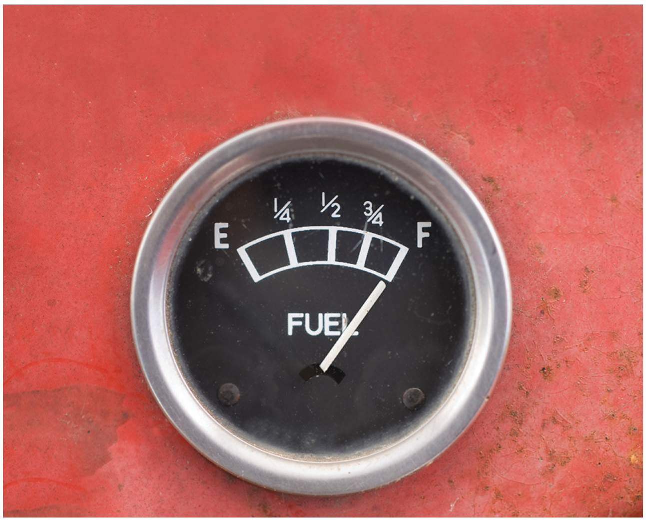


Figure 5 A fuel gauge in a car

It is worth saying that many measuring tools are digital and provide a measurement in digits rather than on a scale. Time measurement provides a common example. We continue to measure time using analogue clocks, where the passage of time is reckoned by the position on the clock face of the hour-hand and the minute-hand (Figure 6). Yet we have become used to digital clocks which provide the time as a series of digits (Figure 7).

Figure 6 An analogue clock

Figure 7 A digital clock

2.2 Degrees of accuracy in measurement

Depending on the context, it may not always be necessary to give an exact answer to a measurement. At other times, it is actually more acceptable to have rounded the amount to an appropriate degree of accuracy. The population of countries is often quoted in millions because the exact figure changes constantly. For example, the population of the UK is ‘about 65 million’.

It can also depend on what you are going to do with the measurement. For example, when measuring a window for curtains, the width to the nearest 5 cm would be more than sufficient. But if you are measuring the same window for a replacement pane of glass, a much higher degree of accuracy would be needed.

It should be understood that measurement is continuous, so lengths, weights, etc. can take any value on the number line including decimals and fractions. This is in contrast to amounts which are counted, such as the number of sweets in a jar, which can only take whole number values. However, large numbers of items to be counted may act as continuous measures, such as national population sizes.

Accuracy is often quoted using the number of decimal places. If you need to round a measurement to 1 decimal place that implies to the nearest one-tenth. If you need to round a measurement to 2 decimal places that implies to the nearest one-hundredth.

A number such as 3.8567 lies in between 3.8 and 3.9; numbers with 1 decimal place. You might remember the video on decimal places which explains how the digits after the decimal point denote decreasing powers of ten: tenths, hundredths, thousandths, and so on. Therefore, the only useful digit in deciding whether to round 3.8567 to 3.8 or 3.9 is the second number after the decimal point, i.e. the 5 in this example. You could call this the first unwanted number. Since that number is 5 or more, convention dictates that the number is rounded up to 3.9.

Similarly, with large numbers such as millions, you need to consider the first unwanted number to decide whether to round up or down. So, 65 million people in the UK may have been rounded from, say, 65 345 211. This number lies between 65 million and 66 million but is clearly closer to 65 million. The first digit in the column below one million (i.e. in the 100 000 column) is 3, which is less than 5, and so you round down.

Activity 4 Rounding numbers

Allow 5 minutes

Now have a go at rounding these numbers to the degree specified.

- 62 to the nearest 10.
- 14.237 to the nearest 1 decimal place.
- 14.237 to the nearest 2 decimal places.
- 465 to the nearest 100.

Discussion

- 60
- 14.2
- 14.24
- 500

Finally: measurement as a human activity is not an exact science. We are bound by our own skills with measuring and the tools we use for measuring. If we are using a school ruler, for example, we can measure accurately to one decimal place but not more than that. Suppose we do a calculation such as the ones mentioned earlier for the length of the diagonal of a rectangle. We are likely to get a calculator answer with many figures after the decimal point. We cannot measure more accurately than one decimal place and so it makes sense to round the answer to 1 decimal place.

3 Teaching angle measures and angle facts

Learners struggle with the idea of angle being a measurement of turn and also a measure of the gradient between two lines. In particular, learners can be distracted by the lengths of the lines which make up the angles. The angles between the two sets of lines shown in Figure 8 may be considered to be different even though the angle of turn from one line to the other is the same.

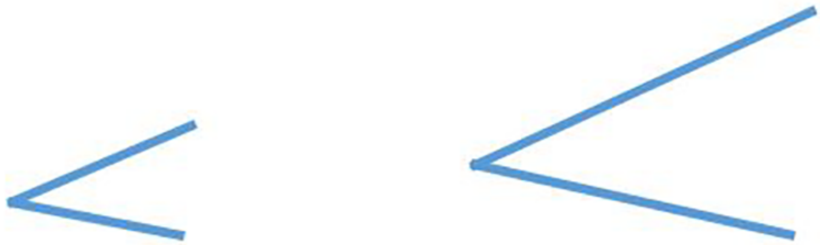


Figure 8 Two angles of the same size

3.1 Using a protractor to measure angles

The half-circle protractor is commonly used to measure angles (Figure 9). The inner number scale rises from 0 to 180, from right to left, and the outer scale goes from 0 to 180, left to right. Learners commonly make mistakes because they are not using the correct scale when measuring the angle between two lines. The point to remember when measuring angles is that you need to start at zero on one of the lines and read round on that scale to see the number on the second line.

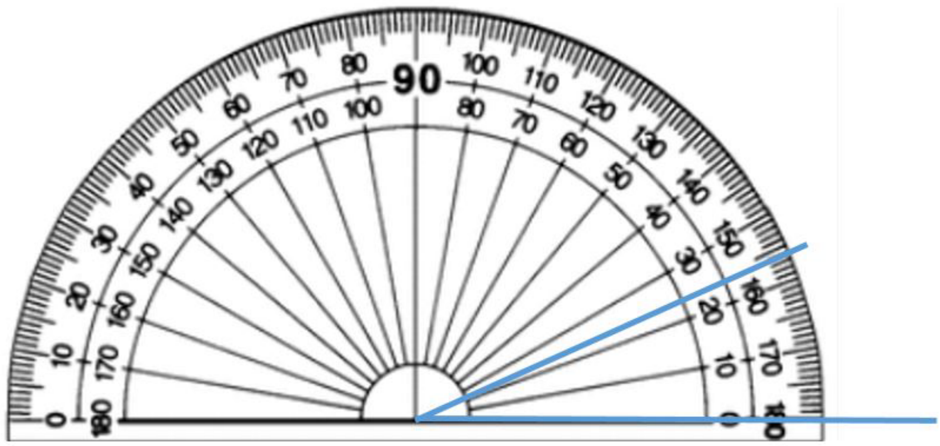


Figure 9 Using a protractor to measure an angle

In order to read the number from the protractor, it may help to use a ruler to make the lines longer so that they meet the scales on the protractor. Make sure that one of the lines in the angle sits on the zero line. Then, using the scale which starts at zero (the inner scale in this case), read off where the second line meets the scale (24º for this particular angle). If a learner gets 156º for the angle, it is obvious that they have made a mistake. Why is this obvious?

The angle is smaller than a right angle (acute), so the number of degrees must be less than 90.

Degrees in a circle

As you can see from the half-circle protractor displaying 180º, it is commonly accepted that there are 360ºin a circle. No one is entirely sure where this originated, possibly with the Babylonians [2000-1600 BC]. It is certainly worth having at least one plausible explanation ready when teaching this in order to answer the question 'Why?'

3.2 Angle sum of triangles

The angles of any triangle sum to 180º. This can be proved in several ways and can easily be shown to convince learners as a classroom activity.

Activity 5 Demonstrating the angle sum of a triangle

Allow 15 minutes

You will need paper, pen, ruler and scissors.

Draw any triangle and mark the angles at the corners.

Cut out the triangle.

Tear off the corners then put the corner angles together. They should all form a straight line.

If we accept that there are 360º in a circle and 180º on a straight line then we have shown that the three corners add up to 180º.

Discussion

This activity can be done in the classroom. If 30 learners can do this activity using 30 different triangles then they will have demonstrated the angle sum of a triangle in 30 different ways. This is not a proof, merely a demonstration that it works.

It is much more fun for learners if they discover angle facts for themselves, rather than being simply told them.

Think of other ways for learners to discover the angle facts using drawing and cutting, and measuring using a protractor.

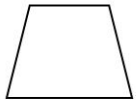
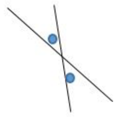
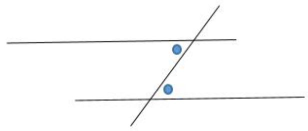
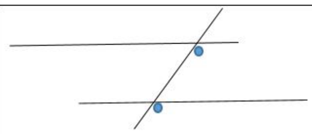
Angles in a quadrilateral add up to 360 degrees	
Opposite angles are equal X angles	
Alternate angles *inside parallel lines* are equal Z angles	
Corresponding angles *inside parallel lines* are equal F angles	

Figure 10 Simple angle facts

Activity 6 A problem using angle facts

Allow 15 minutes

Below is a design using 8 congruent (i.e. exactly the same size and shape) rhombuses. Calculate the two different angles in the rhombuses.

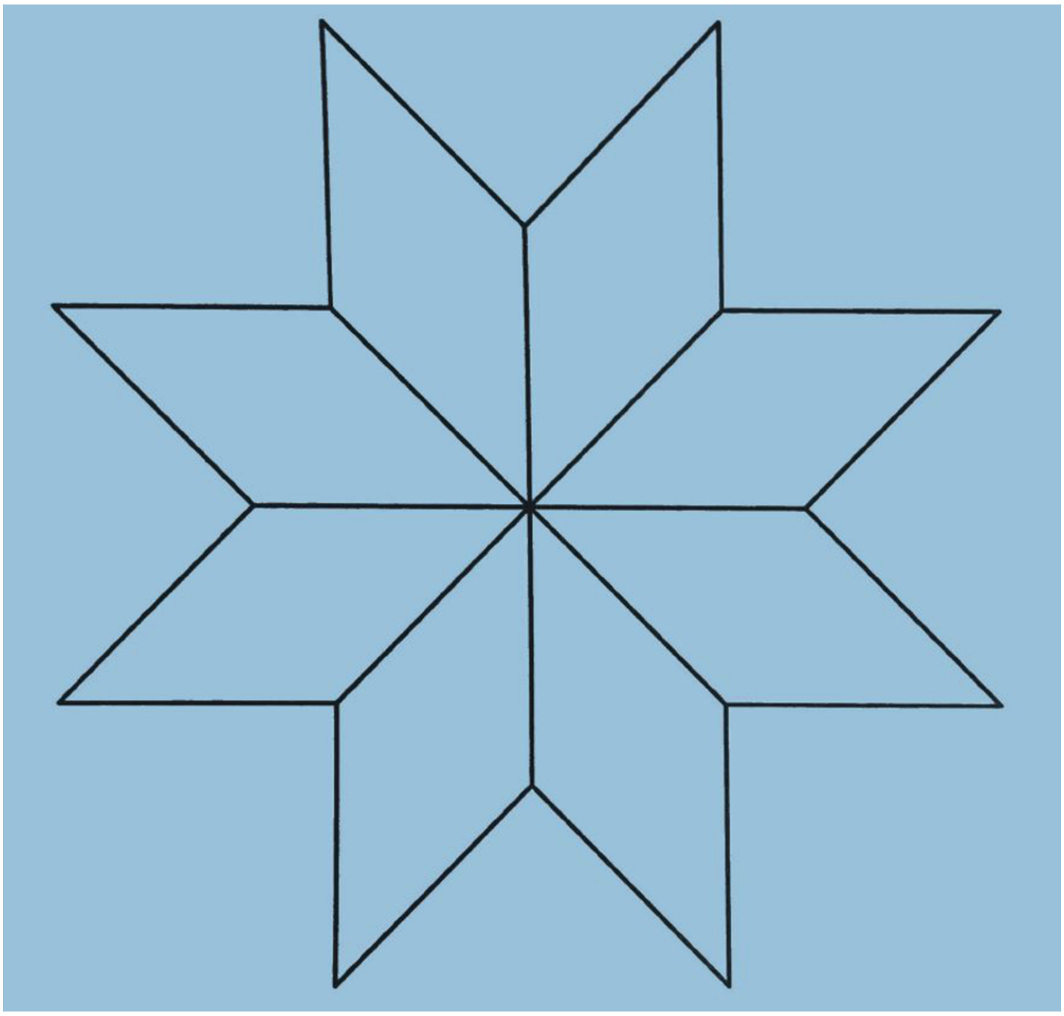


Figure 11 Design created using eight rhombuses

Discussion

Think about the number of rhombuses which meet in the middle of the design. Their sum must be 360º.

Think about the rhombus as a quadrilateral. Its angles must sum to 360º. A rhombus has two pairs of equal angles.

Use compasses and a protractor to recreate this design. Begin by drawing a circle and marking off 8 equally spaced points on the circumference. Use these points to draw the bottom halves of the 8 rhombuses.

To find the top points for each rhombus use the compasses opened to the same radius as when you drew the circle. Draw 2 arcs (parts of circles) from each point which will intersect in pairs above the rhombus halves. These intersections are the top points of the rhombuses. Join these to the circumference to complete the rhombuses.

4 Teaching about area and volume

Area is a measurement of the amount of space inside a 2D shape and is reckoned by the number of unit squares inside the shape. Common metric units of area are square centimetres (cm^2), square metres (m^2) or hectares (ha) for measuring land areas. A hectare is defined as 100 m × 100 m or 10 000 m^2. Imperial measures of area are square inches, square feet and acres for measuring land areas.

4.1 Area

It is best to introduce the topic of area by considering the inside of shapes as being divided into unit squares with area being recorded as the number of squares and the units of area (e.g. 10 cm^2). In contrast, the perimeter of a shape is the total length of its boundary but this is often confused with area by learners.

The next stage in learning about area is to find the area of rectangles which have sides measured as whole numbers. It is straightforward for learners to note that the rectangles contain an array of unit squares (Figure 12). This leads to the general formula for the area of a rectangle = length × width.

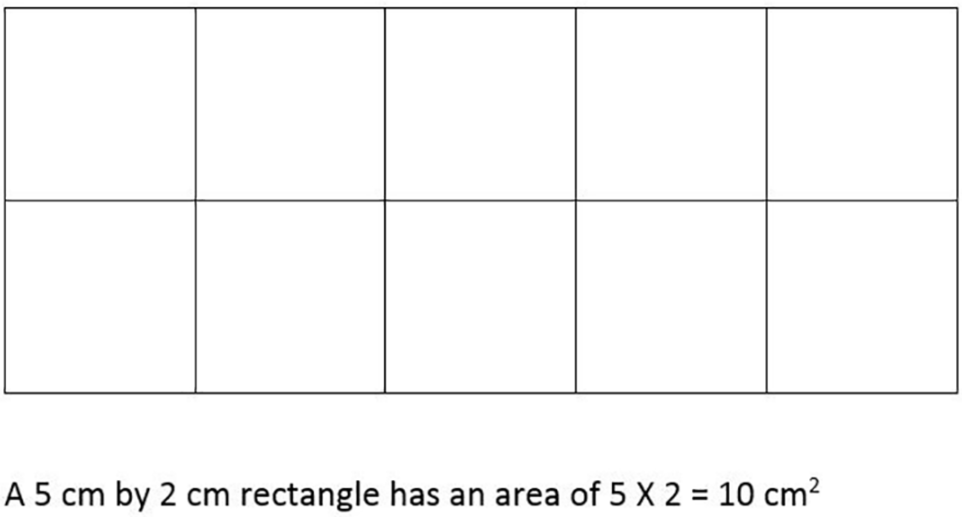


Figure 12 A 5 cm by 2 cm rectangle divided into square units

Using the area formula for a rectangle to derive other area formulas

The area formulas for parallelograms and triangles can be easily derived from the area formula of a rectangle (see Video 1).

Watch this video to see how the area formula of a rectangle can be used to derive the area of a parallelogram and then a triangle.

Video content is not available in this format.

Video 1 Deriving the area of a parallelogram and the area of a triangle

Deriving the area of the parallelogram from the area of the rectangle

Since the parallelogram has been made by moving parts of the original rectangle without losing anything, the area of the parallelogram must be the same as the area of the rectangle.

It is worth bearing in mind that if the area formulas are promoted too quickly, before learners have conceptualised area as being a measure of the inside of a shape, then area measurement becomes a calculation without reference to the geometric situation.

4.2 Converting square metres into square centimetres

If you imagine a tile which is one square metre, then 100 square centimetre tiles would line up along its edge.

You could fit 100 rows of these hundred square centimetre tiles inside the metre square.

That’s 100 × 100 = 10 000 square centimetres in one square metre (Figure 13).

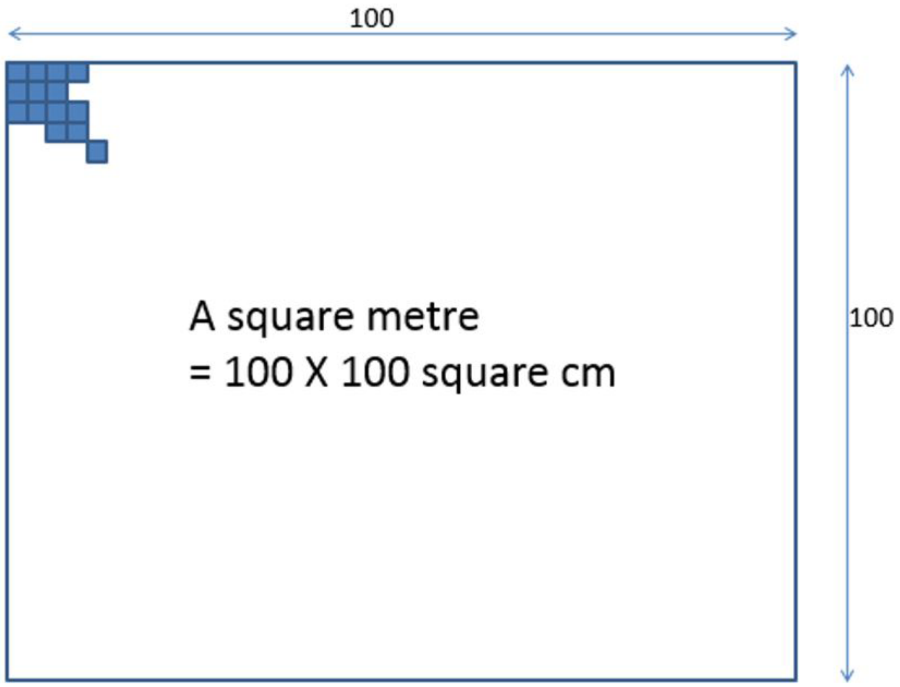


Figure 13 A square metre

4.3 Volume

Volume is a measurement of the amount of space inside a 3D shape and is reckoned by the number of unit cubes inside the shape. Common metric units of volume are cubic centimetres (cm^3) or cc, and cubic metres (m^3). Examples of imperial measures of volume are cubic inches and cubic feet.

It is best to begin the topic of volume by considering the inside of 3D shapes as being divided into unit cubes with volume being recorded as the number of cubes and the units of volume (e.g. 12 cm^3). Volume can be investigated practically using 1 × 1 × 1 cm multilink cubes.

For example, learners can be asked to use multilink cubes to make a cuboid which is 3 cm by 4 cm by 2 cm and to work out how many cubes are needed to make it (Figure 14).

Investigating different sized cuboids should lead to the formula for the volume of a cuboid as

Volume = length × width × height.

As with the area formulas, the volume of other more complicated 3D shapes can be derived from the formula of the volume of a cuboid.

Figure 14 A 4 × 3 × 2 cuboid made of 24 unit cubes

4.4 Converting cubic centimetres into cubic metres

If you imagine a box which is one cubic metre then 100 cubic centimetres would line up along its edge.

You could fit 100 rows of these 100 cubic centimetres in one layer.

There would be 100 layers of cubic centimetres in the cubic metre.

That’s 100 × 100 × 100 = 1 000 000 cubic centimetres in one cubic metre (Figure 15).

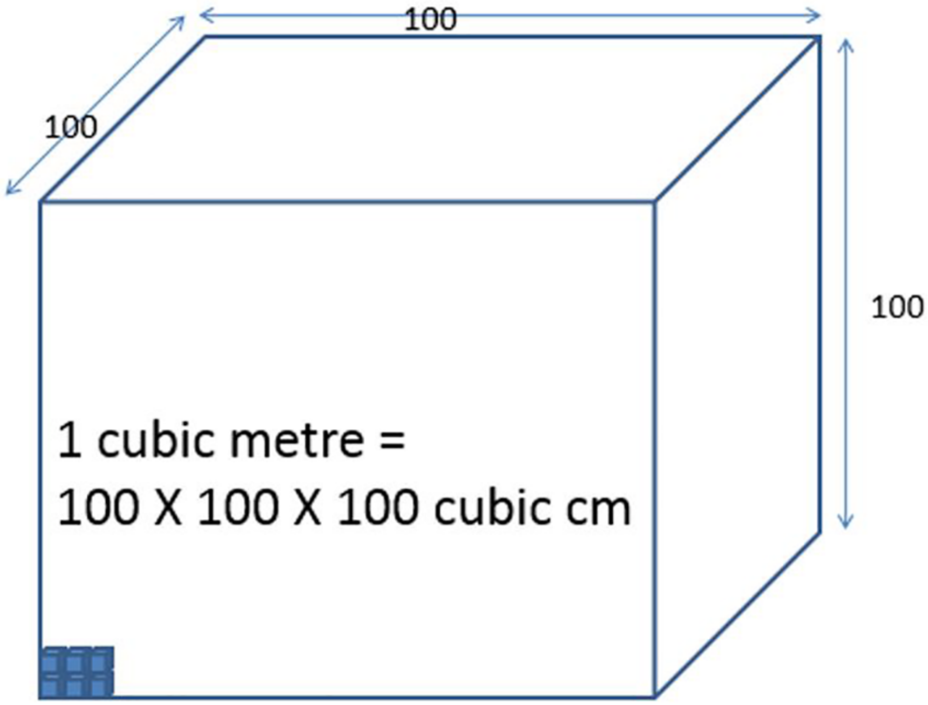


Figure 15 A cubic metre

5 This week's quiz

Check what you've learned this week by doing the end-of-week quiz.

Week 7 practice quiz

Open the quiz in a new tab or window and come back here when you've finished.

6 Summary of Week 7

In Week 7 you have seen how it is important to be aware of different measuring scales and the various types of measuring instruments. Taking a measurement involves human judgement which is therefore limited to a particular degree of accuracy. For example, we can usually only measure lengths correct to the nearest millimetre.

You have also read a brief summary of the development of measuring systems such as the imperial system and the metric system. Giving learners a history of how measures were developed can be interesting and demonstrate how the use of measures drives from human activity.

You should now be able to:

- check that your learners use measuring tools such as rulers, appropriately, for example making sure they put the zero on the ruler at the beginning of the line they are attempting to measure. When measuring an angle, checking that they place the protractor correctly and read from the appropriate scale
- help learners understand that area is a measurement (by unit squares) of the amount of space within a 2D shape and that volume is a measurement (by unit cubes) of the amount of space within a 3D shape
- derive the area formulas of a parallelogram and a triangle from the area formula of a rectangle.

Next week you will focus on data handling, specifically statistics and probability.

You can now go to Week 8

Week 8: Working with data and uncertainty

1 Collecting data and illustrating it in charts

In this section you will consider different methods for collecting and recording data.

In Sections 2 and 3 we will look at analysing and interpreting data.

1.1 Collecting data

Data is a collective name for information recorded for statistical purposes. There are many different types of data, which we will discuss in this section.

Data can be collected in several ways. These include face-to-face or telephone interviews, or asking participants to complete questionnaires in person or by post. Data can also be collected online via an internet search or through other more active data-collecting methods, such as counting the number of passengers on a bus at a specific time of day.

When data is collected from an original source for a specific purpose, this is known as **primary data**. For example, a learner who is investigating the most popular make and model of cars in their town might count cars which pass a particular set of traffic lights. When data is not collected from an original source, this data is **secondary**. For example, a learner might find data on goals scored in the football Premier League last year in order to investigate whether strikers from a particular country score more goals on average. Since the data was not collected for this purpose originally and was not collected by the learner themselves, this data is secondary.

1.2 Population and sampling

The **population** of a survey is everyone who can be questioned in relation to that survey. For example, if a learner wanted to find out the most popular meal in their school canteen, the population of the survey would be all of the students, and possibly also the teachers, in the school.

A **sample** is a small selection of the population, for example just one class in a school.

There are advantages and disadvantages to using entire populations and samples.

Table 1 Advantages and disadvantages of sampling

	Advantages	Disadvantages
Population	Every opinion is included	Time consuming
	Representative and therefore reliable	Expensive
Sample	Quick	Only some opinions
	More cost effective	Selection method could cause bias

Activity 1 Reflecting

Allow 5 minutes

How could a selection method mean that the results of a sample could cause bias?

Discussion

When data is collected from the entire population, all opinions are taken into consideration. When data is collected from a sample of the population, it is important that the sample is representative of the population.

For example: Joe wants to find out whether motorbikes or cars are more popular, so he has asked 20 people at a motor bike convention which they prefer.

Comment: Joe's data selection method is clearly flawed as he is much more likely to find that people prefer motorbikes within this sample of the population.Thus they are unlikely to be representative of the wider population.

Considering potential bias when sampling data is part of having **data sense**.

Sampling

When collecting data from a sample of the population, decisions have to be made about how the sample is chosen. It is important to think about how to ensure the sample is representative of the wider population.

There are many different sampling methods. Here are a few examples.

Convenience sampling is the most simple and straightforward way to collect a sample of data since participants are chosen in the most convenient way and are often asked to volunteer. Statistical investigations carried out by young learners are likely to use this form of sampling, for example when investigating favourite pets earlier in this section, the learners used their own classmates for convenience.

Random sampling can be done by assigning a number to each member of the population and then using a random number generator to select the required number of individuals.

Systematic sampling involves choosing individuals at regular intervals. For instance, if a sample of 20 is needed from a population of 100, every 5th individual would be used in the sample. This sample will obviously vary depending on how the population is ordered. This is a convenient and simple way to collect a sample but may result in some bias as some smaller groups within the population could be omitted from the sample.

A **stratified sample** takes into consideration different groups and their proportions within the overall population, using the same proportions in the sample. For example, if a school has more students in year 7 and fewer students in year 11, the sample would take a greater number of participants from year 7 and fewer from year 11. This is done using percentages.

A **quota sample** does not take into consideration the proportion of each group in the population. Instead, a quota sample may require there to be 20 male and 20 female participants in the sample, regardless of whether there are many more of either gender in the wider population.

Sampling methods are not generally covered in the middle school curriculum. But it is important for learners to be aware that their sampling choices will affect the reliability of their findings when conducting statistical investigations.

1.3 Questionnaires

Questionnaires can be a useful tool when collecting primary data, as they can be quick and cheap to produce. It is important, however, to consider the way questions are written in order to ensure the data collected is representative and unbiased.

Questionnaire questions can require:

- yes/no responses
- tick boxes with different options
- numbered responses
- single word responses, or
- a sentence to be written.

When writing questionnaire questions, it is important that they:

- are unambiguous and easy to understand
- do not lead respondents to give particular answers
- cover every possible answer
- are appropriate for the investigation.

Activity 2 Good or bad questions?

Allow 10 minutes

Look at the questionnaire below.

Can you identify an issue with any of the questions?

If so, rewrite the question to address the issue.

Question 1: What is your favourite colour?

- Red
- Green
- Blue
- Yellow

Provide your answer...

Question 2: Do you agree that mathematics is the best subject?

Provide your answer...

Question 3: How many pieces of fruit do you eat in a week?

- 0–5
- 5–10
- 10–15
- 15–20
- More than 20

Provide your answer...

Discussion

Question 1 provided tick boxes for the respondent but not all the possible answers were provided. If the respondent's favourite colour was purple, they could not respond to this question accurately.

When rewriting this question, you could have included an 'other' box. Here you could also have added a space for the respondent to write their favourite colour.

Question 2 is a leading question.

When rewriting this question, you would need to ensure it is unbiased and not leading. You could ask 'What is your favourite subject?' and provide options or an empty space for respondents to write in the subject.

Question 3 gives ambiguous options. It is not clear which box to tick when the respondent's answer is 5 or 10 or 15.

When rewriting this question, change the option boxes so there is no overlap. For example, 0-5, 6-10, 11-15, 16-20, more than 20.

1.4 Tally charts and frequency charts

Tallying is a way of recording data in groups of five.

Recording frequencies in this way means totalling the number of tally marks made (figure 1).

Figure 1 Tally marks

Tally charts can be used to collect and organise **primary data**.

Look at the tally chart in Figure 2.

Biscuit	Tally	Frequency
Digestive	卌	5
Jammy dodger	卌 卌 |||	13
Chocolate bourbon	||||	4
Custard cream	卌 |||	8
other	||	2

Figure 2 A tally chart

The first column contains all of the possible data options. Note that there is an ‘other’ box in case any of the answers do not fit into the first four categories. As with questionnaire questions, it is important that every possible answer can be recorded in a data collection table such as this one.

The second column is for a tally, so that results can be recorded and added to.The final column is the frequency. This is the total for each data category once all the data has been collected.

It is also possible to have **a frequency chart** without a tally column, like the example shown below.

Table 2 What is your favourite ice cream flavour?

Flavour	Frequency
Chocolate	14
Vanilla	5
Strawberry	7
Other	20

Activity 3 Reflecting

Allow 10 minutes

Did you notice that the 'Other' category in the example above had the highest frequency?

What conclusions could you draw from this?

Discussion

Sometimes when collecting data, the results can be surprising. Participants sometimes give answers you do not expect.

It may be that there are popular icecream flavours that have not been listed.

In some instances, it is worth introducing further categories once some initial data has been collected.

1.5 Grouping data

When there are a large number of possible outcomes, it may be necessary to group the data.

For example, if a learner is investigating how many DVDs each of their classmates own, the values could vary from 0 to 100 (or more).

Introducing 100+ different possible outcomes into a table is going to be very time consuming and difficult to plot in any kind of graph. Instead, grouping the data in a sensible way can make data collection and analysis much more straightforward.

Below you will see a **grouped frequency table** which could be used to investigate how many DVDs the learner's classmates own.

Table 3 How many DVDs do you own?

Number of DVDs	Tally	Frequency
0–5		
6–10		
11–20		
21–40		
Over 40		

Activity 4 Reflecting

Allow 5 minutes

Did you notice that there was no overlap in the class intervals and that all possible outcomes were included?

Did you notice that the class intervals in the example above are not equal?

Discussion

If there is no upper bound for a numerical category, it is common to use ‘over’ or ‘more than’ in order to cover all of the possible responses. Similarly, if there is no lower bound, it is common to use ‘less than’, so the first category could have been ‘less than 5’.

Depending on how the data is going to be used, it might be necessary to use classes with equal widths. We will discuss this later in this week.

1.6 Two-way tables

When data falls into more than one category or a learner is interested in investigating more than one variable, data cannot be organised easily in a frequency table.

Two-way tables are a way of sorting data so that the frequency of each category can be seen quickly and easily.

For example, a learner has asked 20 people about whether they like football and whether they like rugby. The results can be seen in the two-way table below.

From the table, it can be seen that there were 9 people who liked both rugby and football and 4 people who liked neither sport. It can also be seen that there were people who liked rugby but not football, and 5 people who liked football but not rugby.

Table 4 Do people who like football also like rugby?

	Like football	Do not like football
Like rugby	9	2
Do not like rugby	5	4

Activity 5 Interpreting a two-way table

Allow 5 minutes

Use the information presented in the two-way table below to answer the following questions.

1. How many right-handed students are there in the class?
2. How many students are there in the class in total?

Table 5 Do left-handed students like art?

	Left-handed	Right-handed
Likes art	2	12
Do not like art	1	10

Discussion

1. There are 22 right-handed students in the class as this is the sum of the 'right-handed' column.
2. There are 25 students in total. This is the sum of the 'left-handed' and 'right-handed' columns.

1.7 Discrete and continuous data

Earlier this week we discussed **primary** and **secondary** types of data. Numerical data can be further categorised into **discrete** or **continuous** data.

- **Discrete data** is numerical data that can only take certain values. The number of people on a fair ground ride, the score on a pair of dice, or a shoe size are all examples of discrete data.
- **Continuous data** is numerical data that can take any value within a given range. The heights of a group of adults, the lengths of some leaves, or a dog's weight are examples of continuous data.

Discrete data is counted and continuous data is measured.

Activity 6 Are these measures discrete or continuous?

Allow 5 minutes

1. A child's foot length
2. A child's shoe size
3. The time taken to run a race
4. The number of runners in a race
5. The number of matches in a box
6. The speed of a car
7. Air temperature

8. Time displayed on a dial watch
9. Time displayed on a digital watch
10. Annual salaries of teachers

Discussion

The discrete measures are: examples (2), (4), (5) and (9) since the number of people or matches cannot be anything other than a whole number and the time displayed on a digital watch is restricted to hours, minutes and seconds.

Note that in the case of example (2), a discrete variable need not be restricted to whole numbers (shoes in the UK can be in half sizes).

The continuous measures are examples(1), (3), (6), (7) and (8) since length, time, speed, temperature and time can be measured to any degree of accuracy (there is no limit to the number of decimal places that could be included in the measurement).

Example (10) is rather ambiguous. Strictly speaking, it is a discrete variable, given that there is a basic unit of 1p below which measurements cannot be taken. However, 1p is so small in relation to an annual salary (even the annual salary of a teacher) that in practice it is treated like a continuous variable.

1.8 Grouping continuous data

Earlier you used a grouped frequency table to organise the number of DVDs owned by school students. In a similar way, grouped frequency tables are needed for continuous data.

There is a slight difference in the way the class intervals are organised for continuous data. Since continuous data can take any values within a given range, you need to allow for this in your table, as follows.

Table 6 How long do you spend on your mobile phone each week?

Hours (h) spent on phone	Tally	Frequency
$0 \leq h < 10$		
$10 \leq h < 20$		
$20 \leq h < 30$		
More than 30		

In this grouped frequency table we are using ‘less than’ and ‘less than or equal to’ notation.

Earlier, when discussing the discrete data example of DVDs, we discussed the importance of having no overlap in the categories. For this reason, the categories were 0–5, 6–10, and so on.

In this example, since time is continuous, we need to allow for values such as 10.5 hours. The first category includes 0 but does not include 10. This means that any value less than 10 hours will be included in the first row. This would include values such as 9.5 or 9.7.

In the second row, 10 is included but 20 is not. This means that any value which is between 10 (including 10) and 20 (not including 20) will go here.

This concept is covered towards the end of lower secondary school but learners in primary school will have seen the notation <, to mean 'less than' and ≤ to mean 'less than or equal' to. So, we can write $10 \leq h < 20$.

1.9 Stem and leaf diagrams

As well as recording data in frequency tables, results can be recorded and presented in a **stem and leaf diagram.** This visual representation allows for the data to be grouped in classes and it shows the shape of the data.

A group of students challenged their teachers to a speed-texting competition. The 21 participants (12 students and 9 teachers) were each required to send a short text message on a mobile phone. Their times are given in Figure 3 in order of size.

Student times (secs)	19, 19, 21, 24, 24, 25, 27, 29, 30, 30, 33, 41
Teacher times (secs)	18, 27, 28, 31, 33, 33, 36, 47, 51

Figure 3 Speed-texting scores

Creating a stem and leaf diagram involves finding the 'stems', in this case the tens, and then recording the 'leaves' in order.

The stem and leaf diagram for the student times is shown in Figure 4. You can see that 1|9 represents 19 and 4|1 represents 41. There are no values between 0 and 10 which is shown as there are no leaves next to the stem 0. The greatest value in this data set is 41, so there are no values next to the stem 5 either.

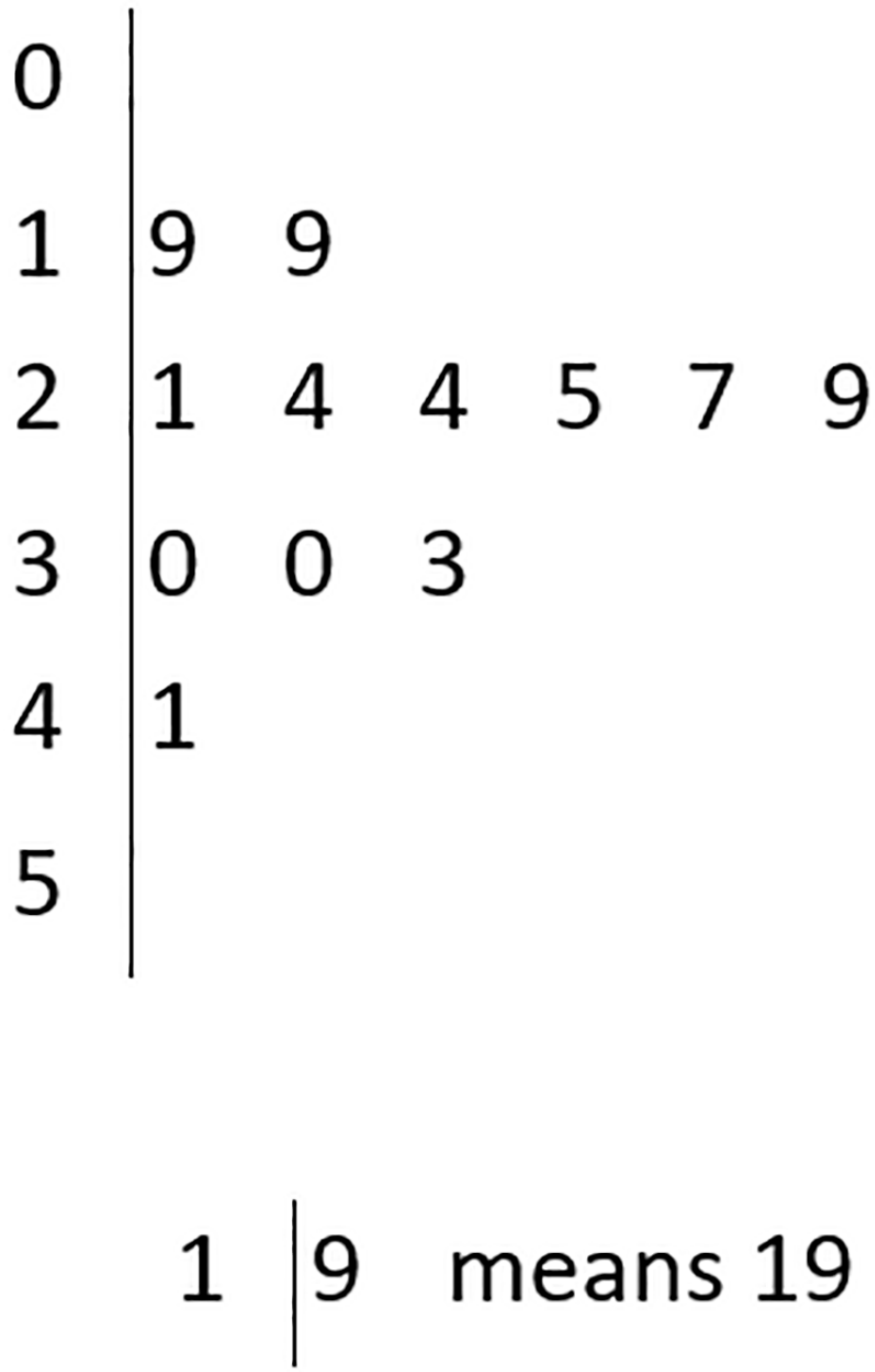


Figure 4 An ordered stem and leaf diagram

Activity 7 Creating a stem and leaf diagram

Allow 10 minutes

On paper, have a go at creating an ordered **stem and leaf diagram** for the teacher's data from Figure 3.

Discussion

Your diagram should look like Figure 5.

0				
1	8			
2	7	8		
3	1	3	3	6
4	7			
5	1			

1 | 8 means 18

Figure 5 Completed stem and leaf diagram

2 Developing data sense and forming sensible conclusions

So far this week, we have discussed how data can be collected and recorded in frequency tables Later, in Section 3, you will look at how data can be presented in different graphs and charts.

In this section you will look how learners can develop data sense. You will look at how data can be analysed using appropriate statistical measures and consider how sensible conclusions can be made from these statistics.

2.1 Statistical thinking and the PCAI cycle

Statistical thinking involves more than just calculating averages and plotting graphs. In the English National Curriculum (2014), students are expected to ‘interpret and present data’ from 7 years of age. Students in lower secondary school, aged 11–14, are expected to ‘describe, interpret and compare’ data and to ‘describe simple mathematical relationships between two variables’.

In order for learners to develop statistical thinking, they need to be encouraged to discuss the meaning behind their calculations and to interpret measures and graphs within relevant contexts. Statistics lessons must therefore involve more than just teaching techniques and methods.

Outside a school setting, statistics are rarely calculated in isolation. Data is collected, analysed and interpreted with a purpose, to investigate something or to answer a research question.

Learners can be introduced to the concept of a statistical investigation using the **PCAI cycle** (Figure 6).

PCAI stands for: **pose** (a question), **collect**, **analyse** and **interpret** (the data).

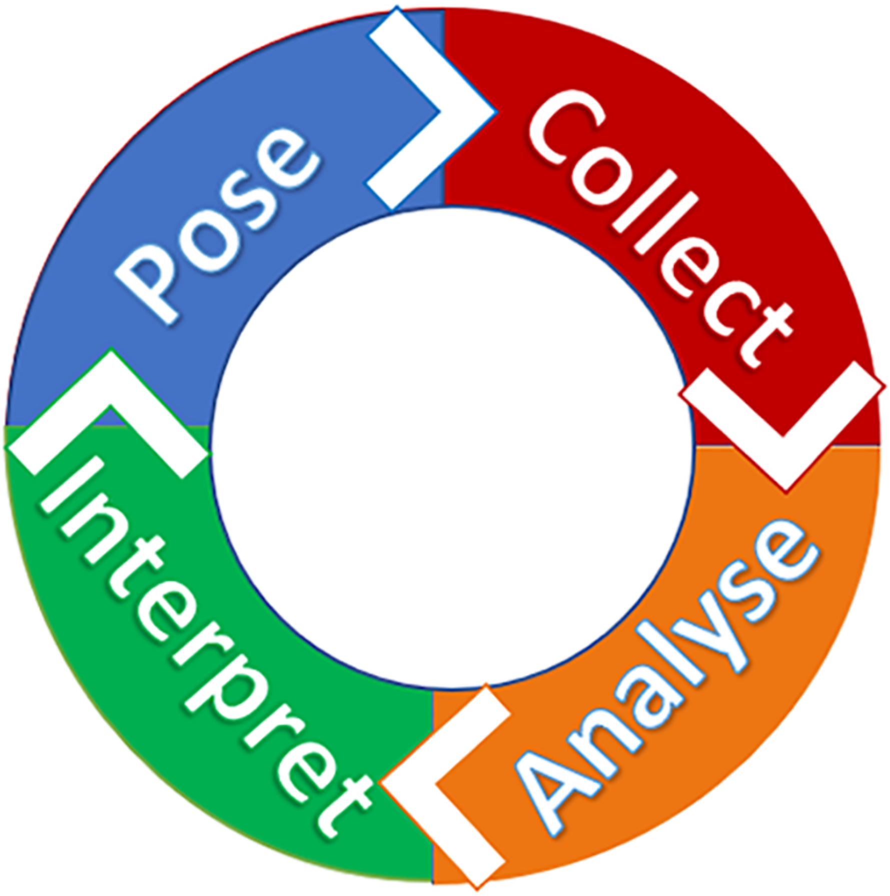


Figure 6 The PCAI cycle

Example

Here is an example of a simple statistical investigation with young learners.

Pose: learners want to find out which type of pet is favoured by their classmates (Figure 7). The question that they will investigate is: '*What is the most popular pet?'*

This could be further refined as: *'What is the most popular pet in our class?'*

Figure 7 A miscellany of pets

Collect: learners will ask each of their classmates what their favourite pet is, and record the results in a tally chart, similar to the one in Figure 8.

Favourite pet	Tally	Frequency
cat	卌 |||	8
dog	卌 卌		11
Fish	||	2
rabbit	卌	5
other	||||	4

Figure 8 A tally and frequency chart

Analyse: learners will add up the totals in their tally charts, to compare the popularity of each pet. They will then produce a bar chart to present this data visually (Figure 9).

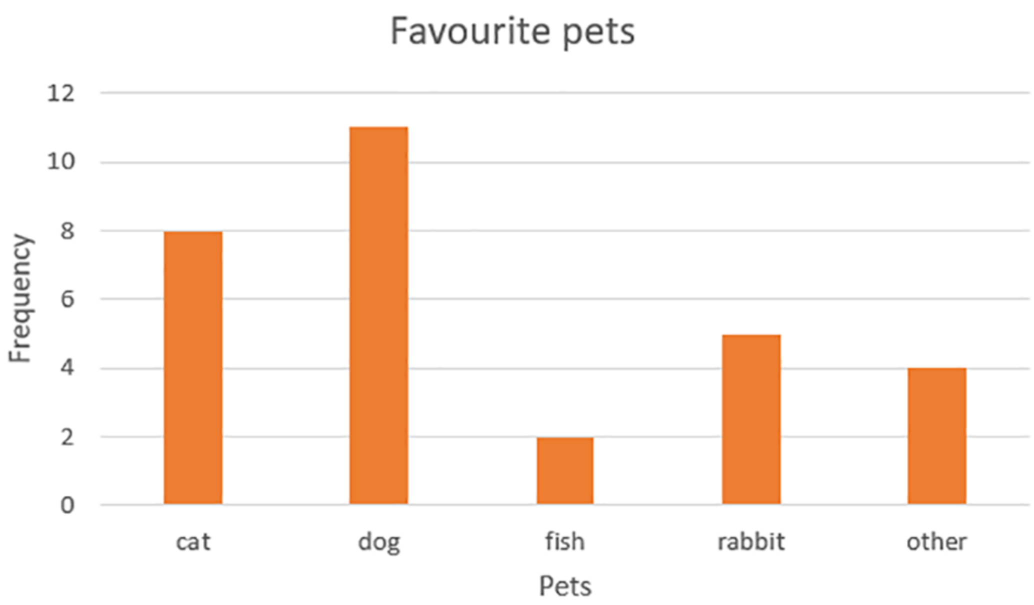


Figure 9 A bar chart

Interpret: learners will look at their bar charts and totals from their table to decide which is the most popular pet in their class.

This is a very simple example of a statistical investigation, but the PCAI cycle can be used with learners of all ages and experiences as the question posed, data collection method, data analytics and interpretation of data can all be varied in levels of sophistication.

For instance, in the UK, older students may use a large data set from the Office for National Statistics to investigate employment levels, using box plots to compare employability in different ages groups.

2.2 Analysing data using measures of central tendency and spread

The **mean**, **median** and **mode** are **measures of central tendency**. Each of these **averages** gives an indication of the typical or central value in the data set.

The **range** is a **measure of spread**. It tells us how much a data sample is spread out or scattered.

These measures can be used to summarise a data set and help us to make conclusions about what the data shows.

The mean

The mean is the most common measure of average. To calculate the mean, add the numbers together and divide the total by the amount of numbers:

Mean = sum of numbers ÷ amount of numbers

Activity 8 Sharing sweets fairly

Allow 5 minutes

Three friends have each been given some sweets (Figure 10).

Peter has 3 sweets, Aisha has 8 sweets and Holly has 1 sweet.

How could the sweets be shared fairly between the three friends?

Figure 10 Sharing sweets with friends

Discussion

This question is asking for the mean average. Between the 3 children there are 12 sweets. Dividing this by 3 would mean that each friend receives 4 sweets.

The mean average is 4.

Activity 9 Reflecting

Allow 5 minutes

How would you present the question above to your learners?

How would you help them to understand what the mean represents?

Discussion

Using concrete resources such as counters, cubes or coins can help learners to understand the concept of the mean average (Figure 11). They can physically move the objects to share them out evenly.

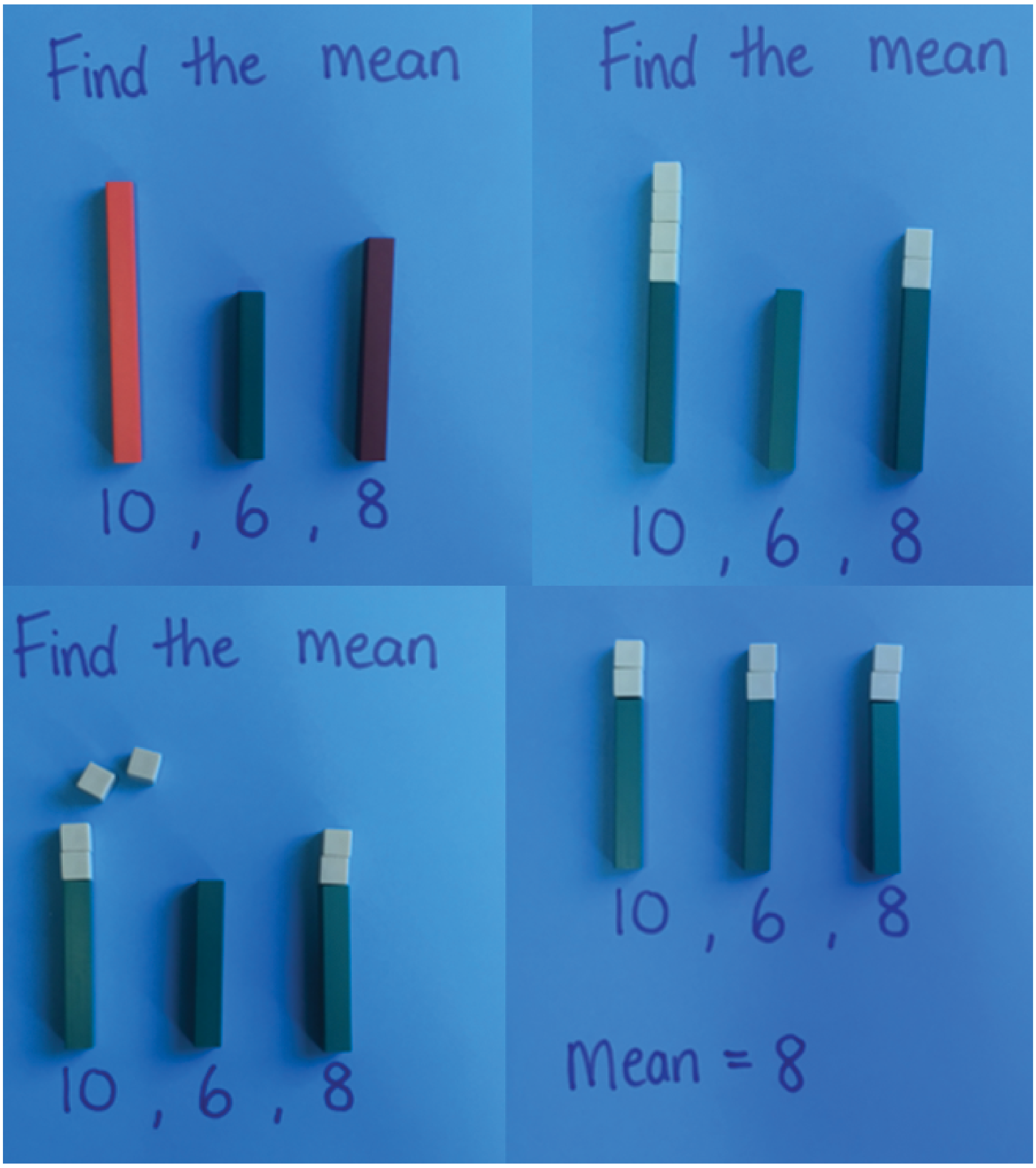


Figure 11 Finding the mean with concrete resources

This can become more challenging when learners are required to find fractions of amounts.

For example:

If four learners have 3, 5, 4 and 2 cubes respectively, what is the mean number of cubes?

3 + 5 + 4 + 2 = 14

14 ÷ 4 = 3½

IIn this example, cubes could be used to show that each learner gets 3 cubes and that there are 2 cubes left over. Depending on the age and experience of the learners, they can then record the mean as 3 remainder 2, or they can discuss how to share the remaining two cubes between four.

It can be helpful to have objects which can be split in half, when working on problems like this for the first time, including edible resources!

Another way would be to use **pictorial representations** of the problem (see Figures 12 to 14).

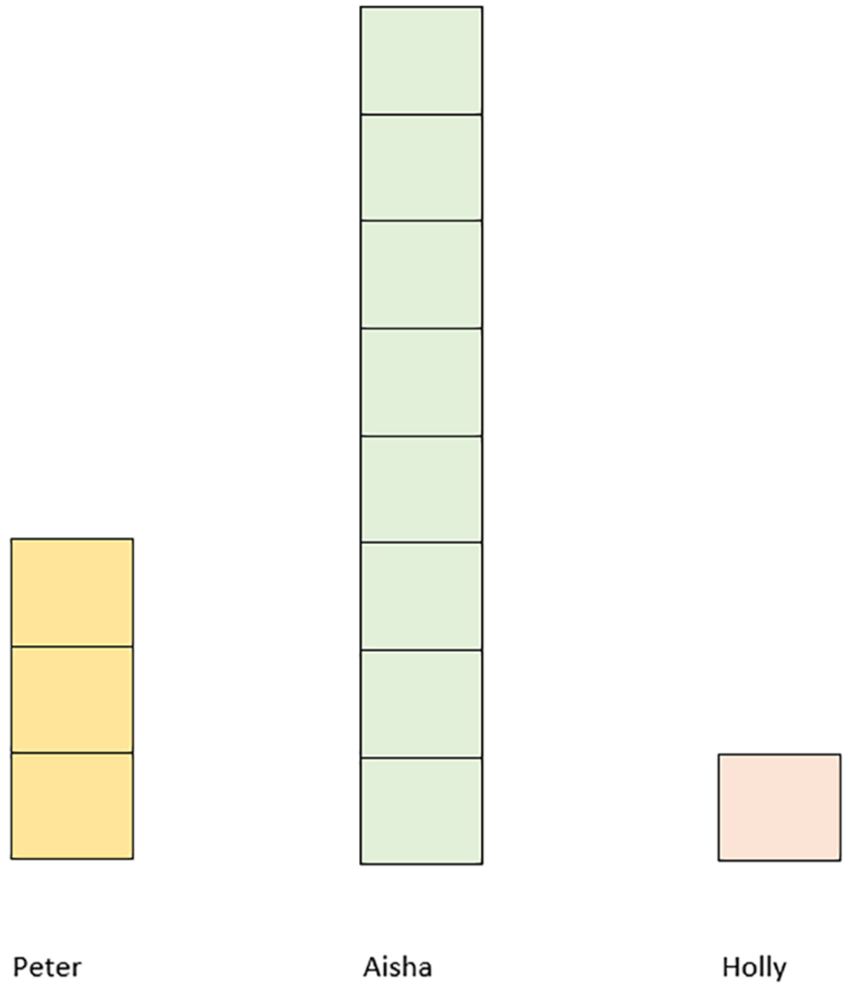


Figure 12 A diagram which represents the number of sweets each child has