MST124

Essential mathematics 1

Differentiation

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Introduction

This unit is the first of three that together introduce the fundamentally important topic of *calculus*. Calculus provides a way of solving many mathematical problems that can't be solved using algebra alone. It's the basis of essential mathematical models in areas such as science, engineering, economics and medicine, and is a fascinating topic in its own right.

In essence, calculus allows you to work with situations where a quantity is continuously changing and its rate of change isn't necessarily constant. As a simple example, imagine a man walking along a straight path, as shown in Figure 1. His displacement from his starting point changes continuously as he walks. As you've seen, the rate of change of displacement is called *velocity*.

Figure 1 A man's displacement along a path

Since the man's displacement and velocity are along a straight line, they are one-dimensional vectors, and hence can be represented by scalars, with direction along the line indicated by the signs (plus or minus) of the scalars, as in Subsection 2.4 of Unit 2, and as mentioned near the start of Subsection 5.1 of Unit 5. In the calculus units in this module, we'll work with displacement and velocity only along straight lines, and hence we'll always represent these quantities by scalars.

If the man's velocity is constant, then it's straightforward to work with the relationship between the time that he's been walking and his displacement from his starting point. As you saw in Unit 2, you just use the equation

$$\text{displacement} = \text{constant velocity} \times \text{time}.$$

For example, if the man walks at a constant velocity of 6 kilometres per hour, then after two hours his displacement is

$$(6 \times 2)\,\text{km} = 12\,\text{km}.$$

However, suppose that the man doesn't walk at a constant velocity. For example, he might become more tired as he walks, and hence gradually slow down. Then the relationship between the time that he's been walking and his displacement is more complicated. Calculus allows you to deal with relationships like this.

Calculus applies to all situations where one quantity changes smoothly with respect to another quantity. In the example above, the displacement of the walking man changes with respect to the time that he's been walking. Similarly, the temperature in a metal rod with one end near a heat source changes with respect to the distance along the rod. Similarly again, the air pressure at a point in the Earth's atmosphere changes with respect to the height of the point above the Earth's surface, the concentration of a prescription drug in a patient's bloodstream changes with respect to the time since the drug was administered, and the total cost of manufacturing many copies of a product changes with respect to the quantity of the product manufactured. You can see from the diversity of these examples just how widely applicable calculus is.

Basic calculus splits into two halves, known as **differential calculus** and **integral calculus**. (In this context 'integral' is pronounced with the emphasis on the 'int' rather than on the 'eg'.) Roughly speaking, in differential calculus, you start off knowing the values taken by a changing quantity throughout a period of change, and you use this information to find the values taken by the rate of change of the quantity throughout the same period. For example, suppose that you're interested in modelling the man's walk as he gradually slows down. You might have worked out a formula that expresses his displacement at any moment during his walk in terms of the time that he's been walking. Differential calculus allows you to use this information to deduce his *velocity* (his rate of change of displacement) at any moment during his walk.

In integral calculus, you carry out the opposite process to differential calculus. For example, suppose that you've modelled the man's walk by finding a formula that expresses his velocity at any moment during his walk in terms of the time that he's been walking. Integral calculus allows you to use this information to deduce his *displacement* at any moment during his walk.

Surprisingly, you can also use integral calculus to solve some types of problem that at first sight seem to have little connection with rates of change. For example, you can use it to find the exact area of a shape whose boundary is a curve or is made up of several curves.

This unit, together with the first half of Unit 7, introduces differential calculus, while the second half of Unit 7, together with Unit 8, introduces integral calculus. You'll also use calculus in Unit 11, *Taylor polynomials*.

The name ‘calculus’ is actually a shortened version of the historical name given to the subject, which is ‘the calculus of infinitesimals’. The word ‘calculus’ just means a system of calculation. The word comes from Latin, in which ‘calculus’ means ‘stone’ – the link is in the use of stones for counting.

The calculus of infinitesimals became so overwhelmingly important compared to other types of calculus that the word ‘calculus’, used alone, is now always understood to refer to it.

An ‘infinitesimal’ was regarded as an infinitely small part of something. When you consider an object’s velocity, for example, in the calculus of infinitesimals, you don’t consider its *average velocity* over some period of time, but rather its ‘instantaneous velocity’ – the velocity that it has during an infinitely small interval of time.

Although the ideas behind calculus are explained in quite a lot of detail in this unit, and in the later calculus units in the module, many of the explanations aren’t as mathematically precise and rigorous as it’s possible to make them, and the proofs of some results and formulas aren’t given at all. For example, the idea of a *limit* of an expression is introduced, but this idea is described in an intuitive way: the unit doesn’t give a precise mathematical definition.

The reason for this is that it’s quite complicated to make the ideas of calculus absolutely precise. At this stage in your studies it’s not appropriate for you to learn how this can be done, because the necessary profusion of small details would make it harder for you to understand the main ideas. Instead, the precise mathematical ideas behind calculus are covered in the subject area known as *real analysis*. You might study this subject at Level 2, depending on your chosen study programme.

The fundamental ideas of calculus were developed in the 1600s, independently by Isaac Newton in England and Gottfried Wilhelm Leibniz in Germany. (‘Leibniz’ is pronounced as ‘Libe-nits’.) Neither Newton nor Leibniz made their ideas rigorous – this work was done later by other mathematicians. There’s more about Newton and Leibniz later in the unit.

1 What is differentiation?

This first section introduces you to the fundamental ideas of differential calculus.

1.1 Graphs of relationships

Throughout Units 6, 7 and 8, you'll learn the ideas of basic calculus by thinking in terms of graphs. As you've seen, the relationship between two quantities, such as the time that a man has been walking and his displacement, can be represented by a graph. For example, the two displacement–time graphs in Figure 2 represent a man's walk along a straight path, on two different occasions.

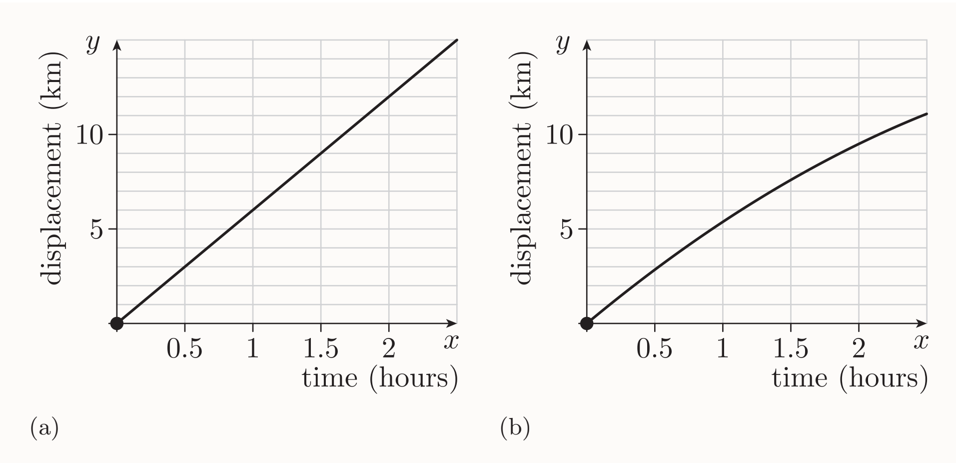


Figure 2 Displacement–time graphs for a man walking along a straight path (a) with constant velocity (b) with decreasing velocity

As you can see, the first graph is a straight line. You saw in Unit 2 that if a graph representing the relationship between two quantities is a straight line, then the quantity on the vertical axis is changing at a constant rate with respect to the quantity on the horizontal axis, and the gradient of the graph is this constant rate of change. So the first graph represents a walk in which the man's rate of change of displacement with respect to time, that is, his velocity, is constant. The gradient of the graph is his constant velocity – you can see that it is about $6\,\mathrm{km\,h^{-1}}$.

The second graph is curved. This graph represents a walk in which the man gradually slows down as time goes on – you can see that the longer he's been walking, the less his displacement changes in a given length of time.

Activity 1 *Working with a curved displacement–time graph*

Estimate from the graph in Figure 2(b) roughly how far the man walks in the first half-hour and in the final half-hour of his walk.

If a graph that represents the relationship between two quantities is curved, then there's no single gradient value that applies to the whole graph. However, the graph has a gradient at *each point* on the graph – you'll learn in the next subsection how this is defined. As with a straight-line graph, the gradient at each point is the rate of change of the quantity on the vertical axis with respect to the quantity on the horizontal axis; the only difference is that it takes different values at different points. For example, in Figure 2(b), the gradient of the graph at each point is the man's velocity at a moment in time, and it gradually decreases as time goes on.

So in this unit you'll begin your study of calculus by looking at how the idea of gradient applies to curved graphs.

All the relationships between two variables that you'll consider in the calculus units in this module are those in which one variable is a *function* of the other. (For instance, in the example of the man's walk, his displacement is a function of the time that he's been walking.) So, in every graph in these units, every value on the horizontal axis corresponds to at most one value on the vertical axis. As in Unit 3, we'll use the word 'function' to mean 'real function'; that is, a function whose input and output values are real numbers. This is the only type of function that we'll deal with in the calculus units in this module.

1.2 Gradients of curved graphs

As mentioned in earlier, in this section you'll learn about gradients of curved graphs. We'll begin with a quick reminder about gradients of straight lines, as these ideas will be crucial in what follows.

As you saw in Unit 2, the gradient of a straight line is a measure of how steep it is. To calculate the gradient of a straight line, you choose any two points on it and find the *run* and the *rise* from the first point to the second point. The *run* is the change in the x-coordinates, and the *rise* is the change in the y-coordinates, as illustrated in Figure 3. Then

$$\text{gradient} = \frac{\text{rise}}{\text{run}}.$$

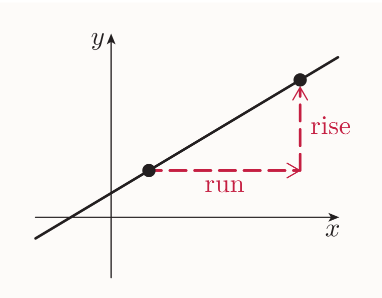


Figure 3 The run and the rise between two points on a straight-line graph

The run and the rise from one point to another on a straight line can be positive, negative or zero, depending on whether the relevant coordinates increase, decrease or stay the same from the first point to the second point. If the *run* is zero, which happens when the line is vertical, then the gradient of the line is undefined, since division by zero isn't possible. Vertical lines are the only straight lines that don't have gradients.

If the two points that you choose to calculate the gradient of a line are (x_1, y_1) and (x_2, y_2), as illustrated in Figure 4, then

$$\text{run} = x_2 - x_1 \quad \text{and} \quad \text{rise} = y_2 - y_1.$$

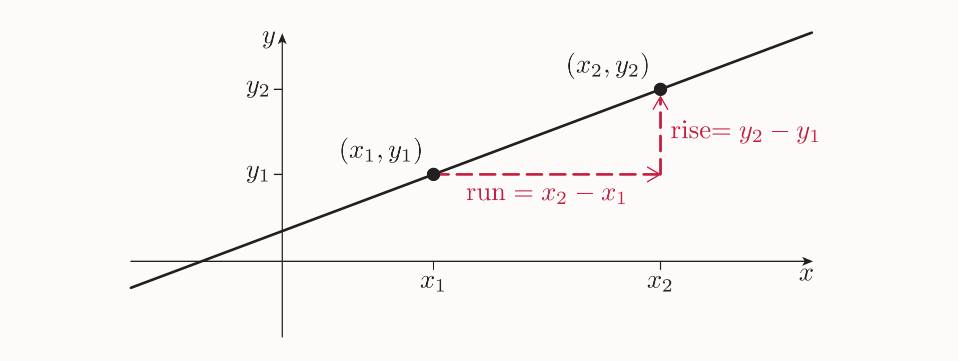


Figure 4 The run and the rise in terms of the coordinates of the two points

These equations give the formula below.

Gradient of a straight line

The gradient of the straight line through the points (x_1, y_1) and (x_2, y_2), where $x_1 \neq x_2$, is given by

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Remember that when you use this formula to calculate the gradient of a straight line, it doesn't matter which point you take to be the first point, (x_1, y_1), and which you take to be the second point, (x_2, y_2), as you get the same result either way. For example, using the formula to calculate the gradient of the line through the points $(1, 8)$ and $(5, 2)$, which is shown in Figure 5, gives either

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{2-8}{5-1} = \frac{-6}{4} = -\frac{3}{2}$$

or

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{8-2}{1-5} = \frac{6}{-4} = -\frac{3}{2}.$$

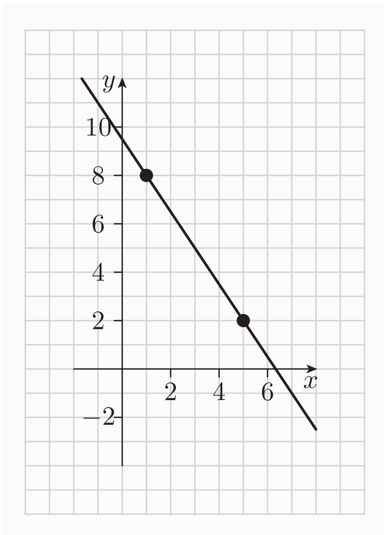


Figure 5 The straight line through the points $(1, 8)$ and $(5, 2)$

You saw the following facts in Unit 2. A straight-line graph that slopes *down* from left to right has a *negative* gradient, one that's *horizontal* has a gradient of *zero*, and one that slopes *up* from left to right has a *positive* gradient. The steeper the graph, the larger the magnitude of the gradient.

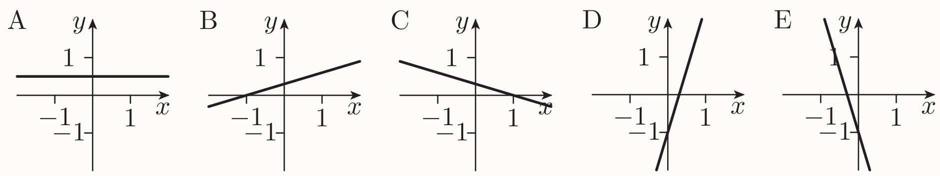
Activity 2 *Relating gradients to graphs*

Match each of the following descriptions to the appropriate graph.

(a) Large negative gradient (b) Small negative gradient

(c) Zero gradient (d) Small positive gradient

(e) Large positive gradient



Now let's look at how the idea of gradient applies to curved graphs. For example, consider the graph of the equation $y = x^2$, which is shown in Figure 6.

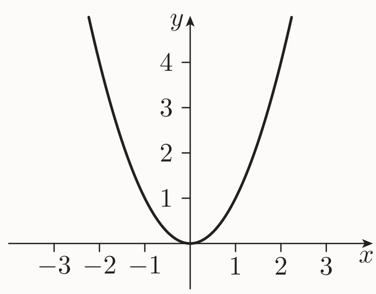


Figure 6 The graph of $y = x^2$

This graph has no single gradient value, since it's not a straight line, but if you choose any particular point on it, then it has a gradient at that point.

The word 'tangent' comes from the Latin word 'tangere', which means 'to touch'. The English word 'tangible', which means 'capable of being touched', comes from the same Latin word. You might remember from Unit 1 that the word 'integer' also comes from this Latin word (an integer is a whole, or 'untouched', number).

To understand what's meant by the gradient of a curved graph at a particular point, consider the point on the graph of $y = x^2$ marked in Figure 7(a). Imagine that you're tracing your pen tip along the graph, but when it reaches the marked point you just carry on moving it in the direction in which it's been moving, instead of following the graph. Then it will move along the straight line drawn in Figure 7(b). No matter whether you trace your pen tip along the graph towards the point from the left or the right, you'll end up moving it along the same straight line, just in different directions. This straight line 'just touches' the graph at the marked point, and is called the **tangent** to the graph at that point.

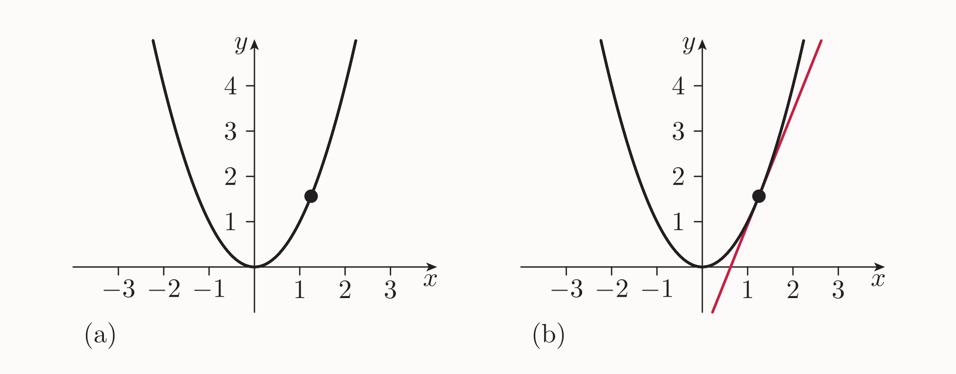


Figure 7 (a) A particular point on the graph of $y = x^2$ (b) the tangent to the graph of $y = x^2$ at that point

The tangent to any curved graph at a particular point can be defined in a similar way. Because the tangent to a graph at a point has the same steepness as the graph at that point, we make the following definition.

The **gradient** of a graph at a particular point is the gradient of the tangent to the graph at that point.

Although this definition is made with curved graphs in mind, it also applies to straight-line graphs. To see this, consider a point on a straight-line graph, as illustrated in Figure 8. If you trace your pen tip along the graph towards the point and continue moving it in the same direction when you get to the point, then it will just continue moving along the graph. So the tangent to the graph at that point is just the straight-line graph itself, and hence the gradient of the graph at the point is just the gradient of the whole straight line, as you'd expect.

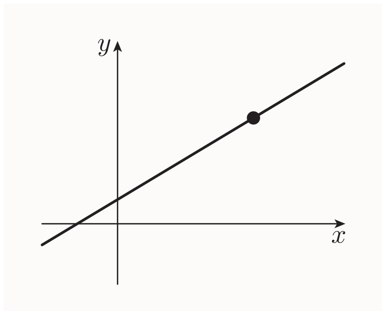


Figure 8 A point on a straight-line graph

It's possible for a graph to have no gradient at a particular point. For example, consider the graph of $y = |x|$, which is shown in Figure 9(a). Imagine tracing your pen tip along this graph towards the origin, and continuing to move it in the same direction when you get to the origin. If you trace it along the graph towards the origin from the left, then you'll end up moving it along a *different* straight line than if you trace it along the graph towards the origin from the right. So there's no tangent to the graph at the origin, and hence the graph doesn't have a gradient at this point. In general, if a graph has a 'sharp corner' at a point, then it has no gradient at that point.

As another example, consider the graph in Figure 9(b). You can trace your pen tip along the graph to the origin from the left, but you can't do the same from the right, because you can't reach the origin that way. So this graph has no tangent at the origin, and hence has no gradient at the origin. In general, if a graph has a 'break' (known as a **discontinuity**) at a particular point, then the graph has no tangent, and hence no gradient, at this point.

As a third example, consider the graph in Figure 9(b). It has a tangent at the origin, but the tangent is vertical, and hence itself has no gradient. So this graph has no gradient at the origin. In general, if a graph has a vertical tangent at a point, then the graph has no gradient at that point.

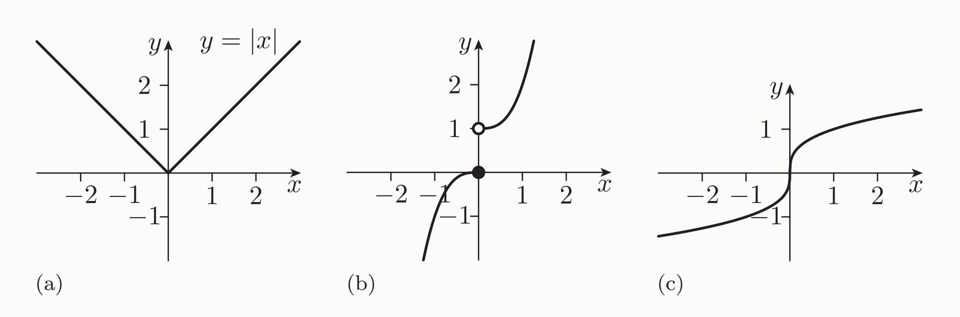


Figure 9 Graphs with points at which there is no gradient

Now let's think about how you could find the gradient of a curved graph at a particular point at which it does have a gradient. An obvious thing to do would be to draw the tangent at that point, choose two points on it and use them to calculate the gradient in the usual way, but this wouldn't be a very accurate method, as it's difficult to draw a tangent accurately by eye.

The next subsection shows you a better method. It considers the particular example of the graph of $y = x^2$.

1.3 Gradients of the graph of $y = x^2$

The graph of $y = x^2$ has a gradient at *every* point, because it has a tangent, which is not vertical, at every point.

Let's try to find the gradient of this graph at the point $(1, 1)$, which is shown in Figure 10. That is, we want to find the gradient of the tangent to the graph at this point, which is also shown in Figure 10.

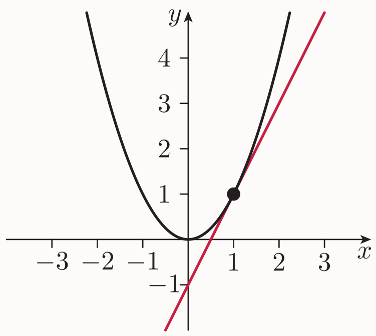


Figure 10 The point $(1, 1)$ on the graph of $y = x^2$ and the tangent at this point

The way to find the gradient at $(1, 1)$ is to begin by thinking about how you could find an *approximate* value for this gradient. Here's how you can do that. You choose a second point on the graph of $y = x^2$, fairly close to $(1, 1)$, as illustrated in blue in Figure 11. The straight line that passes through both $(1, 1)$ and the second point is an approximation for the tangent to the graph at $(1, 1)$. So the gradient of this line, which you can calculate using the two points on the line in the usual way, is an approximation for the gradient of the graph at $(1, 1)$.

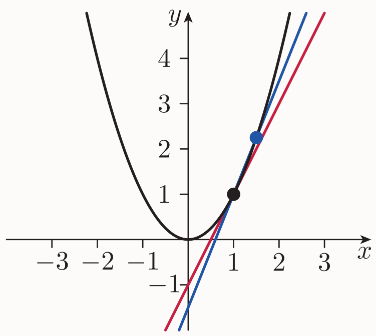


Figure 11 An approximation to the tangent to the graph of $y = x^2$ at $(1, 1)$

The closer to $(1, 1)$ you choose the second point to be, the better the approximation will be. For example, the second point in Figure 12 will give a better approximation than the second point in Figure 11.

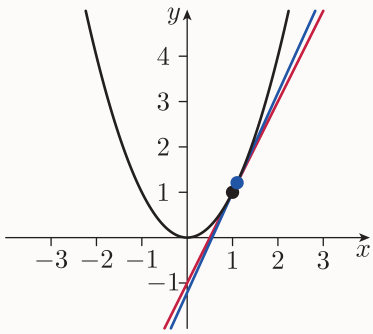


Figure 12 Another approximation to the tangent to the graph of $y = x^2$ at $(1, 1)$

As an example, let's calculate the approximation to the gradient at $(1, 1)$ that you get by choosing the second point to be the point with x-coordinate 1.1, which is the point shown in blue in Figure 12. Since the equation of the graph is $y = x^2$, the y-coordinate of this second point is $1.1^2 = 1.21$. So the second point is $(1.1, 1.21)$. The gradient of the line through $(1, 1)$ and $(1.1, 1.21)$ is

$$\frac{\text{rise}}{\text{run}} = \frac{1.21 - 1}{1.1 - 1} = \frac{0.21}{0.1} = 2.1.$$

So an approximate value for the gradient of the graph at $(1, 1)$ is 2.1.

The second point that you choose on the graph can lie either to the left or to the right of $(1, 1)$. In the next activity, you're asked to calculate the approximation to the gradient that you get by choosing a second point on the graph that lies to the left of $(1, 1)$, as illustrated in Figure 13.

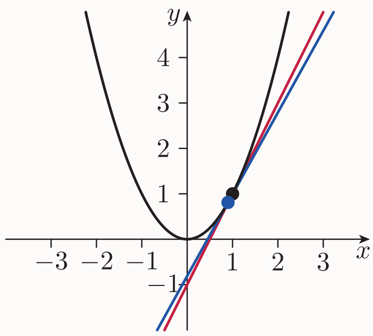


Figure 13 Yet another approximation to the tangent to the graph of $y = x^2$ at $(1, 1)$

Activity 3 *Calculating an approximation to the gradient at a point*

Calculate the approximation to the gradient of the graph of $y = x^2$ at the point with x-coordinate 1 that you get by taking the second point on the graph to be the point with x-coordinate 0.9.

Notice that although you can take the second point to be as close to $(1, 1)$ as you like, either on the left or on the right, you can't take it to be *actually equal* to $(1, 1)$. This is because you can't apply the formula for gradient to the points $(1, 1)$ and $(1, 1)$, since that would involve division by zero, which is meaningless.

In the next activity, you can use a computer to see the gradients of some more lines that pass through $(1, 1)$ and a second point close to $(1, 1)$, calculated by using the coordinates of the two points.

Activity 4 *Calculating more approximations to the gradient at a point*

Open the applet *Approximations to the gradient at a point*. Make sure that the function is set to $f(x) = x^2$ and that the point on its graph is set to the point with x-coordinate 1. Move the second point closer and closer to the original point, and observe the gradients of the resulting lines, which are calculated by the computer using the coordinates of the two points. What do you think is the gradient of the tangent at $(1, 1)$?

Table 1 gives the gradients of some of the lines that you might have seen in Activity 4.

Table 1 The gradients of lines through $(1, 1)$ and a second point close to $(1, 1)$ on the graph of $y = x^2$

x-coordinate of second point	0.8	0.9	0.99	0.999	1.001	1.01	1.1	1.2
Gradient of line	1.8	1.9	1.99	1.999	2.001	2.01	2.1	2.2

From Activity 4 and Table 1, it looks as if the closer the second point is to $(1, 1)$, the closer the gradient of the line through $(1, 1)$ and the second point is to 2. So it looks as if the gradient of the graph at $(1, 1)$ might be exactly 2.

Neither Activity 4 nor Table 1 shows this for certain, however. For example, Table 1 doesn't rule out the possibility that the gradient at $(1, 1)$ might be some other number between 1.999 and 2.001. However, you can confirm that the gradient is *exactly* 2 by using an algebraic version of the method above.

Let's use the variable h to denote the increase in the x-coordinate from the original point $(1, 1)$ to the second point. The value of h can be either positive or negative, but not zero, since the second point can be either to the right or to the left of $(1, 1)$, but can't be $(1, 1)$ itself. The x-coordinate of the second point is $1 + h$, so, since the equation of the graph is $y = x^2$, the y-coordinate of the second point is $(1 + h)^2$. These coordinates are shown in Figure 14.

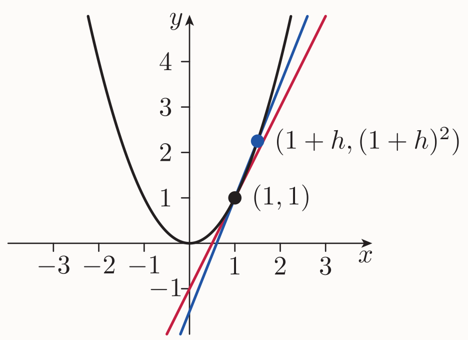


Figure 14 The points $(1, 1)$ and $(1 + h, (1 + h)^2)$ on the graph of $y = x^2$, and the line through them

The gradient of the line that passes through the two points is

$$\frac{\text{rise}}{\text{run}} = \frac{(1 + h)^2 - 1}{(1 + h) - 1}.$$

This expression can be simplified as follows:

$$\begin{aligned}
\text{gradient} &= \frac{(1 + h)^2 - 1}{(1 + h) - 1} \\
&= \frac{1 + 2h + h^2 - 1}{1 + h - 1} \\
&= \frac{2h + h^2}{h} \\
&= \frac{h(2 + h)}{h} \\
&= 2 + h.
\end{aligned}$$

So the gradient of the line that passes through $(1, 1)$ and $(1 + h, (1 + h)^2)$ is given by the expression $2 + h$. (You can see examples of this in Table 1.)

Now think about what happens to this gradient as the second point gets closer and closer to $(1, 1)$. That is, think about what happens as the value of h gets closer and closer to zero.

As the value of h gets closer and closer to zero, the gradient of the line, $2 + h$, gets closer and closer to 2. So the gradient of the graph at $(1, 1)$ must indeed be exactly 2, as we expected.

The fact that, as h gets closer and closer to zero, the value of the expression $2+h$ gets closer and closer to 2 is expressed mathematically by saying that

$2+h$ **tends** to 2 as h **tends** to 0,

or

the **limit** of $2+h$ as h **tends** to zero is 2.

So we've now succeeded in finding the gradient of the graph of $y=x^2$ at the point $(1,1)$. You could use the same algebraic method to find the gradient of the graph at any other point.

A cleverer thing to do, however, is to use the same method to find the gradient of the graph of $y=x^2$ at a *general point* on the graph, whose x-coordinate is denoted by x. This will give you a formula for the gradient of the graph *at any point*, in terms of the x-coordinate of the point. You can then find the gradient of the graph at any particular point by just substituting into the formula, instead of having to go through the algebraic method again.

So let's now apply the algebraic method to find the gradient of the graph of $y=x^2$ at the general point whose x-coordinate is denoted by x. The situation is shown in Figure 15. The point at which we want to find the gradient is (x,x^2). The second point on the graph is $(x+h,(x+h)^2)$, where h is a positive or negative number, but not zero.

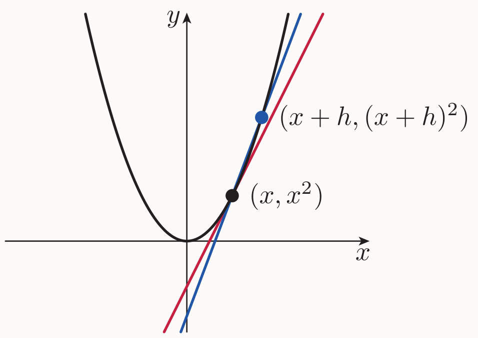


Figure 15 The points (x,x^2) and $(x+h,(x+h)^2)$ on the graph of $y=x^2$, and the line through them

The gradient of the line that passes through the two points is

$$\frac{\text{rise}}{\text{run}}=\frac{(x+h)^2-x^2}{(x+h)-x}.$$

This expression can be simplified as follows:

$$\begin{aligned}\text{gradient} &= \frac{(x+h)^2 - x^2}{(x+h) - x}\\ &= \frac{x^2 + 2xh + h^2 - x^2}{x + h - x}\\ &= \frac{2xh + h^2}{h}\\ &= \frac{h(2x+h)}{h}\\ &= 2x + h.\end{aligned}$$

As the second point gets closer and closer to the original point, the value of h gets closer and closer to 0, and the gradient of the line, $2x + h$, gets closer and closer to $2x$. So the gradient of the graph of $y = x^2$ at the point whose x-coordinate is denoted by x is given by the formula

$$\text{gradient} = 2x. \qquad (1)$$

For example, this formula tells you that at the point with x-coordinate 1,

$$\text{gradient} = 2 \times 1 = 2,$$

as we found earlier.

In the next activity you're asked to use formula (1) to find the gradient of the graph of $y = x^2$ at two more points.

Activity 5 *Finding the gradients of the graph of $y = x^2$ at particular points*

Use formula (1) to find the gradient of the graph of $y = x^2$ at the point with x-coordinate 3, and at the point with x-coordinate -1.5.

Figure 16 shows the tangents to the graph of $y = x^2$ at the points with x-coordinates 3 and -1.5. You can see, by estimating the rise and run between two points on each tangent, that the gradients of these tangents do seem to be at least roughly the values calculated in the solution to Activity 5.

In general, if you have the equation of a graph, then it's often possible to use an algebraic method similar to the one that you've seen in this subsection to find a formula for the gradient of the graph. You'll see some more examples in the next subsection.

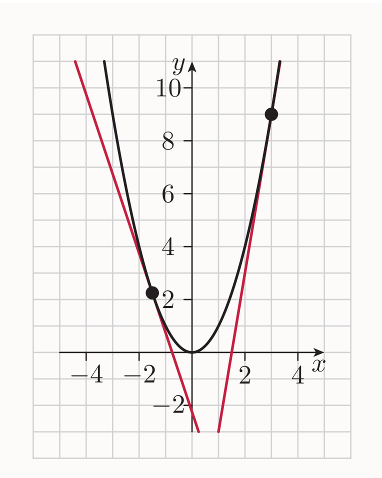


Figure 16 Two tangents to the graph of $y = x^2$

1.4 Derivatives

In Subsection 1.3 you saw how to find a formula for the gradients of the graph of $y = x^2$. In this subsection you'll see how you can use the same method to find formulas for the gradients of some other graphs. Before you do that, it's useful for you to learn some terminology and notation that are used when working with such formulas.

Consider a function f and a particular input value x. The point on the graph of f that corresponds to the input value x is $(x, f(x))$. If the graph of f has a gradient at the point $(x, f(x))$, then we say that f is **differentiable** at x. For example, you've seen that the function $f(x) = x^2$ is differentiable at every value of x. If the graph of f doesn't have a gradient at the point with $(x, f(x))$ (because the graph has no tangent at that point, or because the tangent is vertical), then f isn't differentiable at x. For example, you've seen that the function $f(x) = |x|$ isn't differentiable at 0. Similarly, if a function f isn't even defined at a particular input value x, then it's not differentiable at x.

You've seen that, usually, the gradient of the graph of a function f varies depending on which value of x you're considering. It's convenient to think of these gradients as defining a new function, related to f. The rule of the new function is:

if the input value is x, then the output value is
the gradient of the graph of f at the point with $(x, f(x))$.

This new function is called the **derivative** (or **derived function**) of the function f, and is denoted by f' (which is read as'f prime' or 'f dash' or 'f dashed'). The domain of the derivative consists of all the values at which f is differentiable. The process of finding the derivative of a given function f is called **differentiation**, and when we carry out this process, we say that we're **differentiating** the function f.

For example, in the previous subsection we *differentiated* the function $f(x) = x^2$, and we found that its *derivative* has the rule $f'(x) = 2x$.

The word 'derivative' can be applied to expressions, as well as to functions. For example, rather than saying that

the derivative of $f(x) = x^2$ is $f'(x) = 2x$,

you can simply say that

the derivative of x^2 is $2x$.

The word 'derivative' is also used with a slightly different meaning. The *value* of the derivative of a function f at a particular input value x is called the **derivative of f at x**. For example, you saw in the last subsection that the derivative of the function $f(x) = x^2$ at 1 is $f'(1) = 2$.

The notation for derivatives can of course be used with letters other than the standard ones, f for the function and x for the input variable. For example, if the function g is given by $g(t) = t^2$, then the gradient of the graph of g at the point $(t, g(t))$ is given by the formula $2t$, and we express this by writing

$$g'(t) = 2t.$$

You should now be ready to see some more examples of finding formulas for the gradients of graphs – that is, formulas for derivatives. We'll begin by setting out the general method that can be used to find such formulas. This is just the algebraic method that you saw in Subsection 1.3, where the formula for the derivative of the function $f(x) = x^2$ was found. It's known as **differentiation from first principles**.

1.4.1 Differentiation from first principles

Suppose that f is any function. Let x denote any value in the domain of f such that f is differentiable at x (that is, such that the graph of f has a gradient at the point $(x, f(x))$, as illustrated in Figure 17. Now consider a second point on the graph, with coordinates $(x + h, f(x + h))$, where h is a positive or negative number, but not zero.

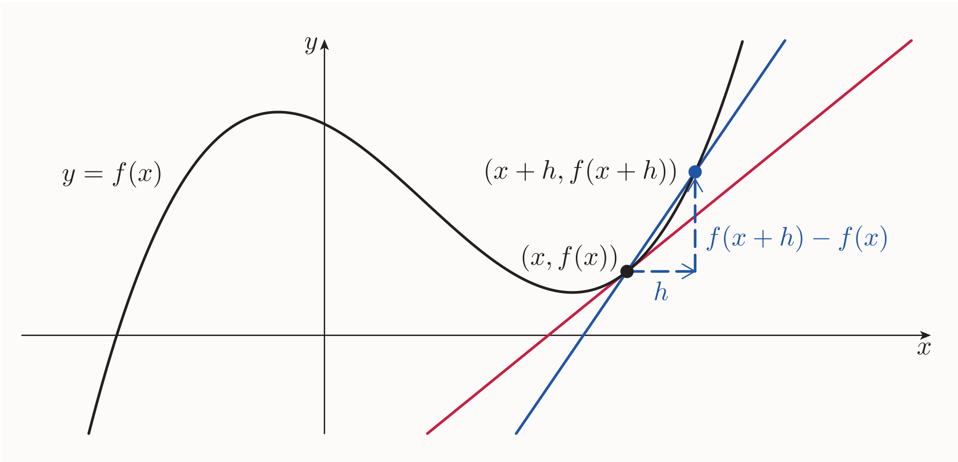


Figure 17 The points $(x, f(x))$ and $(x + h, f(x + h))$ on the graph of $y = f(x)$, and the line through them

The gradient of the line that passes through the two points is

$$\frac{\text{rise}}{\text{run}} = \frac{f(x+h) - f(x)}{(x+h) - x},$$

and this expression can be simplified slightly to give

$$\frac{f(x+h) - f(x)}{h}. \tag{2}$$

This expression is known as the **difference quotient** for the function f at the value x. As the second point $(x + h, f(x + h))$ gets closer and closer to the first point $(x, f(x))$, that is, as the value of h gets closer and closer to zero, the value of the difference quotient gets closer and closer to the gradient of the graph at the point $(x, f(x))$. That is, it gets closer and closer to $f'(x)$. (Remember that the value of h can never actually be 0.)

So, to find a formula for $f'(x)$, you need to consider what happens to the difference quotient for f at x, as h gets closer and closer to zero, taking either positive or negative values as it does so. In other words, you need to find, in terms of x, the *limit* of the difference quotient as h tends to zero. You saw this procedure carried out for the function $f(x) = x^2$ in the last subsection, and in the next example you'll see it carried out for the function $f(x) = x^3$.

Example 1 *Differentiating from first principles*

Differentiate from first principles the function $f(x) = x^3$.

Solution

Write down the difference quotient and use the fact that $f(x) = x^3$.

The difference quotient for the function $f(x) = x^3$ at x is

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h}.$$

Simplify the difference quotient. Start by multiplying out the term $(x+h)^3$ in the numerator.

Multiplying out $(x+h)^3$ gives

$$\begin{aligned}(x+h)^3 &= (x+h)(x+h)^2\\ &= (x+h)(x^2+2xh+h^2)\\ &= x(x^2+2xh+h^2) + h(x^2+2xh+h^2)\\ &= x^3 + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3\\ &= x^3 + 3x^2h + 3xh^2 + h^3.\end{aligned}$$

So

$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}\\ &= \frac{3x^2h + 3xh^2 + h^3}{h}\\ &= \frac{h(3x^2 + 3xh + h^2)}{h}\\ &= 3x^2 + 3xh + h^2.\end{aligned}$$

Work out what happens to the value of the difference quotient as h gets closer and closer to zero.

The second term in the final expression above contains the factor h, and the third term is h^2, so as h gets closer and closer to zero, both of these terms get closer and closer to zero. So the value of the whole expression gets closer and closer to the value of the first term, $3x^2$. That is, the formula for the derivative of the function $f(x) = x^3$ is

$$f'(x) = 3x^2.$$

The formula for the derivative of $f(x) = x^3$ found in Example 1 tells you that, for example, the gradient of the graph of $y = x^3$ at the point with x-coordinate 1 is

$$f'(1) = 3 \times 1^2 = 3,$$

and the gradient of this graph at the point with x-coordinate $\frac{1}{2}$ is

$$f'(\tfrac{1}{2}) = 3 \times \left(\tfrac{1}{2}\right)^2 = \tfrac{3}{4}.$$

You can see from Figure 18 that the gradients of the tangents to the graph at these two points do seem to be the numbers calculated using the formula.

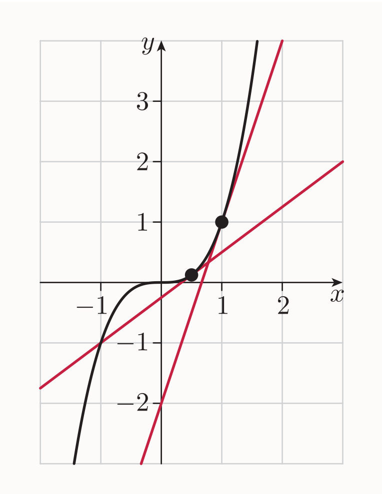


Figure 18 Two tangents to the graph of $f(x) = x^3$

In the next activity, you're asked to use differentiation from first principles to find the derivative of the function $f(x) = x^4$.

Activity 6 *Differentiating from first principles*

(a) Multiply out the expression $(x + h)^4$. To do this, start by writing $(x + h)^4 = (x + h)(x + h)^3$. Then replace the expression $(x + h)^3$ by the expansion of $(x + h)^3$ that was found in the solution to Example 1, remembering to enclose it in brackets. Finally, multiply out the brackets and collect like terms.

(b) Hence differentiate from first principles the function $f(x) = x^4$.

(c) What is the gradient of the graph of the function $f(x) = x^4$ at the point with x-coordinate $\frac{1}{4}$?

In each of the three examples of differentiation from first principles that you've seen so far, the function is differentiable at *every* value of x. If a function isn't differentiable at some values of x, then you can still differentiate it from first principles, but the process won't work for the values of x at which it's not differentiable. For these values of x the value of the difference quotient won't get closer and closer to a particular number as h gets closer and closer to zero (where h can be either positive or negative).

For example, if the graph of a function f has a sharp corner at the x-value x, then the difference quotient for f at x will get closer and closer to a particular value when h is *positive*, but will get closer and closer to a *different* value when h is *negative*. As another example, if the graph of the function has a vertical tangent at $(x, f(x))$, then the magnitude of the difference quotient for f at x will just keep getting larger and larger, without getting closer and closer to any particular value.

In general, saying that a function f is differentiable at a particular value of x is the same as saying that the difference quotient for f at x tends to a limit as h tends to zero. It must tend to the same limit for positive values of h as for negative values.

The method of differentiation from first principles is summarised in the box below. The notation '$\lim\limits_{h\to 0}$' means 'the limit as h tends to zero of'.

Differentiation from first principles

For any function f, the derivative f' of f is given by the equation

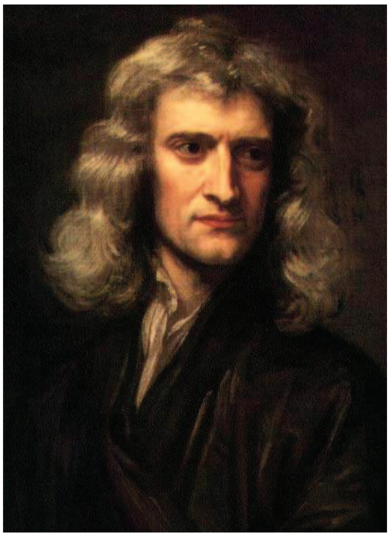
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

for each value of x in the domain of f for which this limit exists.

Differentiation from first principles can be used to find formulas for the derivatives of many of the functions that you'll need to work with. However, it's a laborious process, so usually we don't do it! Instead, in this unit and in the first half of Unit 7 you'll get to know the formulas for the derivatives of a range of standard functions, such as $f(x) = x^2$, $f(x) = x^3$, $f(x) = \sin x$, $f(x) = e^x$, and so on. You'll also learn about ways in which you can combine these formulas to obtain formulas for the derivatives of other, related functions. For example, if you know the formulas for the derivatives of $f(x) = x^2$ and $f(x) = x^3$, then you can combine them to obtain a formula for the derivative of $f(x) = x^2 + x^3$. In these ways you'll be able to find formulas for the derivatives of most of the functions that you'll need to work with.

Of course, the idea of differentiation from first principles is still needed, to find the derivatives of the standard functions, to check that the rules for combining them are valid, and to differentiate functions that aren't standard functions or combinations of standard functions.

Formulas for derivatives are powerful mathematical tools in many different situations, in both pure and applied mathematics. In this module, not only will you learn how to find such formulas, but you'll also be introduced to a few of the ways in which they can be used. You'll see many more uses of them if you go on to study further modules in mathematics, or some modules in areas such as science and economics.



Isaac Newton (1642–1727)

Gottfried Wilhelm Leibniz (1646–1716)

The invention of calculus

The history of calculus goes back to the second half of the seventeenth century, when Isaac Newton in England and Gottfried Wilhelm Leibniz in what is now Germany both independently developed the basic ideas. Newton's ideas were rooted in the applications of mathematics, while Leibniz's were rooted in pure mathematics.

Newton developed the ideas of calculus starting in about 1665. He called his ideas the 'method of fluxions' and wrote a treatise about them in 1671, which was not published in his lifetime, although its contents circulated in manuscript form, and a publication containing the method appeared in 1704. Leibniz then independently developed similar ideas, starting in about 1674. A manuscript that he wrote in 1675 includes the notation used in integral calculus to this day, as well as a standard rule for combining derivatives, the *product rule*, which you'll meet in Unit 7. Leibniz's notation for differential calculus is also still used today, as you'll see shortly. Leibniz first published work on calculus in 1684.

The two men continued to develop their ideas for the next few years. However by the early 1700s Leibniz was being accused by Newton's associates of having plagiarised Newton's work. The allegation was that Leibniz had seen some of Newton's unpublished papers and had merely invented a new notation for Newton's ideas. The ensuing bitter argument led to a Royal Society investigation, which upheld the charge. However, the investigation was largely carried out by Newton's friends, and Newton, who was President of the Royal Society, secretly guided its report. Investigations by modern historians have shown that the accusation against Leibniz was unjust. Newton and Leibniz arrived at equivalent results following different paths of discovery.

Isaac Newton and Gottfried Wilhelm Leibniz

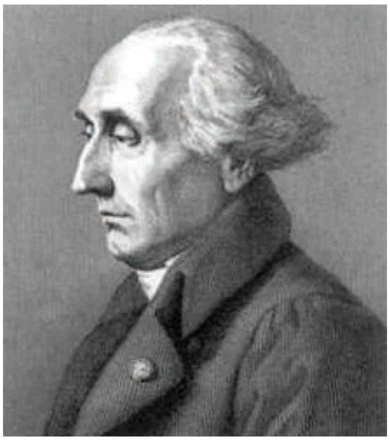
Isaac Newton was born in Lincolnshire, and studied and worked at the University of Cambridge. He was one of the world's greatest physicists, mathematicians and astronomers, and is remembered in particular for his work on classical mechanics. He did much of his initial work on calculus at his family home in Lincolnshire, while Cambridge University was closed due to an outbreak of plague. In his later life Newton largely abandoned physics and mathematics, and wrote theological tracts before becoming Master of the Mint, a highly-paid government official, in London. He also worked on alchemy throughout his life. He was knighted in 1705, but for political reasons rather than for his scientific work or public service.

Gottfried Wilhelm Leibniz was born in Leipzig, and attended university there and in Altdorf. He was a universal thinker who graduated in philosophy and law, and was self-taught in mathematics. He went on to work intensively on mathematics in Paris, before accepting the position of Counsellor and librarian at the court in Hanover, where he remained for the rest of his life. While there, he worked on many different projects, making important contributions to mathematics, philosophy, theology and history. Some of Leibniz's projects were related to his salaried position, but his employers also allowed him to work on other projects of his choosing. He was interested in formalising calculations, and constructed the first mechanical calculator that could add, subtract, multiply and divide.

Before we begin the process of building up a collection of formulas for the derivatives of standard functions, and techniques for combining them, it's useful for you to learn an alternative notation for derivatives.

1.4.2 Leibniz notation

The notation that we've been using so far, in which the derivative of a function f is denoted by f', is called **Lagrange notation** or **prime notation**. ('Lagrange' is pronounced as a French word: 'La-grawnge'.) It was invented by Joseph-Louis Lagrange, about a century after calculus was discovered. The term 'prime notation' arises from the fact that the symbol $'$ in the notation f' is often called 'prime'.



Joseph-Louis Lagrange (1736–1813)

Joseph-Louis Lagrange was an Italian-French mathematician who made important contributions in many areas, including calculus, mechanics, astronomy, probability and number theory. He was appointed as a professor of mathematics at the Royal Artillery School in Turin at the age of only 19, and was a dedicated and prolific mathematician for the rest of his life, working mainly in Turin and Berlin.

However, there's another notation, called **Leibniz notation**, invented by Gottfried Wilhelm Leibniz. (Remember that 'Leibniz' is pronounced as a German word: 'Libe-nits'.) You'll need to become familiar with both notations, as they're both used throughout this module, and throughout mathematics generally.

Each type of notation has different advantages in different situations. Generally, Lagrange notation is used when you're thinking in terms of a variable, and a function of this variable. On the other hand, Leibniz notation is often used when you're thinking more of the relationship between two variables. The distinction will become clearer as you become used to working with the two notations.

To see how derivatives are written in Leibniz notation, consider the equation $y = x^2$, which expresses a relationship between the variables x and y. You've seen that the formula for the gradient of the graph of the equation $y = x^2$ is

$$\text{gradient} = 2x.$$

In Leibniz notation this equation is written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x.$$

So the notation $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ means the same as $f'(x)$, where $y = f(x)$. It's read as 'd y by d x'. When Leibniz notation is being used, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is often referred to as the **derivative of y with respect to x**.

If you want to write the notation $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in a line of text, then you can write it as $\mathrm{d}y/\mathrm{d}x$, just as you would do for a fraction. However, although the notation looks like a fraction, it's important to remember that it isn't a fraction!

Also, you should be aware that the 'd' that's part of Leibniz notation has no meaning outside of it. In particular, although $\mathrm{d}y$ and $\mathrm{d}x$ look like $\mathrm{d} \times y$ and $\mathrm{d} \times x$, respectively, the 'd' is certainly not a factor and must not be cancelled! In many mathematical texts, including this one, the 'd' in Leibniz notation appears in upright type, rather than the italic type used for variables, to emphasise this fact. (You don't need to do anything special when you handwrite Leibniz notation – you should just write the 'd' in the normal way.)

To understand the thinking behind Leibniz notation, consider Figure 19. It's exactly the same as Figure 17 on page 20, which illustrates differentiation from first principles, except that some things are labelled differently. For example, the point at which we want to find the gradient is labelled (x, y) instead of $(x, f(x))$. This is because the emphasis here is on the relationship between the variables x and y, rather than on the idea of f as a function of x. Another difference is that the change in the x-coordinate from the point at which we want to find the gradient to the second point is denoted by δx instead of h, so the x-coordinate of the second point is written as $x + \delta x$ instead of $x + h$.

The symbol δ is the lower-case Greek letter delta, and is read as 'delta'. By convention, when the symbol δ is used as a prefix it indicates 'a small change in', so δx denotes a small change in x. The change in the y-coordinate from the point at which we want to find the gradient to the second point is denoted in a similar way, as δy, which means that the y-coordinate of the second point is $y + \delta y$.

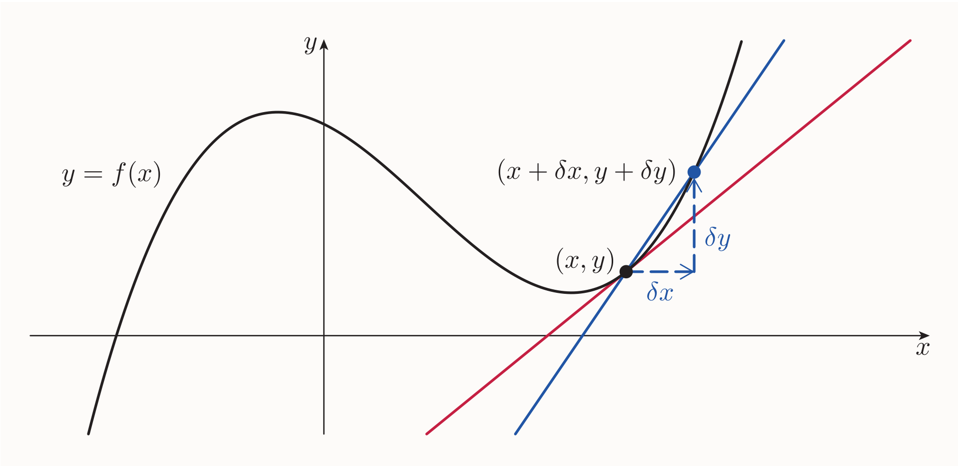


Figure 19 The points (x, y) and $(x + \delta x, y + \delta y)$ on the graph of $y = f(x)$, and the line through them

The gradient of the line that passes through the two points is then

$$\frac{\text{rise}}{\text{run}} = \frac{\delta y}{\delta x}.$$

As the second point gets closer and closer to the first point, the value of $\delta y/\delta x$ gets closer and closer to the gradient of the curve at the point (x, y). In other words, the gradient is the limit of $\delta y/\delta x$ as the second point gets closer and closer to the first point (and so as δx and δy get smaller and smaller). This is why the notation $\mathrm{d}y/\mathrm{d}x$, which looks similar to $\delta y/\delta x$, was chosen to represent the gradient.

The expression $\delta y/\delta x$ can be used instead of expression (2) on page 20 to carry out differentiation from first principles, and you might see this done in some other texts on calculus. The process is exactly the same, just with h replaced by δx, but it can look a bit more complicated at first sight.

Leibniz notation can be used in a variety of ways. For example, the symbol

$$\frac{\mathrm{d}}{\mathrm{d}x}$$

means 'the derivative with respect to x of'. This means that, for example, a concise way to express the fact that the gradient of the graph of the equation $y = x^2$ is given by the formula $2x$ is to write

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x.$$

As with Lagrange notation, Leibniz notation can be used with variables other than the standard ones, x and y. For example,

$$\text{if } s = t^2, \text{ then } \frac{\mathrm{d}s}{\mathrm{d}t} = 2t,$$

and

$$\text{if } p = q^3, \text{ then (by Example 1) } \frac{\mathrm{d}p}{\mathrm{d}q} = 3q^2.$$

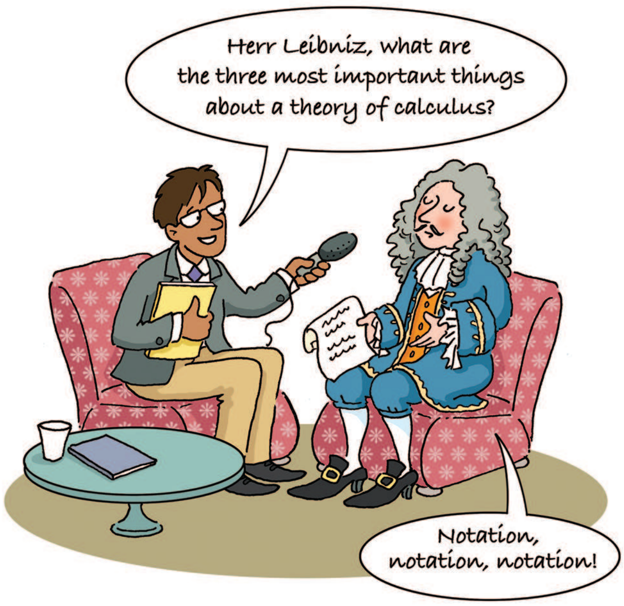
Similarly (by Activity 6),

$$\frac{\mathrm{d}}{\mathrm{d}w}(w^4) = 4w^3.$$

Sometimes, particularly on a computer algebra system, you might see $\mathrm{d}y/\mathrm{d}x$ written as

$$\frac{\mathrm{d}}{\mathrm{d}x}y.$$

Usually, if a function is specified using function notation, then you use Lagrange notation for its derivative, whereas if it's specified using an equation that expresses one variable in terms of another, then you use Leibniz notation. For example, if you know that $f(x) = x^2$, then you write $f'(x) = 2x$, whereas if you know that $y = x^2$, then you write $\mathrm{d}y/\mathrm{d}x = 2x$. However, there are no absolute rules about this, and in fact it's often helpful to mix the two notations. In particular, it's often convenient to use the symbol $\mathrm{d}/\mathrm{d}x$, even when you're mostly using Lagrange notation, as you'll see.



Lagrange notation and Leibniz notation are the two most common notations for derivatives, but there are other useful notations, including one invented by Isaac Newton. You'll meet some of these notations if you go on to study calculus beyond this module, particularly in the area of applied mathematics.

1.4.3 Functions whose domains include endpoints

It's often useful to find the derivative of a function whose domain includes an endpoint. For example, the function $f(x) = x^{3/2}$, whose graph is shown in Figure 20, has domain $[0, \infty)$, which includes 0 as an endpoint.

Any function whose domain includes an endpoint isn't differentiable at this endpoint. For example, consider the function $f(x) = x^{3/2}$. The point on its graph that corresponds to the endpoint 0 of its domain is the origin. You can trace your pen tip along the graph to the origin from the right, but you can't do the same from the left, so the graph doesn't have a tangent at the origin. Hence it doesn't have a gradient at the origin; that is, it's not differentiable at $x = 0$.

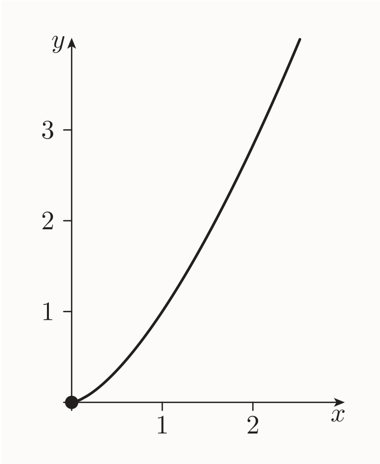


Figure 20 The graph of $f(x) = x^{3/2}$

However, the graph of $f(x) = x^{3/2}$ does have a 'tangent on the right' at the origin, in the sense that you can trace your pen tip along the graph towards the origin from the right, and continue moving it in the direction in which it's been moving when it reaches the origin. It will then move along a straight line, namely the x-axis. So the graph has a 'gradient on the right' at the origin, namely 0, the gradient of the x-axis.

Because of this, we say that the function $f(x) = x^{3/2}$ is **right-differentiable** at $x = 0$, and that its **right derivative** at 0 is 0. Similarly, a function can be **left-differentiable** at a particular x-value, and it will then have a **left derivative** at that x-value.

The left or right derivative of a function at a particular x-value can be found by using differentiation from first principles in much the same way as the usual, two-sided derivatives. The only difference is that instead of the increase in the x-coordinates, h, taking either positive or negative values as it gets closer and closer to zero, for a right-sided derivative, h takes just positive values, and for a left-sided derivative, h takes just negative values.

Saying that a function is differentiable at a particular x-value is the same as saying that it has both a left and a right derivative at that x-value, and the left and right derivatives are equal.

If f is a function whose domain includes one or more endpoints, then we adjust the definition of its derivative f' slightly to allow for these, as follows. We include in the domain of f' not just the values of x at which f is differentiable, but also the values of x that are endpoints of the domain of f and at which f is left- or right-differentiable. The value of f' at each of these endpoints is the appropriate left or right derivative.

All the results about derivatives that you'll meet in this module apply, with the appropriate adjustments, to left and right derivatives as well as to the usual, two-sided derivatives. For simplicity, this isn't stated explicitly for each individual result. For example, in the next subsection you'll learn how to find a formula for the derivative of the function $f(x) = x^{3/2}$. This formula is valid for all values of x in the domain $[0, \infty)$ of this function f, but the value that it gives for $x = 0$ is the right derivative of f at 0, rather than the derivative of f at 0, which doesn't exist.

1.4.4 Summary of important ideas

To finish this section, here's a summary of some of the important ideas that you've met.

Derivatives

The **derivative** (or **derived function**) of a function f is the function f' such that

$$f'(x) = \text{gradient of the graph of } f \text{ at the point } (x, f(x)).$$

The domain of f' consists of the values in the domain of f at which f is **differentiable** (that is, the x-values that give points at which the gradient exists).

If $y = f(x)$, then $f'(x)$ is also denoted by $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

The derivative f' is given by the equation

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The procedure of using this equation to find a formula for the derivative f' is called **differentiation from first principles**.

2 Finding derivatives of simple functions

In this section you'll meet formulas for the derivatives of standard functions of a particular type, and two rules for combining formulas for derivatives.

2.1 Derivatives of power functions

In the previous section you saw formulas for the derivatives of the functions

$$f(x) = x^2, \quad f(x) = x^3 \quad \text{and} \quad f(x) = x^4.$$

Any function of the form

$$f(x) = x^n,$$

where n is a real number, is called a **power function**. In this subsection you'll see how to find the formula for the derivative of any power function, without having to use differentiation from first principles.

The formulas that you saw for the derivatives of the three power functions above can be stated as follows, using Leibniz notation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x, \quad \frac{\mathrm{d}}{\mathrm{d}x}(x^3) = 3x^2, \quad \frac{\mathrm{d}}{\mathrm{d}x}(x^4) = 4x^3.$$

Notice that they all follow the same pattern: in each case, to obtain the derivative, you multiply by the power, then reduce the power by 1, as shown below.

$$x^2 \xrightarrow{\text{multiply by the power}} 2x^2 \xrightarrow{\text{reduce the power by 1}} 2x$$
$$x^3 \longrightarrow 3x^3 \longrightarrow 3x^2$$
$$x^4 \longrightarrow 4x^4 \longrightarrow 4x^3$$

It turns out that the derivative of every power function follows the same pattern. This fact can be stated algebraically as follows.

Derivative of a power function

For any number n,

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}.$$

You can confirm this formula for any particular value of n that's a *positive integer* by differentiating from first principles. You saw this process carried out for the particular power functions $f(x) = x^2$, $f(x) = x^3$ and $f(x) = x^4$ in the last subsection. In general, for the power function $f(x) = x^n$, the process involves multiplying out the expression $(x + h)^n$, which can involve quite a lot of algebraic manipulation. (In Unit 10 you'll meet a quick way to multiply out expressions like this, namely by using the *binomial theorem*.)

However, the formula holds not just for values of n that are positive integers, but for *all* values of n, including negative and fractional values. You'll see a proof of this fact in Unit 7.

As you'd expect, for any particular value of n, the formula holds for all values of x for which the function $f(x) = x^n$ is differentiable. For example, if $n = 2$, then the formula holds for *all* values of x. On the other hand, if $n = \frac{1}{2}$, then the formula holds only for *positive* values of x. The function $f(x) = x^{1/2}$, that is, $f(x) = \sqrt{x}$, whose graph is shown in Figure 21, isn't defined for negative values of x, and isn't differentiable (or even right-differentiable) at $x = 0$.

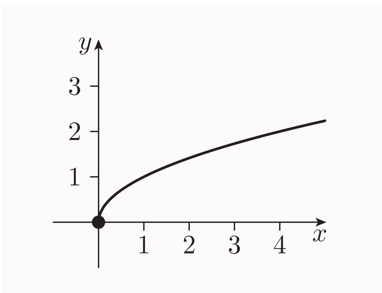


Figure 21 The graph of $f(x) = \sqrt{x}$

More generally, whenever you see a formula for the derivative of a function, such as

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x \quad \text{or} \quad f'(x) = 2x,$$

you can assume that the formula holds for all values of the variable (usually x) for which the function is differentiable, unless it's stated otherwise. You can follow the same convention when you write such formulas yourself. Remember in particular that for a function to be *differentiable* at a particular value of x, it must also be *defined* there. For example, the function $f(x) = \sqrt{x}$ isn't differentiable at $x = -3$, since it isn't even defined there.

The next example shows you how to use the general formula for the derivative of a power function. In parts (b) and (c), the expression on the right-hand side of the rule of the function isn't given in the form x^n, but you can use some of the index laws that you met in Unit 1 to rewrite it in this form. The two index laws that are the most useful for this sort of situation are repeated in the box below.

Two index laws

$$a^{-n} = \frac{1}{a^n} \qquad a^{1/n} = \sqrt[n]{a}$$

For example, you can write $1/x^3$ as x^{-3}, and $\sqrt[3]{x}$ as $x^{1/3}$.

In the solution to the example, Lagrange notation is used where the function is specified using function notation, and Leibniz notation is used where it is specified by an equation expressing one variable in terms of another. You should do likewise in the activity that follows the example.

Example 2 *Differentiating power functions*

Differentiate the following functions.

(a) $f(x) = x^{10}$ (b) $f(x) = \dfrac{1}{x^5}$ (c) $y = \sqrt{x}$

Solution

(a) Multiply by the power, then reduce the power by 1.

$f'(x) = 10x^9$.

(b) First write the function in the form $f(x) = x^n$.

The function is $f(x) = x^{-5}$.

Multiply by the power, then reduce the power by 1.

So $f'(x) = -5x^{-6}$

Simplify the answer.

$$= -\frac{5}{x^6}.$$

(c) First write the function in the form $y = x^n$.

The function is $y = x^{1/2}$.

Multiply by the power, then reduce the power by 1.

So $\dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2}$

Simplify the answer.

$$= \frac{1}{2} \times \frac{1}{x^{1/2}} = \frac{1}{2} \times \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

As illustrated in Example 2, whenever you find the derivative of a function, you should simplify your answer, if possible. As is often the case with algebraic simplifications, there may be no 'right answer' for the simplest form, as any of several different possibilities might do. The derivatives of the functions in Example 2 and the next activity contain indices – you saw some general guidelines for simplifying expressions of this kind in Subsection 4.3 of Unit 1.

Activity 7 *Differentiating power functions*

Differentiate the following functions.

(a) $f(x) = x^8$ (b) $f(x) = x^5$ (c) $f(x) = \dfrac{1}{x^3}$ (d) $f(x) = x^{3/2}$

(e) $f(x) = \dfrac{1}{x}$ (f) $f(x) = x^{5/2}$ (g) $f(x) = x^{4/3}$ (h) $f(x) = \dfrac{1}{x^8}$

(i) $y = \dfrac{1}{x^{1/4}}$ (j) $y = x^{1/3}$ (k) $y = \dfrac{1}{\sqrt{x}}$ (l) $y = \sqrt[3]{x^5}$

(m) $y = x^{2/7}$ (n) $f(x) = x^{-2}$ (o) $f(x) = \dfrac{1}{x^{1/3}}$ (p) $f(x) = \dfrac{1}{x^4}$

Activity 8 *Finding the gradient at a point on the graph of a power function*

Use the solution to Example 2(c) to find the gradient of the graph of $y = \sqrt{x}$ at the point with x-coordinate 4. (The tangent to the graph at this point is shown in Figure 22.)

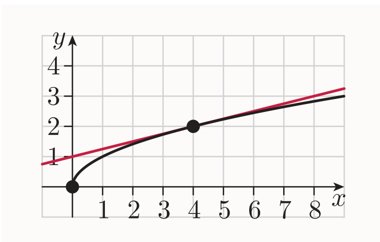


Figure 22 The graph of $y = \sqrt{x}$, and its tangent at the point with x-coordinate 4

Notice that the formula for the derivative of a power function tells you that the function

$$f(x) = x \quad \text{(which is the same as } f(x) = x^1\text{)}$$

has derivative

$$f'(x) = 1 \times x^0; \quad \text{that is,} \quad f'(x) = 1.$$

This is as you'd expect, because the graph of the function $f(x) = x$ (shown in Figure 23(a)) is a straight line with gradient 1, which means that the gradient at every point on the graph is 1.

Another simple power function is the constant function

$$f(x) = 1 \quad (\text{which is the same as } f(x) = x^0).$$

Its graph (shown in Figure 23(b)) is a straight line with gradient 0, so the gradient at every point on its graph is 0. So its derivative is simply

$$f'(x) = 0.$$

Note that to make the general formula work for these two functions when $x = 0$, we have to assume that $0^\circ = 1$. The value of 0° was discussed in Unit 1.

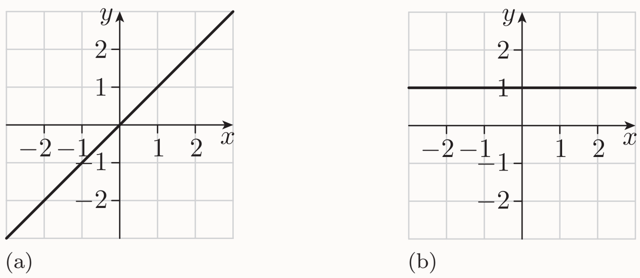


Figure 23 The graphs of (a) $f(x) = x$ (b) $f(x) = 1$

2.2 Constant multiple rule

In this subsection and the next, you'll see two ways in which you can use formulas that you know for the derivatives of functions to find formulas for the derivatives of other, related functions.

First, suppose that you know the formula for the derivative of a particular function, and you want to know the formula for the derivative of a constant multiple of the function. For example, you already know the formula for the derivative of the function $f(x) = x^2$, but suppose that you want to know the formula for the derivative of the function $g(x) = 3x^2$. Let's think about how the formula for the derivative of the second function can be worked out from the formula for the derivative of the first function.

When you multiply a function by a constant, the effect on its graph is that, for each x-value, the corresponding y-value is multiplied by the constant. So the graph is stretched or squashed vertically, and, if the constant is negative, then it's also reflected in the x-axis. As you saw in Unit 3, these effects are called *vertical scalings*. For example, Figure 24 shows the graphs of $y = x^2$, $y = 3x^2$, $y = \frac{1}{2}x^2$ and $y = -x^2$, and the point with x-coordinate 1 on each of these graphs.

You can see the following effects.

- Multiplying the function $f(x) = x^2$ by the constant 3 scales its graph vertically by a factor of 3 (which stretches it).
- Multiplying the function $f(x) = x^2$ by the constant $\frac{1}{2}$ scales its graph vertically by a factor of $\frac{1}{2}$ (which squashes it).
- Multiplying the function $f(x) = x^2$ by the constant -1 scales its graph vertically by a factor of -1 (which reflects it in the x-axis).

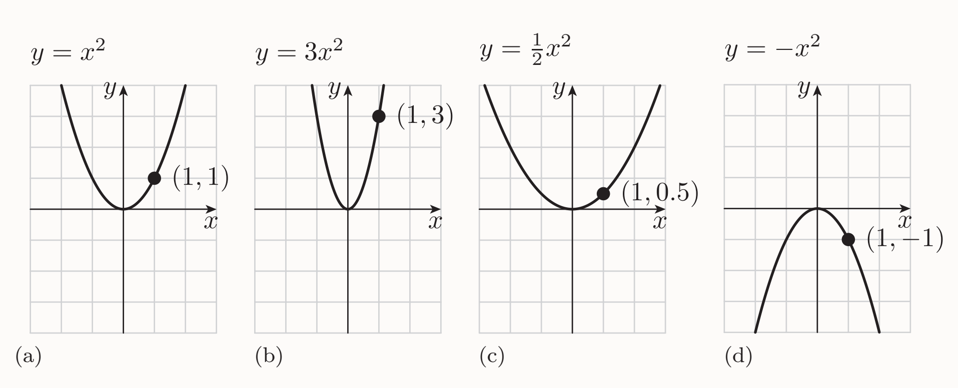


Figure 24 The graphs of (a) $y = x^2$ (b) $y = 3x^2$ (c) $y = \frac{1}{2}x^2$ (d) $y = -x^2$

The stretching or squashing, and possible reflection, of the graph causes the gradient at each x-value to change. For example, you can see that, at the point with x-coordinate 1, the graph of $y = 3x^2$ is steeper than the graph of $y = x^2$.

To see exactly how the gradients change, first consider what happens to the gradient of a straight line when you scale it vertically by a particular factor, say c. The scaled line will go up by c times as many units for every one unit that it goes along, compared to the unscaled line. In other words, its gradient is multiplied by the factor c. For example, Figure 25(a) illustrates what happens when you take a straight line with gradient 1 and scale it vertically by a factor of 3.

The same thing happens for any graph: if you scale it vertically by a particular factor, then its gradient at any particular x-value is multiplied by this factor. For example, Figure 25(b) illustrates that if you take a curve that has gradient 1 at a particular x-value, and scale it vertically by a factor of 3, then the new curve has gradient 3 at that x-value.

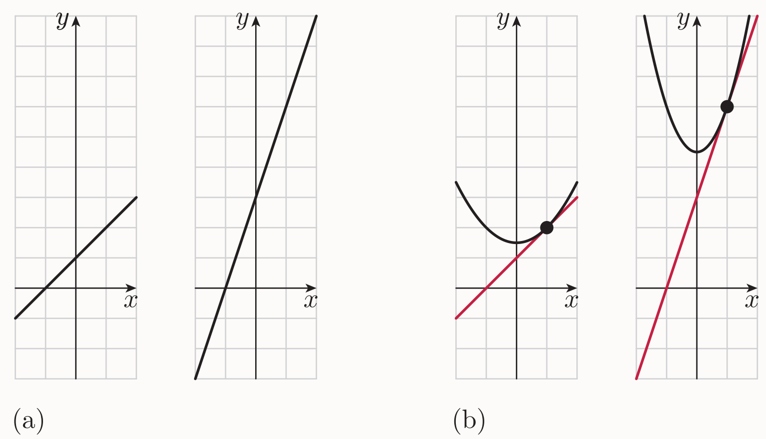


Figure 25 (a) A straight line with gradient 1, and the result of scaling it vertically by a factor of 3 (b) a curve that has gradient 1 at a particular x-value, and the result of scaling it vertically by a factor of 3

So, if you multiply a function by a constant, then its derivative is multiplied by the same constant. This fact can be stated as in the box below.

Constant multiple rule (Lagrange notation)

If the function k is given by $k(x) = af(x)$, where f is a function and a is a constant, then

$$k'(x) = af'(x),$$

for all values of x at which f is differentiable.

For example, since the derivative of $f(x) = x^2$ is $f'(x) = 2x$, it follows by the constant multiple rule that

- the derivative of $g(x) = 3x^2$ is $g'(x) = 3 \times 2x = 6x$
- the derivative of $h(x) = \frac{1}{2}x^2$ is $h'(x) = \frac{1}{2} \times 2x = x$
- the derivative of $p(x) = -x^2$ is $p'(x) = -2x$.

The third of these follows because taking a negative is the same as multiplying by -1. It's useful to remember in general that if f and k are functions such that $k(x) = -f(x)$, then

$$k'(x) = -f'(x),$$

for all values of x at which f is differentiable, by the constant multiple rule.

Like everything involving derivatives, the constant multiple rule can also be stated in Leibniz notation, as follows.

Constant multiple rule (Leibniz notation)

If $y = au$, where u is a function of x and c is a constant, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\frac{\mathrm{d}u}{\mathrm{d}x},$$

for all values of x at which u is differentiable.

(The phrase 'u is differentiable' in the box above is a condensed way of saying that if we write $u = f(x)$ then f is differentiable.)

The constant multiple rule can be proved formally by using the idea of differentiation from first principles, and you'll see this done at the end of this subsection. First, however, you should concentrate on learning to use it. Here's an example.

Example 3 *Using the constant multiple rule*

Differentiate the following functions.

(a) $f(x) = 8x^4$ (b) $f(x) = -\sqrt{x}$ (c) $y = \dfrac{3}{x}$

Solution

(a) The derivative is 8 times the derivative of x^4.

$f'(x) = 8 \times 4x^3 = 32x^3$

(b) The derivative is the negative of the derivative of $\sqrt{x}$.

$f(x) = -x^{1/2}$, so

$$f'(x) = -\frac{1}{2}x^{-1/2} = -\frac{1}{2} \times \frac{1}{x^{1/2}} = -\frac{1}{2x^{1/2}} = -\frac{1}{2\sqrt{x}}$$

(c) The derivative is 3 times the derivative of $1/x$.

$y = 3x^{-1}$, so

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3 \times (-1)x^{-2} = -\frac{3}{x^2}$$

Activity 9 *Using the constant multiple rule*

Differentiate the following functions.

(a) $f(x) = 5x^3$ (b) $f(x) = -x^7$ (c) $f(x) = 2\sqrt{x}$

(d) $f(x) = 6x$ (e) $f(x) = \dfrac{x}{4}$ (f) $f(x) = \dfrac{2}{x}$ (g) $f(x) = -7x$

(h) $y = \dfrac{\sqrt{x}}{3}$ (i) $y = \dfrac{8}{x^2}$ (j) $y = -\dfrac{5}{x}$ (k) $y = \dfrac{4}{\sqrt{x}}$

(l) $y = 4x^{3/2}$ (m) $y = \dfrac{1}{3\sqrt{x}}$ (n) $y = -12x^{1/3}$

Activity 10 *Using the constant multiple rule to find a gradient*

Find the gradient of the graph of the function $f(x) = 3x^2$ at the point with x-coordinate 2.

You saw in previous subsection that the function

$$f(x) = 1$$

has derivative

$$f'(x) = 0.$$

This fact, together with the constant multiple rule, tells you that if a is any constant, then the function

$$f(x) = a \quad \text{(which is the same as } f(x) = a \times 1)$$

has derivative

$$f'(x) = a \times 0 = 0.$$

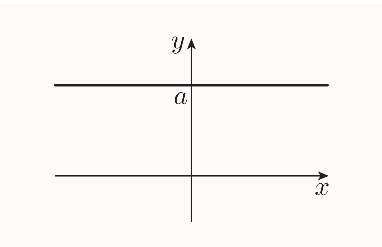


Figure 26 The graph of $f(x) = c$

For example, the function $f(x) = 3$ has derivative $f'(x) = 0$.

This is as you would expect, because the graph of the function $f(x) = a$ (which is illustrated in Figure 26, in the case where a is positive) is a straight line with gradient 0, which means that the gradient at every point on the graph is 0.

This fact about the derivative of a constant function can be stated as follows.

Derivative of a constant function

If a is a constant, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(a) = 0.$$

To finish this subsection, here is a formal proof of the constant multiple rule, using differentiation from first principles. It uses the Lagrange notation form of the constant multiple rule, which is repeated below.

Constant multiple rule (Lagrange notation)

If the function k is given by $k(x) = af(x)$, where f is a function and a is a constant, then

$$k'(x) = af'(x),$$

for all values of x at which f is differentiable.

A proof of the constant multiple rule

Suppose that f is a function and a is a constant. Consider the function k given by $k(x) = af(x)$. Let x be any value at which f is differentiable. To find $k'(x)$, you have to consider what happens to the difference quotient for k at x, which is

$$\frac{k(x+h) - k(x)}{h}$$

(where h can be positive or negative but not zero), as h gets closer and closer to zero. Since $k(x) = af(x)$, the difference quotient for k at x is equal to

$$\frac{af(x+h) - af(x)}{h},$$

which is equal to

$$a\left(\frac{f(x+h) - f(x)}{h}\right).$$

The expression in the large brackets is the difference quotient for f at x, so, as h gets closer and closer to zero, it gets closer and closer to $f'(x)$. Hence the whole expression gets closer and closer to $af'(x)$. In other words,

$$k'(x) = af'(x),$$

which is the constant multiple rule.

2.3 Sum rule

Now suppose that you know the formulas for the derivatives of two functions, and you want to know the formula for the derivative of their sum. For example, you already know the formulas for the derivatives of the functions $f(x) = x^2$ and $g(x) = x^3$, but suppose that you want to know the formula for the derivative of the function $k(x) = x^2 + x^3$.

When you add two functions, the y-coordinates of the points on the two graphs are added. For example, Figure 27 shows the graphs of $f(x) = x^2$,

$g(x) = x^3$ and $k(x) = x^2 + x^3$, and the point with x-coordinate 1 on each of the three graphs.

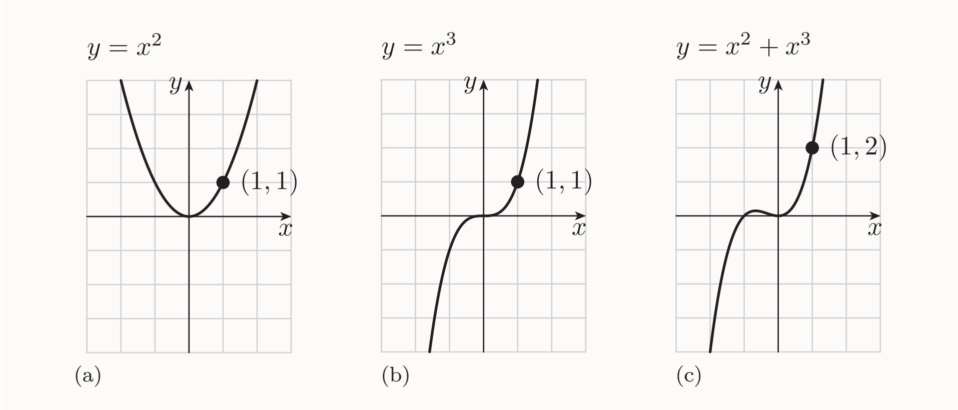


Figure 27 The graphs of two functions, and the graph of the sum of these functions

The effect that adding the functions has on the gradients is quite complicated to think about, but in fact it's just what you might expect: if you add two functions, then the gradient of the sum function at any particular x-value is the sum of the gradients of the two original functions at that x-value.

For example, Figure 28 shows the tangents at the three points marked in Figure 27. You can see that they have gradients 2, 3 and 5 (the sum of 2 and 3), respectively.

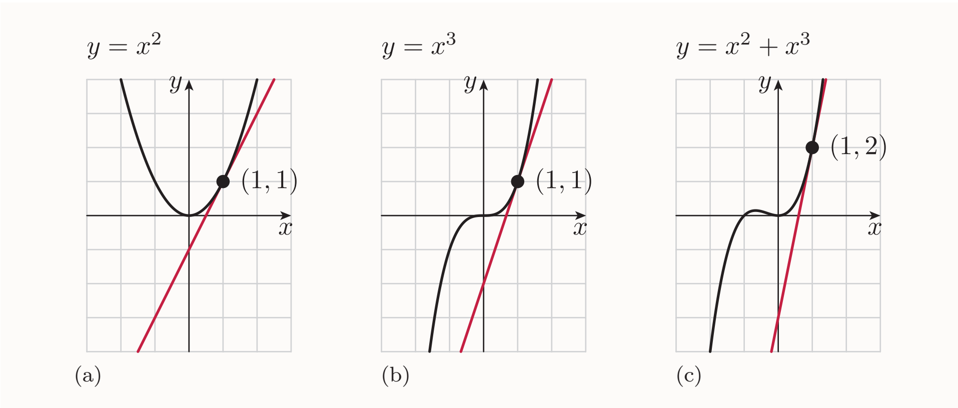


Figure 28 Tangents to the graphs in Figure 27

The general rule can be stated as follows.

Sum rule (Lagrange notation)

If $k(x) = f(x) + g(x)$, where f and g are functions, then

$$k'(x) = f'(x) + g'(x),$$

for all values of x at which both f and g are differentiable.

For example, since the derivative of $f(x) = x^2$ is $f'(x) = 2x$, and the derivative of $g(x) = x^3$ is $g'(x) = 3x^2$, it follows from the sum rule that the derivative of $k(x) = x^2 + x^3$ is

$$k'(x) = 2x + 3x^2.$$

The sum rule extends to sums of larger numbers of functions. For example, if $k(x) = f(x) + g(x) + h(x)$, then

$$k'(x) = f'(x) + g'(x) + h'(x),$$

for all values of x at which all of f, g and h are differentiable.

It also follows from the sum rule, together with the constant multiple rule, that if $k(x) = f(x) - g(x)$, then

$$k'(x) = f'(x) - g'(x),$$

for all values of x at which both f and g are differentiable. You can see this by writing the equation $k(x) = f(x) - g(x)$ as

$$k(x) = f(x) + (-g(x)).$$

Then, since the derivative of $-g(x)$ is $-g'(x)$ by the constant multiple rule, it follows by the sum rule that

$$\begin{aligned} k'(x) &= f'(x) + (-g'(x)) \\ &= f'(x) - g'(x), \end{aligned}$$

as claimed. Again, this fact extends to sums and differences of larger numbers of functions. For example, if $k(x) = f(x) + g(x) - h(x)$, then $k'(x) = f'(x) + g'(x) - h'(x)$ for all values of x at which all of f, g and h are differentiable.

Here is the sum rule expressed in Leibniz notation.

Sum rule (Leibniz notation)

If $y = u + v$, where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx},$$

for all values of x at which both u and v are differentiable.

Like the constant multiple rule, the sum rule can be formally proved by using the idea of differentiation from first principles, and you'll see this done at the end of this subsection. First, however, you need to practise using it.

The next example shows you how to do this. In particular, it illustrates that you can sometimes differentiate a function that isn't expressed as a sum of functions by first rearranging its formula so that it is. It also illustrates how you can use the sum rule together with the constant multiple rule.

Example 4 *Using the sum rule*

Differentiate each of the following functions.

(a) $f(x) = x + 2x^{1/3}$ (b) $f(x) = (x+2)(x-5)$

(c) $y = \dfrac{2x-1}{x}$

(d) $y = 5(x^2 - 2x + 1)$

Solution

(a) By the sum rule, you can differentiate each term individually (and add the results).

$f(x) = x + 2x^{1/3}$, so

$$\begin{aligned} f'(x) &= 1 + 2 \times \frac{1}{3}x^{-2/3} \\ &= 1 + \frac{2}{3x^{2/3}}. \end{aligned}$$

(b) First write the function as a sum of functions, by multiplying out.

$f(x) = (x+2)(x-5) = x^2 - 3x - 10$, so

Differentiate each term individually.

$$\begin{aligned} f'(x) &= 2x - 3 - 0 \\ &= 2x - 3. \end{aligned}$$

(c) First write the function as a sum of functions, by expanding the fraction.

$y = \dfrac{2x-1}{x} = \dfrac{2x}{x} - \dfrac{1}{x} = 2 - \dfrac{1}{x} = 2 - x^{-1}$, so

Differentiate each term individually. Simplify your answer.

$$\begin{aligned} \frac{dy}{dx} &= 0 - (-1)x^{-2} \\ &= x^{-2} = \frac{1}{x^2}. \end{aligned}$$

(d) By the constant multiple rule, the derivative of $5(x^2 - 2x + 1)$ is 5 times the derivative of $x^2 - 2x + 1$. To differentiate $x^2 - 2x + 1$, differentiate each term individually.

$y = 5(x^2 - 2x + 1)$, so

$$\frac{dy}{dx} = 5(2x - 2 + 0) = 10(x - 1).$$

Alternatively, multiply out the brackets and then differentiate each term individually.

$y = 5(x^2 - 2x + 1) = 5x^2 - 10x + 5$, so

$$\frac{dy}{dx} = 5 \times 2x - 10 \times 1 + 0 = 10x - 10.$$

Activity 11 *Using the sum rule*

Differentiate each of the following functions.

(a) $f(x) = 6x^2 - 2x + 1$ (b) $f(x) = \frac{2}{3}x^3 + 2x^2 + x - \frac{1}{2}$

(c) $f(x) = 5x + 1$ (d) $f(x) = \frac{1}{2}x + \sqrt{x}$ (e) $f(x) = (1 + x^2)(1 + 3x)$

(f) $f(x) = (x + 3)^2$ (g) $f(x) = 30(x^{3/2} - x)$

(h) $f(x) = x(x^{3/2} - x)$ (i) $y = \dfrac{(x - 2)(x + 5)}{x}$

(j) $y = \dfrac{x + \sqrt{x}}{x^2}$ (k) $y = (x^{1/3} + 1)(x^{1/3} + 5x)$

In this section you've seen how to differentiate any power function, and you've met the constant multiple rule and the sum rule. So, in particular, you can now differentiate any polynomial function, that is, any function of the form

$f(x) =$ a sum of terms, each of the form ax^n,
where a is a number and n is a non-negative integer.

Of course, you can also differentiate many other functions too, such as some functions involving negative or fractional powers of x.

You've seen that every function of the form $f(x) = x^n$, where n is a positive integer, is differentiable at every value of x and so is every constant function. These facts, together with the constant multiple rule and the sum rule, gives the following useful fact.

Every polynomial function is differentiable at *every* value of x.

To finish this subsection, here's a proof of the sum rule, using differentiation from first principles. It uses the Lagrange notation form of the sum rule, which is repeated below.

Sum rule (Lagrange notation)

If $k(x) = f(x) + g(x)$, where f and g are functions, then

$$k'(x) = f'(x) + g'(x),$$

for all values of x at which both f and g are differentiable.

A proof of the sum rule

Suppose that f and g are functions, and that the function k is given by $k(x) = f(x) + g(x)$. Let x denote any value at which both f and g are differentiable. To find $k'(x)$, you have to consider what happens to the difference quotient for k at x, which is

$$\frac{k(x+h) - k(x)}{h}$$

(where h can be either positive or negative, but not zero), as h gets closer and closer to zero. Since $k(x) = f(x) + g(x)$, this expression is equal to

$$\frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h},$$

that is,

$$\frac{f(x+h) + g(x+h) - f(x) - g(x)}{h},$$

which is equal to

$$\left(\frac{f(x+h) - f(x)}{h}\right) + \left(\frac{g(x+h) - g(x)}{h}\right).$$

The expression in the first pair of large brackets is the difference quotient for f at x, and the expression in the second pair of large brackets is the difference quotient for g at x. So as h gets closer and closer to zero, the values of these two expressions get closer and closer to $f'(x)$ and $g'(x)$, respectively. Hence the whole expression gets closer and closer to $f'(x) + g'(x)$. So

$$k'(x) = f'(x) + g'(x),$$

which is the sum rule.

3 Rates of change

You saw in Unit 2 that the gradient of a straight-line graph is the rate of change of the variable on the vertical axis with respect to the variable on the horizontal axis. For example, if the relationship between the variables x and y is represented by a straight line with gradient 2, then y is increasing at the rate of 2 units for every unit that x increases, as illustrated in Figure 29(a). Similarly, if the relationship between x and y is represented by a straight line with gradient -3, then y is *decreasing* at the rate of 3 units for every unit that x increases, as illustrated in Figure 29(b).

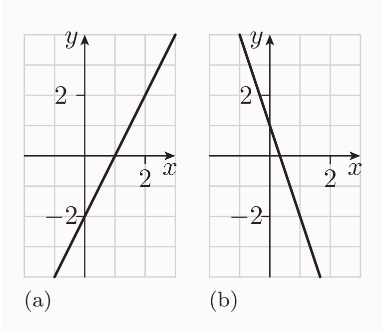


Figure 29 The lines
(a) $y = 2x - 2$
(b) $y = -3x + 1$

As mentioned in Subsection 1.1, the idea of a gradient as a rate of change also applies to curved graphs. For example, you've seen that the graph of the equation $y = x^2$ has gradient 2 at the point with x-coordinate 1. This means that, when $x = 1$, the variable y is increasing at the rate of 2 units for every unit that x increases, as illustrated in Figure 30.

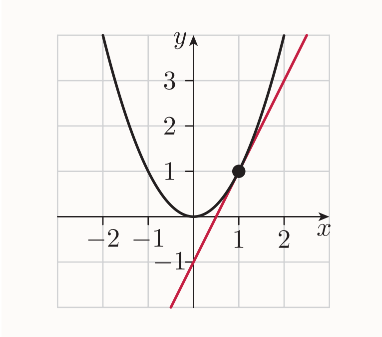


Figure 30 The tangent to the graph of $y = x^2$ at the point with x-coordinate 1

Of course, unlike in Figure 29(a), in Figure 30 the variable y doesn't *actually* increase by two units for every unit that x increases. This rate of change is an 'instantaneous' value, valid only for the x-value 1. For other values of x, the rate of change of y with respect to x (the gradient of the graph) is different.

Since the gradient of any graph of the variable y against the variable x is given by the derivative $\mathrm{d}y/\mathrm{d}x$, another way to think about the derivative $\mathrm{d}y/\mathrm{d}x$ is that it is the rate of change of y with respect to x.

As you've seen, it's particularly helpful to think of a gradient as a rate of change when you're working with a graph that models a real-life situation. In this section you'll look at two types of real-life situation involving rates of change.

3.1 Displacement and velocity

Remember from Subsection 2.4 of Unit 2 that when you model the motion of an object along a straight line, you choose some point on the line to be the reference point, and one of the two directions along the line to be the positive direction. Then the object's **displacement** at any particular time is its distance from the reference point, with a plus or minus sign to indicate its direction from this point. For example, in Figure 31 an object at position A has a displacement of 12 m, while an object at position B has a displacement of -7 m.

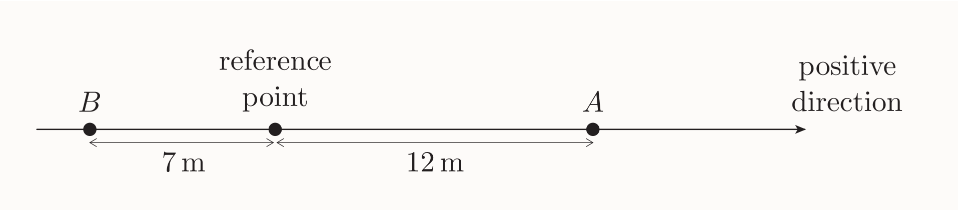


Figure 31 Displacement along a straight line

You saw that if the displacement of an object along a straight line is plotted against time, then the gradient of the resulting graph is the rate of change of the object's displacement with respect to time, which is called its **velocity**. (Remember that, in the calculus units in this module, displacement and velocity are always one-dimensional, and hence are represented by scalars, with direction indicated by the signs of the scalars.)

For example, Figure 32 shows the two displacement–time graphs for a man walking along a straight path that you saw in Subsection 1.1. The reference point on the straight path has been chosen to be the point where the man begins his walk, and the positive direction has been chosen to be the direction in which he walks. Time is measured from the moment when the man begins his walk.

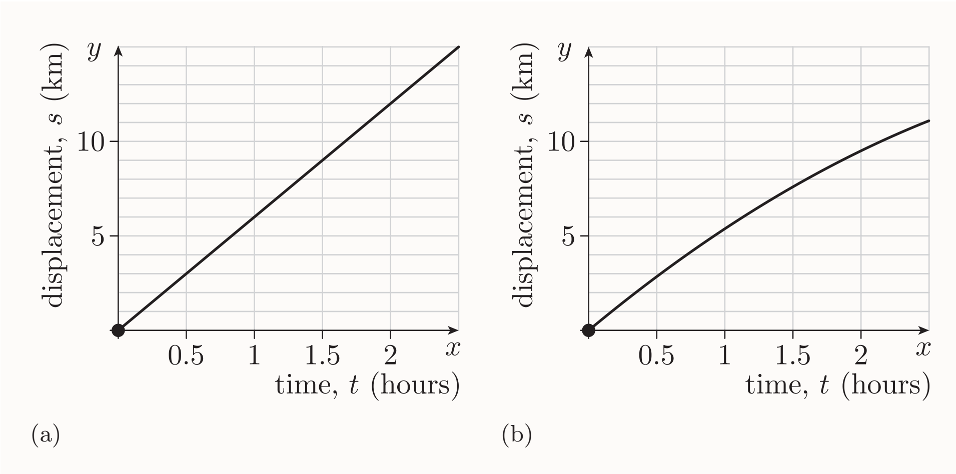


Figure 32 Two displacement–time graphs for a man walking along a straight path

In each of the two graphs in Figure 32, the quantity on the horizontal axis (time) is measured in hours, and the quantity on the vertical axis (distance) is measured in kilometres, so the gradient is measured in kilometres per hour. The graph in Figure 32(a) is a straight line with gradient $6\,\mathrm{km\,h^{-1}}$, which means that the man is walking at a constant velocity of $6\,\mathrm{km\,h^{-1}}$. In the graph in Figure 32(b), the man seems to start off walking at a velocity of about $6\,\mathrm{km\,h^{-1}}$, but his velocity continuously decreases as he walks.

In general, since the gradient of the displacement–time graph of any object that moves along a straight line is the velocity of the object, we have the following important fact.

Suppose that an object is moving along a straight line. If t is the time that has elapsed since some chosen point in time, s is the displacement of the object from some chosen reference point, and v is the velocity of the object, then

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}.$$

(Time, displacement and velocity can be measured in any suitable units, as long as they're consistent.)

An example of a set of consistent units is seconds for time, metres for displacement, and metres per second for velocity. These units are consistent because the units for time and displacement are the same as the units for time and displacement within the derived units for velocity.

Notice that displacement is denoted by the letter s in the box above and in Figure 32. This is the usual choice of letter for this quantity.

One explanation of the fact that displacement (or distance) is usually denoted by the letter s is that it is the initial letter of the Latin word for distance, which is 'spatium'.

The more natural choice of d is usually avoided, because it might cause confusion with the d used in Leibniz notation. Velocity is usually denoted by the letter v, and time by the letter t, as you'd expect.

The fact in the box above tells you that if you have a formula for the displacement of an object moving along a straight line, in terms of time, then you can use differentiation to obtain a formula for the object's velocity in terms of time.

This fact is used in the next example. Here the equation of the displacement–time graph in Figure 32(b), which is

$$s = 6t - \tfrac{5}{8}t^2,$$

is used to obtain an equation for the man's velocity in terms of time.

Example 5 *Using differentiation to find a velocity*

Suppose that a man walks along a straight path, and his displacement s (in kilometres) at time t (in hours) after he began his walk is given by

$$s = 6t - \tfrac{5}{8}t^2.$$

Let the man's velocity at time t be v (in kilometres per hour).

(a) Find an equation expressing v in terms of t.

(b) Hence find the man's velocity one hour into his walk.

Solution

(a) The man's displacement s at time t is given by

$$s = 6t - \tfrac{5}{8}t^2.$$

Hence his velocity v at time t is given by

$$\begin{aligned} v &= \frac{\mathrm{d}s}{\mathrm{d}t} \\ &= 6 - \tfrac{5}{8} \times 2t \\ &= 6 - \tfrac{5}{4}t. \end{aligned}$$

That is, the required equation is $v = 6 - \frac{5}{4}t$.

(b) When $t = 1$,

$$v = 6 - \tfrac{5}{4} \times 1 = 4.75.$$

So the man's velocity after one hour is $4.75\ \mathrm{km\,h^{-1}}$.

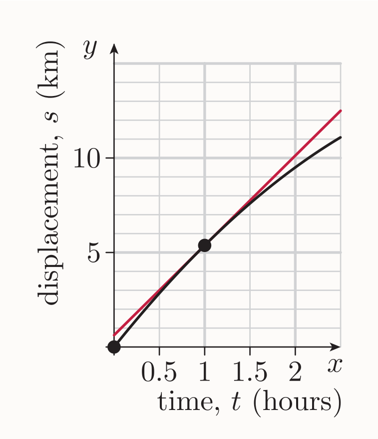


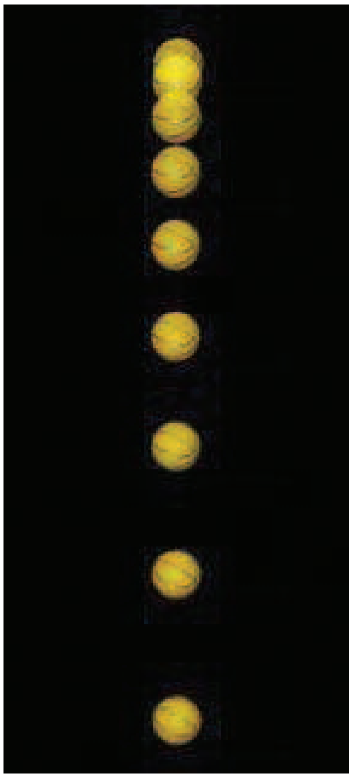
Figure 33 The tangent to the graph of $s = 6t - \frac{5}{8}t^2$ at the point with t-coordinate 1

Figure 33 shows the tangent to the graph in Figure 32(b) at the point one hour after the man begins his walk. You can see that the gradient of the tangent – that is, the man's velocity – does indeed seem to be roughly $4.75\ \mathrm{km\,h^{-1}}$, as calculated in the example above.

Notice that in Example 5 we started off knowing how the man's displacement changed with time – the relationship between these two quantities is given by the equation $s = 6t - \frac{5}{8}t^2$. We used this information to find how the man's velocity (his rate of change of displacement) changed with time – we found that this relationship is given by the equation $v = 6 - \frac{5}{4}t$. This is the sort of calculation that differential calculus allows you to carry out: if you know the values taken by a changing quantity throughout a period of change, then you can use differentiation to find the values taken by the rate of change of the quantity throughout the same period.

You might find it enlightening to think about what the process of differentiation from first principles means when it is applied to a displacement–time graph. In this case the difference quotient at a particular time value represents the *average velocity* over a time interval starting or ending at that time value. As the time interval gets smaller and smaller, the average velocity gets closer and closer to the instantaneous velocity at that time value.

The next activity is about a particular type of motion in a straight line: that of an object falling vertically from rest, near the surface of the Earth. The phrase 'from rest' here means that the object has no initial velocity when it begins falling. If the effect of air resistance is negligible (which it is if the object is fairly compact and the fall is fairly short), then the distance fallen by the object isn't affected by how heavy it is, for example, but depends only on the time that it has been falling. In fact, the total distance fallen is proportional to the square of the time that it has been falling. If time is measured in seconds and distance is measured in metres, then the constant of proportionality is about 4.9, so at time t (in seconds) after the object began falling it will have fallen a total distance of about $4.9t^2$ (in metres).



A falling ball, photographed at equal time intervals

When we model the motion of a falling object using displacement rather than distance, we often take the positive direction along the line of motion to be upwards. So, if we also take the reference point to be the point from which the object starts to fall, then the displacement s (in metres) of the object at time t (in seconds) after it starts to fall is given by the equation

$$s = -4.9t^2.$$

The graph of this equation is shown in Figure 34.

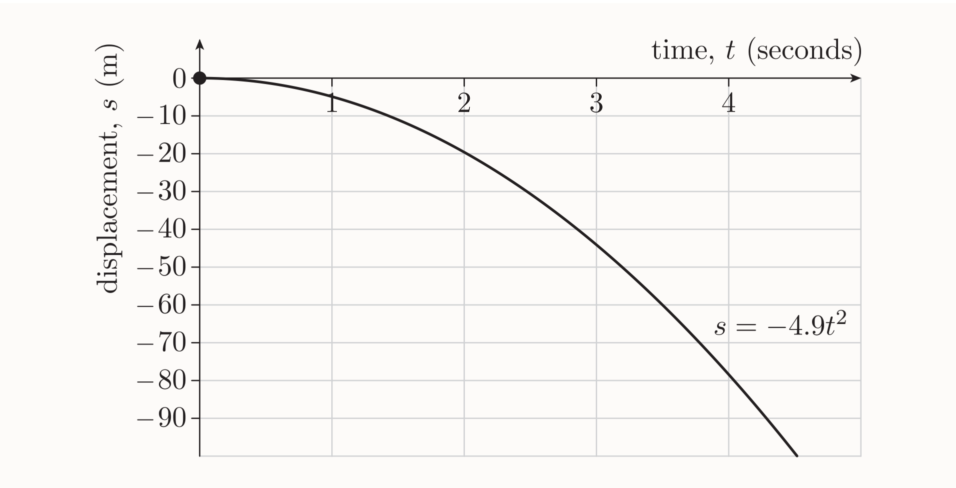


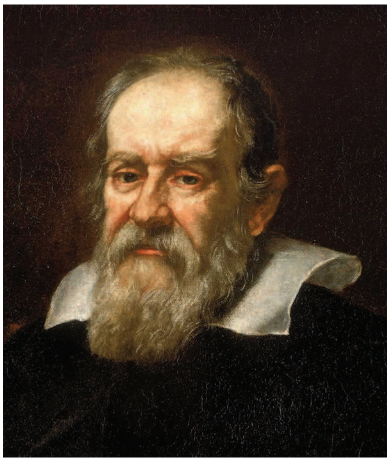
Figure 34 The displacement–time graph for a falling object

Activity 12 *Finding the velocity of a falling object*

(a) You saw above that the displacement s (in metres) of an object falling from rest, at time t (in seconds) since it began falling, is given by the equation $s = -4.9t^2$. Use this equation to find an equation for the velocity v (in $\mathrm{m\,s^{-1}}$) of the object at time t.

(b) Hence find the velocity of such an object three seconds into its fall. Give your answer in metres per second, to one decimal place.

(c) If an object is dropped from the top of a tower, how long does it take to reach a speed of $15\,\mathrm{m\,s^{-1}}$? Give your answer in seconds, to one decimal place.



Galileo Galilei (1564–1642)

The fact that the distance travelled by a falling object is proportional to the square of the time that it has been falling was determined by the Italian physicist Galileo Galilei in the sixteenth century, using experiments that he carried out in his workshop. At the time, there were no clocks that could measure intervals of time short enough to allow him to determine the motion of a falling object, but instead he rolled balls down a sloping groove, and reasoned that their motion would be similar to that of a falling object, but slower.

3.2 Total cost and marginal cost

Here's another example of a situation where it's useful to think of the gradient of a graph (and hence the derivative of a function) as a rate of change. This example shows you one way in which the idea of differentiation can be used in economics.

Suppose that a small confectionery company has found that the weekly cost of making its milk chocolate consists of a fixed cost of £3000 (to pay for the rent and maintenance of its premises and equipment, for example), plus £8 per kilogram of chocolate made (to pay for the ingredients and staff time, for example). In other words, the weekly cost, £c, is modelled by the equation

$$c = 3000 + 8q, \qquad (3)$$

where q is the amount of chocolate made in the week, in kilograms. The graph of this equation is shown in Figure 35.

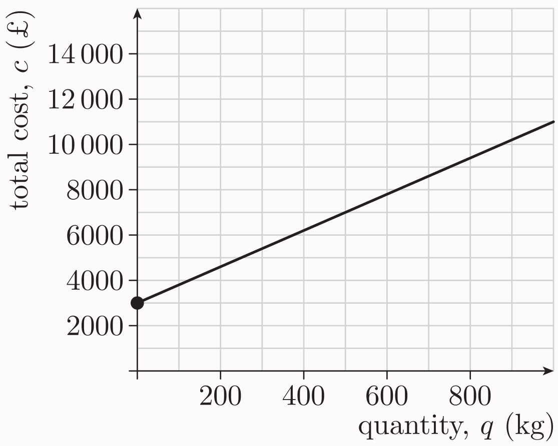


Figure 35 The graph of the equation $c = 3000 + 8q$, a model for the total weekly cost c (in £) of making milk chocolate in terms of the quantity q (in kg) of chocolate made

Suppose that the company is currently making a particular quantity of milk chocolate per week, but is thinking of increasing its production. Then the additional weekly cost of making the extra chocolate will, of course, be £8 per extra kilogram made. This is known as the **marginal cost** per kilogram of making the extra chocolate. It's different from the real cost per kilogram of making the chocolate (usually called the **unit cost** or **average cost**), which is equal to the total cost, £$(3000 + 8q)$, divided by the quantity in kilograms of chocolate made, q.

In fact, the marginal cost of making the extra chocolate is the gradient of the graph in Figure 35. This is because the gradient of the graph is the rate at which the quantity on the vertical axis (the total cost, in £) increases as the quantity on the horizontal axis (the quantity of chocolate made, in kg) increases. The fact that the gradient is 8 pounds per kilogram, which is normally written as £8 per kilogram, tells you that the total cost is increasing at a rate of £8 for every kilogram by which the quantity of chocolate increases.

Often an equation that models the total cost of making a quantity of a product doesn't have a straight-line graph. For example, suppose that the confectionery company has found that a better model for the total weekly cost c (in £) of making a quantity q (in kg) of milk chocolate is

$$c = 3000 + 4q + \tfrac{1}{125}q^2.$$

The graph of this equation is shown in Figure 36.

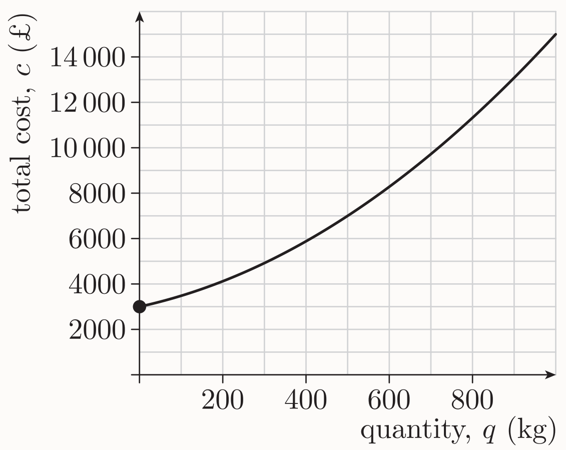


Figure 36 The graph of the equation $c = 3000 + 4q + \frac{1}{125}q^2$, an alternative model for the total weekly cost c (in £) of making milk chocolate in terms of the quantity q (in kg) of chocolate made

As before, the gradient of the graph tells you the rate at which the total weekly cost increases as the amount of chocolate made increases. In other words, the gradient of the graph is the marginal cost per kilogram of making extra chocolate. You can see that the gradient of the graph – the marginal cost per kilogram – *increases* as the quantity of chocolate made increases. This tells you that the more chocolate the company makes, the larger is the cost of making an extra kilogram of chocolate.

There are various reasons why this might be the case. For example, to increase its production of chocolate the company might have to pay its staff at higher rates, for overtime or non-standard hours.

For any particular quantity q (in kg) of chocolate being made, the marginal cost per kilogram of making any extra chocolate is the gradient of the graph at that value of q, which as you know is given by $\mathrm{d}c/\mathrm{d}q$. Of course, this is an 'instantaneous' value, valid only for that value of q.

Activity 13 *Working with marginal cost*

Suppose that the confectionery company discussed above has decided that the second equation above, namely

$$c = 3000 + 4q + \tfrac{1}{125}q^2,$$

is an appropriate model for the total weekly cost of making its milk chocolate. Here q is the quantity of chocolate made (in kg), and c is the total cost (in £). Let the marginal cost per kilogram of making any extra chocolate be m (in £).

(a) Use differentiation to find an equation for the marginal cost m in terms of the quantity q.

(b) What is the marginal cost per kilogram of making extra chocolate when the amount of chocolate already being made is 300 kilograms?

(c) The company sells all the chocolate that it makes at a price of £16 per kilogram. It decides to keep increasing its weekly production of chocolate until the marginal cost is equal to the price at which it sells the chocolate. (This is because if it increases its production any further, then it will cost more money to make the extra chocolate than the company will obtain by selling it.) By writing down and solving a suitable equation, find the weekly quantity of chocolate that the company should make.

The fact that marginal cost is the derivative of total cost is used in many economic models.

4 Finding where functions are increasing, decreasing or stationary

Because the derivative of a function tells you the gradient at each point on the graph of the function, it gives you information about the shape of the graph. In this section, you'll see how you can use the derivative of a function to deduce useful facts about its graph.

4.1 Increasing/decreasing criterion

You saw the following definitions in Unit 3.

Functions increasing or decreasing on an interval

A function f is **increasing on the interval** I if for all values x_1 and x_2 in I such that $x_1 < x_2$,

$$f(x_1) < f(x_2).$$

A function f is **decreasing on the interval** I if for all values x_1 and x_2 in I such that $x_1 < x_2$,

$$f(x_1) > f(x_2).$$

(The interval I must be part of the domain of f.)

Informally, a function is increasing on an interval if its graph slopes up on that interval, and is decreasing on an interval if its graph slopes down on the interval.

For example, the function $f(x) = 2x^3 - 3x^2 - 36x$ is increasing on the interval $(-\infty, -2)$, decreasing on the interval $(-2, 3)$, and increasing on the interval $(3, \infty)$, as illustrated in Figure 37.

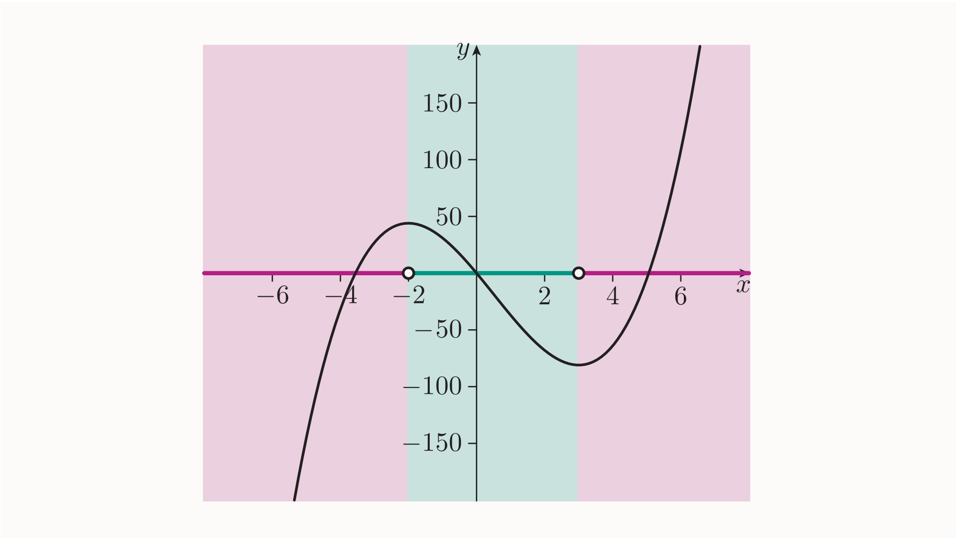


Figure 37 The graph of the function $f(x) = 2x^3 - 3x^2 - 36x$

Of course, you can't tell from the graph alone that this function stops increasing and starts decreasing when x is *exactly* -2, or that it stops decreasing and starts increasing when x is *exactly* 3. You'll see shortly how to confirm that the function is increasing and decreasing on the intervals mentioned above, exactly.

You could also say that the function in Figure 37 is increasing on the interval $(-\infty, -2]$, decreasing on the interval $[-2, 3]$, and increasing on the interval $[3, \infty)$. However, when we discuss intervals on which a function is increasing or decreasing, it's often helpful to consider intervals that don't overlap, so we usually consider *open* intervals.

Since positive gradients correspond to a graph sloping up, while negative gradients correspond to a graph sloping down, the derivative of a function tells you the intervals on which the function is increasing or decreasing, as set out below.

Increasing/decreasing criterion

If $f'(x)$ is positive for all x in an interval I, then f is increasing on I.

If $f'(x)$ is negative for all x in an interval I, then f is decreasing on I.

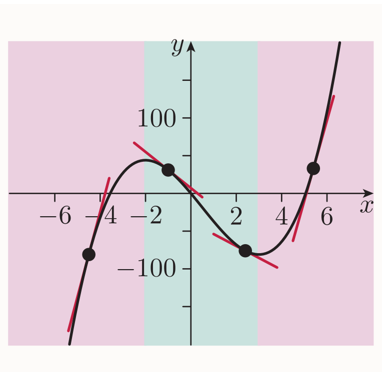


Figure 38 Some tangents to the graph of $f(x) = 2x^3 - 3x^2 - 36x$

For example, Figure 38 shows parts of the tangents to some points on the graph of the function $f(x) = 2x^3 - 3x^2 - 36x$. It illustrates that positive gradients correspond to intervals on which the graph is increasing, and negative gradients correspond to intervals on which the graph is decreasing.

It's sometimes useful to determine whether a particular function that you're working with is increasing on a particular interval, or decreasing on a particular interval. (For example, this can help you sketch its graph.) You can often use the increasing/decreasing criterion to do this, as illustrated in the next example. This example confirms that the function in Figure 37 is increasing and decreasing on the intervals mentioned earlier.

Example 6 *Using the increasing/decreasing criterion*

Show that the function $f(x) = 2x^3 - 3x^2 - 36x$ is increasing on each of the intervals $(-\infty, -2)$ and $(3, \infty)$, and decreasing on the interval $(-2, 3)$.

Solution

Find the derivative.

The derivative is

$$f'(x) = 6x^2 - 6x - 36.$$

Show that the derivative is always positive when x is in the interval $(-\infty, -2)$ or in the interval $(3, \infty)$ and always negative when x is in the interval $(-2, 3)$. Factorising is often helpful in such arguments.

Factorising this gives

$$f'(x) = 6(x^2 - x - 6) = 6(x + 2)(x - 3).$$

When x is less than -2, the values of $x + 2$ and $x - 3$ are both negative, and hence the value of $f'(x) = 6(x + 2)(x - 3)$ is positive. Therefore, by the increasing/decreasing criterion, the function f is increasing on the interval $(-\infty, -2)$.

Similarly, when x is greater than 3, the values of $x + 2$ and $x - 3$ are both positive, and hence the value of $f'(x) = 6(x + 2)(x - 3)$ is also positive. Therefore, by the increasing/decreasing criterion, the function f is increasing on the interval $(3, \infty)$.

When x is in the interval $(-2, 3)$, the value of $x + 2$ is positive and the value of $x - 3$ is negative, and hence the value of $f'(x) = 6(x + 2)(x - 3)$ is negative. Therefore, by the increasing/decreasing criterion, the function f is decreasing on the interval $(-2, 3)$.

Activity 14 *Using the increasing/decreasing criterion*

Consider the function $f(x) = \frac{2}{3}x^3 - 8x^2 + 30x - 36$.

(a) Find the derivative $f'(x)$, and factorise it.

(b) Show that f is increasing on each of the intervals $(-\infty, 3)$ and $(5, \infty)$.

(c) Show that f is decreasing on the interval $(3, 5)$.

Activity 15 *Using the increasing/decreasing criterion again*

Consider the function $f(x) = x^3 - 3x^2 + 4x + 3$.

(a) Find the derivative $f'(x)$, and complete the square on this expression. (You saw how to complete the square in Unit 2.)

(b) Hence show that f is increasing on the interval $(-\infty, \infty)$ (that is, on the whole of its domain).

If you look back to the statement of the increasing/decreasing criterion on page 54, you'll see that the first part begins

> If $f'(x)$ is positive for all x in an interval I ....

A slightly more concise way to express the same thing is to say

> If f' is positive *on* an interval I ....

Similarly, the beginning of the second part of the increasing/decreasing criterion,

> If $f'(x)$ is negative for all x in an interval I ...,

can be expressed as

> If f' is negative *on* an interval I ....

These more concise forms are used in the next subsection.

4.2 Stationary points

Consider again the function

$$f(x) = 2x^3 - 3x^2 - 36x,$$

which was discussed in the previous subsection. Its graph is shown again in Figure 39. As shown earlier, the gradient of the graph is positive on the interval $(-\infty, -2)$, negative on the interval $(-2, 3)$ and positive again on the interval $(3, \infty)$.

Between these open intervals, there are points at which the gradient of the graph is zero, as shown in Figure 39. A point at which the gradient of a graph is zero is called a **stationary point**. So the graph in Figure 39 has stationary points at the points with x-coordinates -2 and 3. If you work out the corresponding y-coordinates (by substituting the x-coordinates into the rule of f in the usual way), then you find that the stationary points are $(-2, 44)$ and $(3, -81)$.

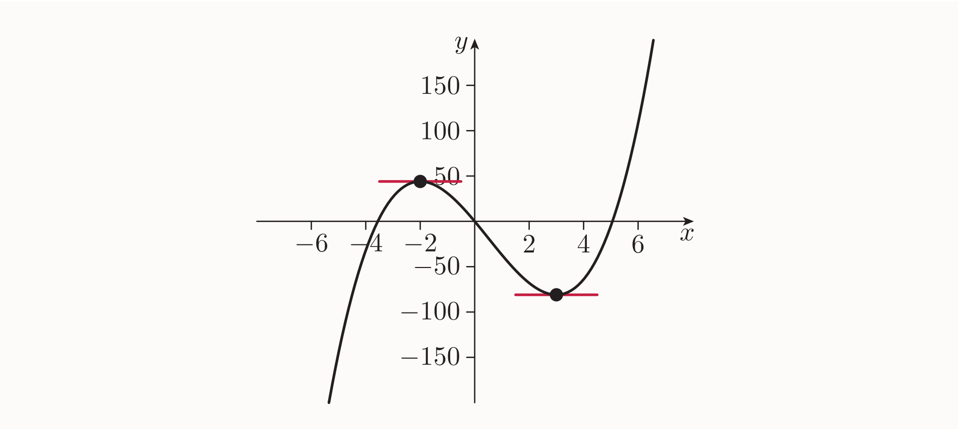


Figure 39 The stationary points on the graph of $f(x) = 2x^3 - 3x^2 - 36x$

The term 'stationary point' is used to refer to the x-coordinate of a stationary point, as well as to the point itself. This is because real numbers are sometimes called *points* – you can think of them as points on the number line. (You've already seen that a real number at one end of an interval is known as an *endpoint*.) So, for example, we can say that the stationary points of the function $f(x) = 2x^3 - 3x^2 - 36x$ are -2 and 3.

At the left-hand stationary point in Figure 39, the value taken by the function is larger than at any other point nearby, so we say that the function has a **local maximum** at this point. Similarly, at the right-hand stationary point, the value taken by the function is smaller than at any other point nearby, so we say that the function has a **local minimum** at this point.

Notice that the value taken by a function at a local maximum or local minimum isn't necessarily the greatest or least value that the function takes overall. For example, you can see from Figure 39 that the function $f(x) = 2x^3 - 3x^2 - 36x$ takes a larger value when $x = 6$, say, than it does at its local maximum, at $x = -2$. This is why the word 'local' is used.

A point where a function has a local maximum or local minimum is called a **turning point**, because if you imagine tracing your pen tip along the graph of the function from left to right, then at a local maximum or minimum it stops going up or down, and turns to go the other way.

Notice that at a local maximum, the graph of a function is increasing on the left and decreasing on the right, and at a local minimum, it is decreasing on the left and increasing on the right.

There are other types of stationary point, apart from turning points. For example, Figure 40(a) shows the graph of the function $f(x) = x^3$. The derivative of this function is $f'(x) = 3x^2$, so the gradient of the graph when $x = 0$ is

$$f'(0) = 3 \times 0^2 = 0.$$

So the function $f(x) = x^3$ has a stationary point at $x = 0$. However, this stationary point isn't a turning point. The graph of the function $f(x) = x^3$ is increasing on *both* sides of the stationary point, and just levels off momentarily at the stationary point itself.

Similarly, the graph of the function $g(x) = -x^3$ is *decreasing* on both sides of its stationary point, as illustrated in Figure 40(b).

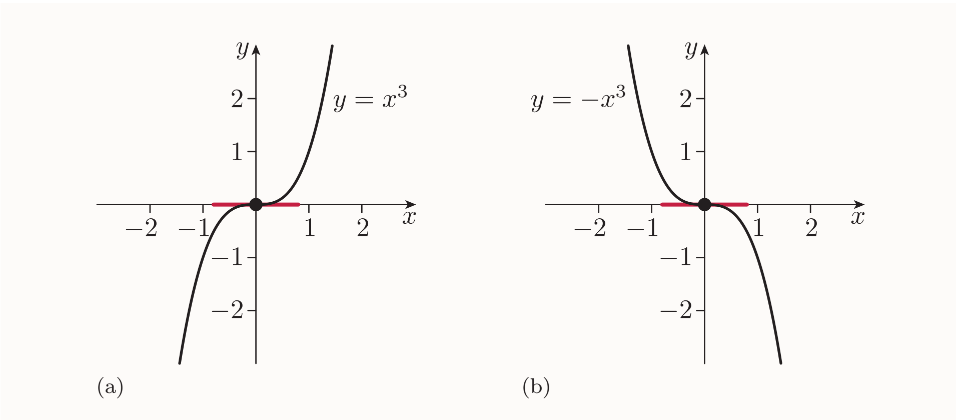


Figure 40 The graphs of (a) $f(x) = x^3$ (b) $g(x) = -x^3$

A stationary point of either of the types shown in Figure 40 is called a **horizontal point of inflection**. You'll see the reason for this term later in the unit. Notice that the tangent to a curve at a horizontal point of inflection crosses the curve at that point.

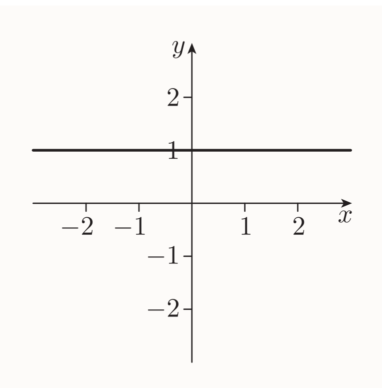


Figure 41 The graph of $y = 1$

Some stationary points are neither turning points nor horizontal points of inflection. For example, every point on the graph of the equation $y = 1$ (see Figure 41), or on any horizontal line, is a stationary point that is neither a turning point nor a horizontal point of inflection.

When you're working with a function, it's sometimes useful to find its stationary points, and, if possible, determine the nature of each stationary point – that is, determine whether it is a local maximum, local minimum or horizontal point of inflection. In the rest of this subsection you'll learn how to do this, and in the next subsection you'll see some examples of why it can be a useful thing to do.

You've seen that the stationary points of a function are the values of x at which the gradient of its graph is zero. So the method for finding the stationary points of a function is as follows.

Strategy:
To find the stationary points of a function f

Solve the equation $f'(x) = 0$.

In the next example, this strategy is used to confirm that the function

$$f(x) = 2x^3 - 3x^2 - 36x,$$

which we considered at the beginning of this subsection and in the last subsection, has stationary points at $x = -2$ and $x = 3$.

Example 7 *Finding the stationary points of a function*

Find the stationary points of the function

$$f(x) = 2x^3 - 3x^2 - 36x.$$

Solution

Find the derivative $f'(x)$. Factorise it if possible.

The derivative is

$$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x + 2)(x - 3).$$

Solve the equation $f'(x) = 0$.

Solving the equation $f'(x) = 0$ gives

$$6(x + 2)(x - 3) = 0;$$

that is,

$$x = -2 \quad \text{or} \quad x = 3.$$

Hence the stationary points are -2 and 3.

Activity 16 *Finding the stationary points of a function*

Find the approximate values, to two decimal places, of the stationary points of the function

$$f(x) = x^3 - x^2 - 2x.$$

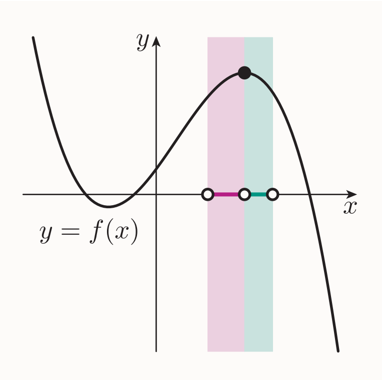


Figure 42 Open intervals immediately to the left and right of a local maximum

To determine whether a stationary point of a function f is a local maximum, local minimum or horizontal point of inflection, you can use the following facts. If there's an open interval immediately to the left of the stationary point, and an open interval immediately to the right of the stationary point, such that

- f is increasing on the left interval and decreasing on the right interval, then the stationary point is a local maximum
- f is decreasing on the left interval and increasing on the right interval, then the stationary point is a local minimum
- f is increasing on both intervals or decreasing on both intervals, then the stationary point is a horizontal point of inflection.

For example, Figure 42 shows the graph of a function with a local maximum, and open intervals immediately to the left and right of the local maximum. The function is increasing on the left interval, and decreasing on the right interval.

Combining the facts above with the increasing/decreasing criterion (stated on page 54) gives the following useful test for determining the nature of a stationary point. It's known as the *first derivative test* – the reason for the word 'first' is explained in Section 5.

First derivative test (for determining the nature of a stationary point of a function f)

If there are open intervals immediately to the left and right of a stationary point such that

- $f'(x)$ is positive on the left interval and negative on the right interval, then the stationary point is a local maximum
- $f'(x)$ is negative on the left interval and positive on the right interval, then the stationary point is a local minimum
- $f'(x)$ is positive on both intervals or negative on both intervals, then the stationary point is a horizontal point of inflection.

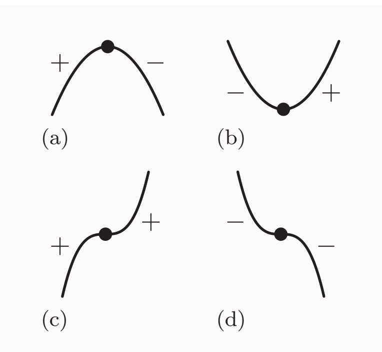


Figure 43 The signs of the derivative to the left and right of different types of stationary point

This test is illustrated in the diagram in Figure 43. The signs show whether the derivative is positive or negative on intervals immediately to the left and right of the stationary point.

Here's an example showing how you can use the first derivative test to determine the nature of the stationary points of a function. The example uses the technique of drawing up a table of signs, which you met in Unit 3.

Example 8 *Determining the nature of stationary points using the first derivative test*

Consider the function

$$f(x) = \tfrac{4}{3}x^3 + 5x^2 - 6x - 2.$$

(a) Find the stationary points of f.

(b) Determine whether each stationary point of f is a local maximum, local minimum or horizontal point of inflection.

Solution

(a) Find $f'(x)$, and factorise it, if possible.

The derivative is

$$\begin{aligned} f'(x) &= 4x^2 + 10x - 6 \\ &= 2(2x^2 + 5x - 3) \\ &= 2(2x - 1)(x + 3). \end{aligned}$$

Solve the equation $f'(x) = 0$.

Solving the equation $f'(x) = 0$ gives

$$2(2x - 1)(x + 3) = 0;$$

that is,

$$x = \tfrac{1}{2} \quad \text{or} \quad x = -3.$$

Hence the stationary points of f are -3 and $\frac{1}{2}$.

(b) Construct a table of signs to help you find intervals on which $f'(x)$ is negative or positive. In the column headings, write the values where $f'(x) = 0$ (that is, the stationary points of f), in increasing order, and also the intervals to the left and right of, and between, these values. In the row headings, write the factors of $f'(x)$, and then $f'(x)$ itself. Find the signs in the way that you learned in Subsection 5.4 of Unit 3.

x	$(-\infty,-3)$	-3	$(-3,\frac{1}{2})$	$\frac{1}{2}$	$(\frac{1}{2},\infty)$
2	$+$	$+$	$+$	$+$	$+$
$2x-1$	$-$	$-$	$-$	0	$+$
$x+3$	$-$	0	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	0	$+$

Apply the first derivative test. You might find it helpful to add a further row to the table, in which you use the signs of f' to determine whether f is increasing, decreasing or stationary, and indicate this with sloping or horizontal lines, as shown.

slope of f	$\nearrow$	$-$	$\searrow$	$-$	$\nearrow$

The stationary point -3 is a local maximum, and the stationary point $\frac{1}{2}$ is a local minimum.

Activity 17 *Determining the nature of stationary points using the first derivative test*

Consider the function

$$f(x) = 3x^4 - 2x^3 - 9x^2 + 7.$$

(a) Find the stationary points of f.
Hint: to factorise $f'(x)$, first notice that it has x as a factor.

(b) Determine whether each stationary point of f is a local maximum, local minimum or horizontal point of inflection.

There's an alternative way to apply the first derivative test, which you can use instead of constructing a table of signs, if you prefer. It's described in the following box. The different possibilities in step 3 are illustrated in Figure 44.

Strategy:
To apply the first derivative test by choosing sample points

1. Choose two points (that is, two x-values) fairly close to the stationary point, one on each side.
2. Check that the function is differentiable at all points between the chosen points and the stationary point, and that there are no other stationary points between the chosen points and the stationary point.
3. Find the value of the derivative of the function at the two chosen points.
   - If the derivative is positive at the left chosen point and negative at the right chosen point, then the stationary point is a local maximum.
   - If the derivative is negative at the left chosen point and positive at the right chosen point, then the stationary point is a local minimum.
   - If the derivative is positive at both chosen points or negative at both chosen points, then the stationary point is a horizontal point of inflection.

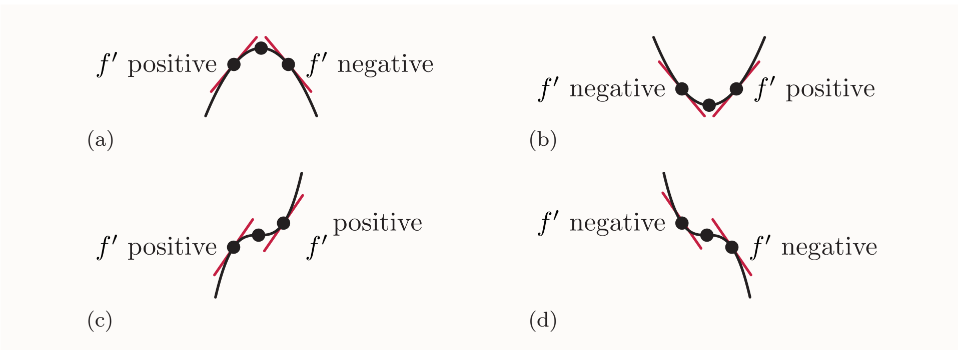


Figure 44 How to recognise, by considering points on each side of a stationary point, whether the stationary point is (a) a local maximum (b) a local minimum or (c), (d) a horizontal point of inflection

When you use this alternative way of applying the first derivative test, it doesn't matter how far the two points that you choose are from the stationary point, as long as the conditions in step 2 are satisfied. The reason why these conditions are important is explained after the next example. This example repeats Example 8, but uses the alternative way of applying the first derivative test.

Example 9 *Determining the nature of stationary points using the alternative way of applying the first derivative test*

Consider the function

$$f(x) = \tfrac{4}{3}x^3 + 5x^2 - 6x - 2.$$

(a) Find the stationary points of f.

(b) Determine whether each stationary point of f is a local maximum, local minimum or horizontal point of inflection.

Solution

(a) This part is the same as in Example 8.

The derivative is

$$\begin{aligned} f'(x) &= 4x^2 + 10x - 6 \\ &= 2(2x^2 + 5x - 3) \\ &= 2(2x - 1)(x + 3). \end{aligned}$$

Solving the equation $f'(x) = 0$ gives

$$2(2x - 1)(x + 3) = 0;$$

that is,

$$x = \tfrac{1}{2} \quad \text{or} \quad x = -3.$$

Hence the stationary points of f are -3 and $\frac{1}{2}$.

(b) Apply the method in the box above. There are many possible choices of points in step 1; choose them to be simple as possible.

Consider the values -4, 0 and 1. The values -4 and 0 lie on each side of the stationary point -3, and the values 0 and 1 lie on each side of the stationary point $\frac{1}{2}$.

The function f is differentiable at all values of x (as is every polynomial function). Also, there are no stationary points between -4 and -3 or between -3 and 0. Similarly, there are no stationary points between 0 and $\frac{1}{2}$ or between $\frac{1}{2}$ and 1.

Since $f'(x) = 4x^2 + 10x - 6$, we have

$$\begin{aligned} &f'(-4) = 4(-4)^2 + 10(-4) - 6 = 64 - 40 - 6 = 18, \\ &f'(0) = -6, \\ &f'(1) = 4 + 10 - 6 = 8. \end{aligned}$$

Since f' is positive at -4 and negative at 0, the stationary point -3 is a local maximum.

Since f' is negative at 0 and positive at 1, the stationary point $\frac{1}{2}$ is a local minimum.

Figure 45 illustrates the reasons for the restrictions mentioned in the box above. Each of the three graphs shows a situation where you might be misled into thinking that a stationary point is a local minimum, whereas in fact it is a local maximum.

In Figure 45(a), the function isn't differentiable at all points between the chosen points and the stationary point, because it isn't even *defined* at all such points. It is undefined (and has vertical asymptotes) at $x = 4$ and $x = 8$.

In Figure 45(b), the function isn't differentiable at all points between the chosen points and the stationary point. It is not differentiable at $x = 4$ and $x = 8$.

Finally, in Figure 45(c), there are further stationary points between the chosen points and the stationary point of interest. There are stationary points at $x = 4$ and $x = 8$.

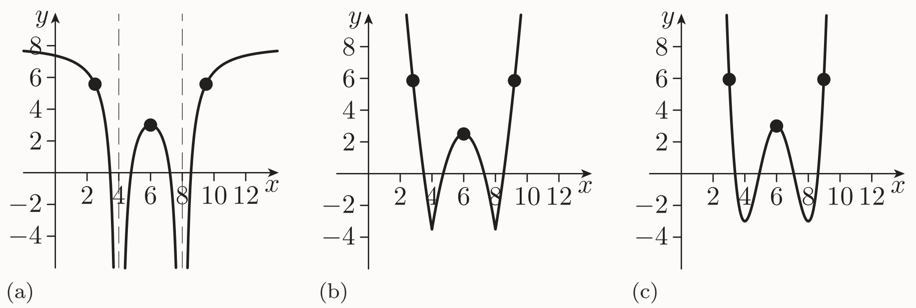


Figure 45 The graphs of three functions

You can try using the alternative way of applying the first derivative test in the next activity.

Activity 18 *Determining the nature of stationary points using the first derivative test, again*

Consider the function

$$f(x) = x^4 - 2x^3.$$

(a) Find the stationary points of f.
Hint: to factorise $f'(x)$, first notice that it has x^2 as a factor.

(b) By finding the value of f' at appropriate points, determine whether each stationary point of f is a local maximum, local minimum or horizontal point of inflection.

4.3 Uses of finding stationary points

In this subsection you'll see some situations where it can be useful to find, and sometimes determine the nature of, the stationary points of a function.

4.3.1 Finding the vertex of a parabola

In Unit 2 you saw some methods for finding the vertex of a u-shaped or n-shaped parabola. An alternative method is to use the fact that the vertex of a parabola is a stationary point. You're asked to do this in the next activity, for a parabola that models a particular economic situation.

The type of economic graph in this activity is different from the type that you considered in the 'chocolate manufacturer' example in Subsection 3.2. There you considered graphs of total cost against quantity of product made, whereas here you'll consider a graph of total *profit* against quantity of product made.

Activity 19 *Using differentiation to find the vertex of a parabola*

A bakery has found that its daily profit, £p, from making and selling cakes of a particular type can be modelled by the equation

$$p = -0.006q^2 + 3.3q,$$

where q is the quantity of cakes sold.

(Since the coefficient of q^2 is negative, the graph of p against q is an n-shaped parabola, so, as the quantity of cakes sold increases, the profit first increases and then decreases. The decrease might be caused by the fact that, when the quantity of cakes increases beyond a certain number, the limits of the bakery's facilities means that it becomes more expensive to make them – for example, the bakery may have to pay its staff at higher rates, for overtime or non-standard hours.)

(a) Find the formula for $\dfrac{dp}{dq}$.

(b) Use your answer to part (a) to find the stationary point of the parabola that is the graph of p against q. Hence write down the number of cakes that the bakery should make and sell per day if it wants to earn the maximum profit.

(c) Find the daily profit earned by the bakery from these cakes if it sells the number of them found in part (b).

4.3.2 Sketching the graph of a function

Another situation where it can be useful to find, and in this case determine the nature of, the stationary points of a function is when you want to sketch its graph. If you plot the stationary points, and indicate their nature, then you can often draw a reasonable sketch of the graph.

It's a good idea also to include on your sketch any x- or y-intercepts that you can find and plot easily. It's usually straightforward to find the y-intercept, if there is one, but often not so easy to find any x-intercepts.

As an illustration, we'll consider *cubic* functions, that is, functions of the form

$$f(x) = ax^3 + bx^2 + cx + d,$$

where a, b, c and d are constants, with $a \neq 0$. The graph of every cubic function, in common with the graph of every polynomial function that isn't a constant function, tends to plus or minus infinity at the left and right. You can find its shape between these extremes by finding its stationary points, if it has any.

The derivative of every cubic function is a quadratic function, because when you differentiate the formula for a cubic function you reduce the power of x in each term by 1. So to find the stationary points of a cubic function you have to solve a quadratic equation, and hence there are at most two stationary points.

Example 10 *Sketching the graph of a cubic function*

Sketch the graph of the function $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$.

Solution

Find the stationary points, including the y-coordinates.

The derivative is

$$f'(x) = x^2 - 2x - 3 = (x + 1)(x - 3).$$

Solving the equation $f'(x) = 0$ gives

$$(x + 1)(x - 3) = 0;$$

that is,

$$x = -1 \quad \text{or} \quad x = 3.$$

When $x = -1$,

$$\begin{aligned} y &= f(-1) \\ &= \tfrac{1}{3}(-1)^3 - (-1)^2 - 3(-1) + 2 \\ &= -\tfrac{1}{3} - 1 + 3 + 2 \\ &= \tfrac{11}{3}. \end{aligned}$$

When $x = 3$,

$$\begin{aligned} y &= f(3) \\ &= \tfrac{1}{3} \times 3^3 - 3^2 - 3 \times 3 + 2 \\ &= 9 - 9 - 9 + 2 \\ &= -7. \end{aligned}$$

So the stationary points are $(-1, \frac{11}{3})$ and $(3, -7)$.

Determine the nature of the stationary points.

x	$(-\infty, -1)$	-1	$(-1, 3)$	3	$(3, \infty)$
$x + 1$	$-$	0	$+$	$+$	$+$
$x - 3$	$-$	$-$	$-$	0	$+$
$f'(x)$	$+$	0	$-$	0	$+$
slope of f	/	—	\	—	/

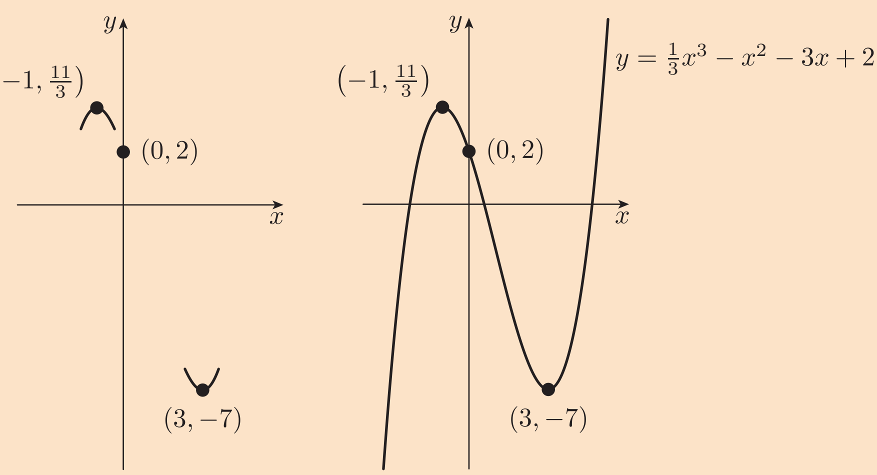
By the first derivative test, $(-1, \frac{11}{3})$ is a local maximum, and $(3, -7)$ is a local minimum.

Find the y-intercept. Don't try to find the x-intercepts in this case, as this isn't easy, because it involves solving the equation $\frac{1}{3}x^3 - x^2 - 3x + 2 = 0$.

When $x = 0$,

$$y = f(0) = 2.$$

Mark the stationary points and y-intercept on a pair of labelled axes, and indicate the nature of the stationary points, as shown on the left below. Hence sketch the graph, as shown on the right. Label the points that you've marked with their coordinates, and label the graph with its equation.



You've now seen the graphs of several cubic functions in this unit. These include the one in the example above, and the graphs of $f(x) = x^3$ and $g(x) = -x^3$ in Figure 40 on page 58. Before you go on, it's useful for you to get to know the general shapes of the graphs of cubic functions. You can do that in the next activity.

Activity 20 *Investigating the graphs of cubic functions*

Open the *Investigating cubic functions* applet. Experiment with changing the values of the constants a, b, c and d in the equation $y = ax^3 + bx^2 + cx + d$, and observe the shapes of the resulting graphs. Try to answer the following questions.

(a) How does the sign of a, the coefficient of x^3, affect the properties of the graph?

(b) How many x-intercepts can a cubic function have?

Finally, read through the list of properties of graphs of cubic functions given below this activity, and check that these properties seem to describe the graphs that you've seen.

In Activity 20 you should have seen evidence of the following facts.

Properties of graphs of cubic functions

The graph of every cubic function has the following properties.

- There are two, one or no stationary points.
- Apart from at any stationary points and in the interval between them if there are two,
  - the graph slopes up from left to right if the coefficient of x^3 in the rule of the function is positive
  - the graph slopes down from left to right if the coefficient of x^3 is negative.
- If there are two stationary points, then one is a local maximum and the other is a local minimum, and the graph slopes the other way in the interval between them.
- If there is one stationary point, then it is a horizontal point of inflection.
- There are one, two or three x-intercepts.
- There is one y-intercept.
- The graph tends to plus or minus infinity for large positive and large negative values of x.

Whether a cubic function has two, one or no stationary points depends on whether the quadratic equation that you solve to find the stationary points has two, one or no solutions.

It's useful to keep the properties in the box above in mind when you're sketching the graph of a cubic function.

Activity 21 *Sketching the graphs of cubic functions*

Sketch the graphs of the following cubic functions.

(a) $f(x) = -\frac{1}{3}x^3 - 4x^2 - 12x - 4$

(b) $f(x) = \frac{1}{9}x^3 - x^2 + 3x$

Hint for part (a): factorise the derivative $f'(x)$ into a product of *three* factors, one of which is -1.

4.3.3 Finding the greatest and least values of a function on an interval of the form $[a, b]$

A third situation where it can be useful to find the stationary points of a function is when you want to find the greatest or least value that a function takes on a particular interval of the form $[a, b]$, where a and b are real numbers. This is sometimes a helpful thing to do when the function models a real-life situation. You'll see some examples of this in Unit 7.

Usually the function that you're working with is **continuous** on the interval – this means that it has no discontinuities in the interval, or, informally, that you can draw its graph over the whole interval without taking your pen off the paper. (In particular, for a function to be continuous on an interval, it must also be *defined* for all values in the interval.)

If the function is also differentiable at all values in the interval except possibly the endpoints, then its greatest or least value on the interval occurs either at a stationary point in the interval or at an endpoint of the interval. For example, in Figure 46 the greatest value of the function on the interval shown occurs at the right endpoint of the interval, and the least value occurs at the local minimum in the interval.

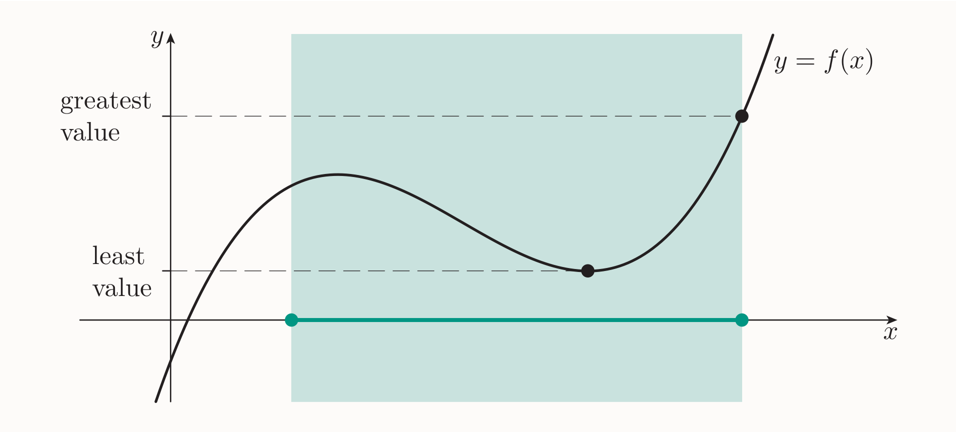


Figure 46 The greatest and least values of a particular function on a particular interval

This fact gives the following strategy, which is demonstrated in Example 11.

Strategy:
To find the greatest or least value of a function on an interval of the form $[a, b]$

(This strategy is valid when the function is continuous on the interval, and differentiable at all values in the interval except possibly the endpoints.)

1. Find the stationary points of the function.
2. Find the values of the function at any stationary points inside the interval, and at the endpoints of the interval.
3. Find the greatest or least of the function values found.

Example 11 *Finding the greatest and least values of a function on an interval of the form $[a, b]$*

Find the greatest and least values of the function

$$f(x) = -\tfrac{1}{3}x^3 + 2x^2 + 5x$$

on the interval $[-7, 7]$.

Solution

The function is

$$f(x) = -\tfrac{1}{3}x^3 + 2x^2 + 5x.$$

Find its stationary points.

Its derivative is

$$\begin{aligned} f'(x) &= -x^2 + 4x + 5 \\ &= -(x^2 - 4x - 5) \\ &= -(x-5)(x+1). \end{aligned}$$

So its stationary points are -1 and 5.

Find the values of f at any stationary points inside the interval, and at the endpoints of the interval.

Both stationary points are inside the interval, and the values of f at these points are

$$f(-1) = -\tfrac{1}{3}(-1)^3 + 2(-1)^2 + 5(-1) = -\tfrac{8}{3} = -2\tfrac{2}{3}$$
$$f(5) = -\tfrac{1}{3} \times 5^3 + 2 \times 5^2 + 5 \times 5 = \tfrac{100}{3} = 33\tfrac{1}{3}.$$

The values of f at the endpoints of the interval are

$$f(-7) = -\tfrac{1}{3}(-7)^3 + 2(-7)^2 + 5(-7) = \tfrac{532}{3} = 177\tfrac{1}{3}$$
$$f(7) = -\tfrac{1}{3} \times 7^3 + 2 \times 7^2 + 5 \times 7 = \tfrac{56}{3} = 18\tfrac{2}{3}.$$

Find the greatest and least of the values of f found.

The greatest value of f on the interval $[-7, 7]$ is $177\frac{1}{3}$, and the least value is $-2\frac{2}{3}$.

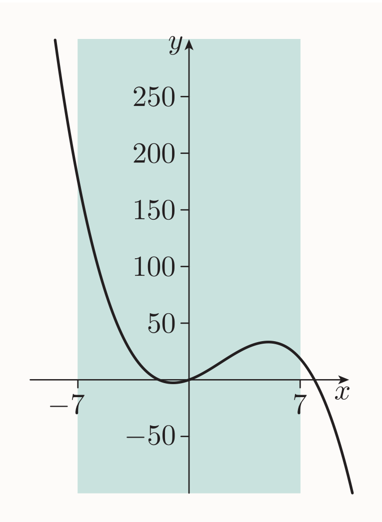


Figure 47 The graph of the function $f(x) = -\frac{1}{3}x^3 + 2x^2 + 5x$

The graph of the function in Example 11 is shown in Figure 47. You can see that its greatest and least values on the interval $[-7, 7]$ do seem to be the values found in the example, at least roughly.

When you use the strategy in the box above to find the greatest or least value of a function on an interval of the form $[a, b]$, it's important to make sure that any stationary points that you consider are *inside the interval*! For example, if Example 11 had asked for the greatest and least values of the function on the interval $[0, 7]$, rather than on $[-7, 7]$, then the solution would have considered the values of the function at the stationary point 5 and at the endpoints of the interval, and would have ignored the stationary point -1.

Activity 22 *Finding the greatest and least values of functions on intervals of the form* $[a, b]$

Find the greatest and least values of each of the following functions on the stated interval. Check that your answers seem reasonable by plotting the graphs of the functions using the module computer algebra system.

(a) $f(x) = \frac{1}{3}x^3 - x^2 - 8x + 1$ on the interval $[-3, 2]$

(b) $f(x) = 3x^2 - 2x + 5$ on the interval $[-1, 4]$

Activity 23 *Thinking about the strategy*

Give an example of a function and a closed interval such that the strategy in the box on page 71 doesn't work, because the function isn't differentiable (but is defined) at a value in the interval. Draw a sketch-graph to illustrate your answer.

Hint: some graphs of functions with values at which they aren't differentiable are given in Figure 9 on page 12.

5 Differentiating twice

In this section you'll learn what it means to differentiate a function more than once, and why this can be a helpful thing to do.

5.1 Second derivatives

As you've seen, the derivative of a function is itself a function, which is sometimes called the *derived function*. This means that the derivative of a function can itself be differentiated. The function that is obtained by differentiating a function twice in this way is called the **second derivative** (or **second derived function**) of the original function. In Lagrange notation, the second derivative of a function f is denoted by f'' (read as 'f double prime' or 'f double dash' or 'f double dashed').

For example, consider the function $f(x) = x^3$. The derivative of f is

$$f'(x) = 3x^2.$$

Differentiating this derivative gives the second derivative of f:

$$f''(x) = 6x.$$

Activity 24 *Finding a second derivative*

Find the second derivative of the function $f(x) = 2x^4 + 3x^2 + x$.

In Leibniz notation, the second derivative of y with respect to x is denoted by $\mathrm{d}^2y/\mathrm{d}x^2$ (read as 'd two y by d x squared'). For example, if $y = 4x^3 + 5x^2$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 + 10x, \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 24x + 10.$$

To see the thinking behind this notation for a second derivative, notice that the second derivative of y with respect to x can be written in Leibniz notation as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}y\right).$$

Historically, this was, rather loosely, abbreviated to

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 y, \quad \text{and then to} \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}.$$

The domain of the second derivative of a function consists of all the values of x at which its first derivative is differentiable. We say that the original function is **twice-differentiable** at such values of x.

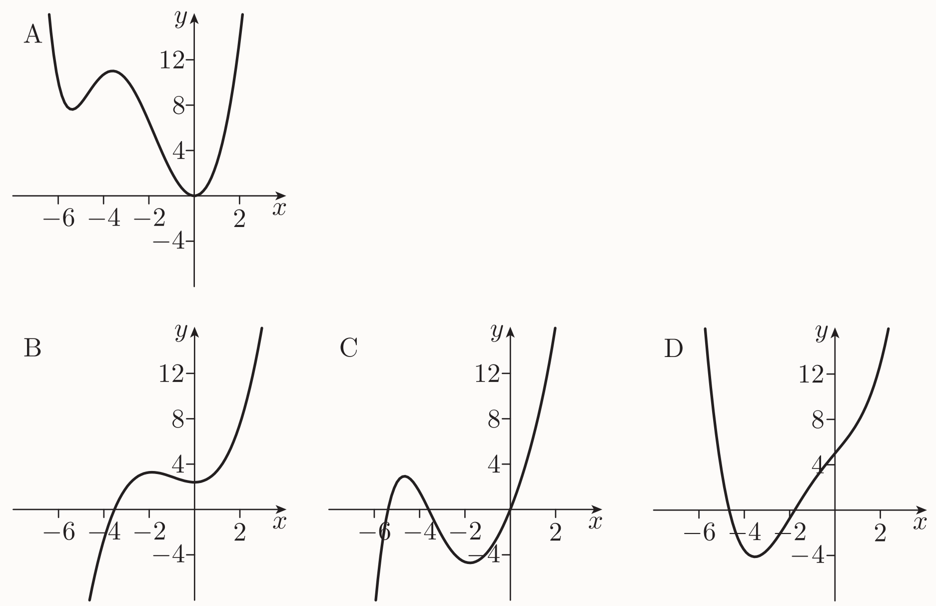
Once you've differentiated a function twice, you can go on to differentiate it three times, and then four times, and so on. The derivatives obtained in this way are called the **third derivative**, the **fourth derivative**, and so on. In Lagrange notation, the third derivative is denoted by f''' or $f^{(3)}$, the fourth derivative is denoted by $f^{(4)}$, and the higher derivatives are denoted in the same way. In Leibniz notation, the third derivative is denoted by $\mathrm{d}^3y/\mathrm{d}x^3$, the fourth derivative is denoted by $\mathrm{d}^4y/\mathrm{d}x^4$, and so on.

The domain of the third derivative of a function consists of all the values of x at which the second derivative of the function is differentiable, that is, all the values of x at which the original function is **three times differentiable**. Similarly a function can be **four times differentiable** at a value of x, and so on. It can also be **differentiable infinitely many times** at a value of x. For example, every polynomial function is differentiable infinitely many times at every value of x.

In Unit 11 you'll see one reason why the third and higher derivatives of functions are useful, but for now you won't need to work with derivatives beyond second derivatives.

Activity 25 *Thinking about derivatives and second derivatives*

Try to work out which of the graphs B, C and D below shows the derivative of the function in graph A, and which shows its second derivative.



The derivative of a function is sometimes called its **first derivative**, to emphasise that it's not one of the higher derivatives. This is the reason for the word 'first' in the name 'first derivative test', which you met on page 60. Later in this section, you'll meet the *second derivative test*, which has the same purpose as the first derivative test – determining the nature of a stationary point – but involves the second derivative of a function.

You know that the derivative of a function tells you the gradients of the graph of the function. So the second derivative of a function tells you the gradients of the graph of the first derivative. This means that the second derivative also gives you information about the shape of the graph of the original function.

To see this, suppose, for example, that there is an interval on which the second derivative of a particular function is *positive*. By the increasing/decreasing criterion, this means that the first derivative is *increasing* on that interval. In other words, the gradient of the graph of the original function is increasing on that interval. This means that the shape of the graph of the original function on the interval must be something like one of the shapes shown in Figure 48. All of these shapes are sections of graphs with increasing gradient. For example, in the second shape the gradient increases from large negative values to small negative values.

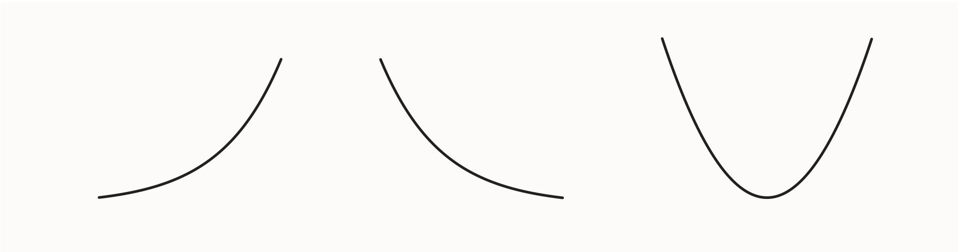


Figure 48 Sections of graphs with increasing gradient

A section of a graph shaped like one of the diagrams in Figure 48 is said to be *concave up*. More precisely, a graph is **concave up** on an interval if the tangents to the graph on that interval lie *below* the graph, as illustrated in Figure 49. (Of course the tangents may cross the graph outside the interval.) Informally, a section of a graph is concave up if it looks like a cross-section of a bowl that's the right way up, or part of such a cross-section. One way to remember the meaning of 'concave up' is to think of the rhyme 'Concave up, like a cup'.

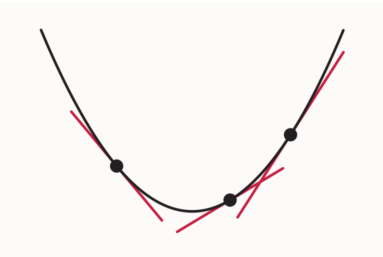


Figure 49 A section of a graph is concave up when the tangents lie below the graph

So, in summary, if the second derivative of a function is *positive* on an interval, then the graph of the function is concave up on the interval.

For example, Figure 50 shows the graphs of the function $f(x) = x^2$, its derivative $f'(x) = 2x$ and its second derivative $f''(x) = 2$. The domain of each of these functions consists of all of the real numbers. The second derivative f'' is positive on the whole domain, which tells you that the first derivative f' is increasing on the whole domain, which in turn tells you that the graph of the function f is concave up on its whole domain.

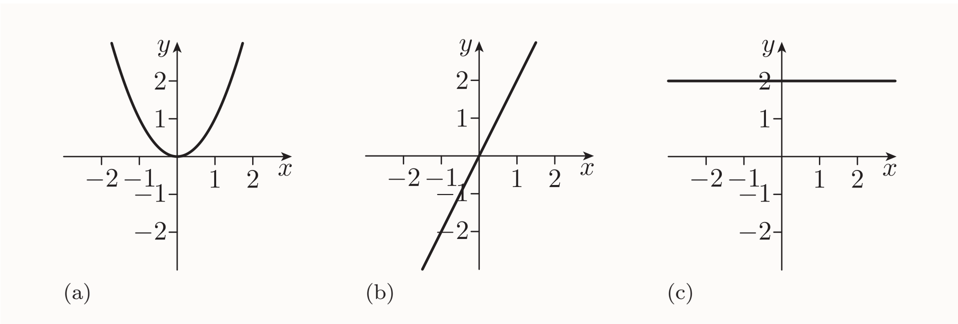


Figure 50 The graphs of (a) $f(x) = x^2$ (b) its derivative, $f'(x) = 2x$ (c) its second derivative, $f''(x) = 2$

Now suppose that there's an interval on which the second derivative of a particular function is *negative*. By the increasing/decreasing criterion, this means that the first derivative is *decreasing* on that interval. In other words, the gradient of the graph of the original function is decreasing on that interval. This means that the shape of the graph of the original function on the interval must be something like one of the shapes shown in Figure 51.

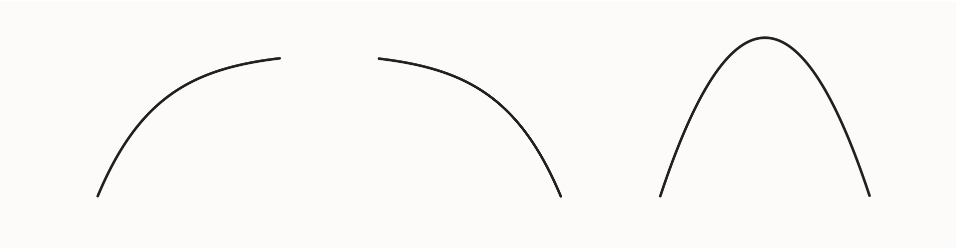


Figure 51 Sections of graphs with decreasing gradient

A section of a graph shaped like one of the diagrams in Figure 51 is said to be *concave down*. More precisely, a graph is **concave down** on an interval if the tangents to the graph on that interval lie *above* the graph, as illustrated in Figure 52. (Again the tangents may cross the graph outside the interval.) Informally, a section of a graph is concave down if it looks like a cross-section of a bowl that's the wrong way up, or part of such a cross-section. One way to remember the meaning of 'concave down' is to think of the rhyme 'Concave down, like a frown'.

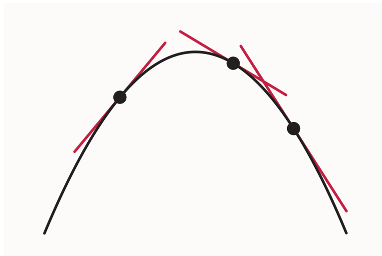


Figure 52 A section of a graph is concave down when the tangents lie above the graph

So, in summary, if the second derivative of a function is *negative* on an interval, then the graph of the function is concave down on the interval.

For example, Figure 53 shows the graphs of the function $f(x) = x^3$, its derivative $f'(x) = 3x^2$ and its second derivative $f''(x) = 6x$. The domain of each of these functions consists of all of the real numbers. The second derivative f'' is negative on the interval $(-\infty, 0)$ and positive on the interval $(0, \infty)$, which tells you that the first derivative f' is decreasing on the interval $(-\infty, 0)$ and increasing on the interval $(0, \infty)$, which in turn tells you that the function f is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.

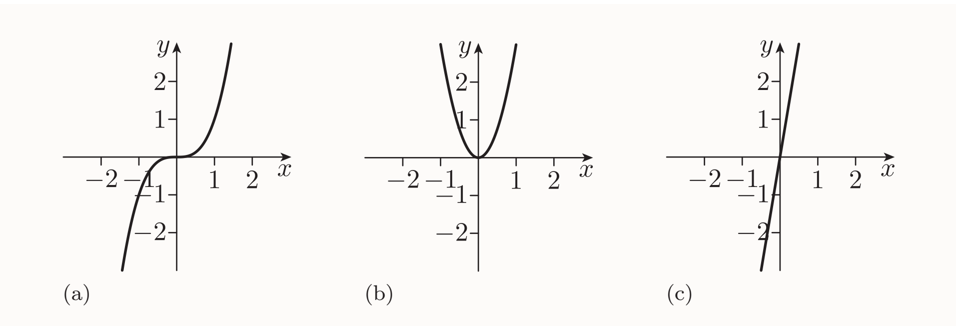


Figure 53 The graphs of (a) $f(x) = x^3$ (b) its derivative, $f'(x) = 3x^2$ (c) its second derivative, $f''(x) = 6x$

In some texts, the words *convex* and *concave* are used instead of concave up and concave down, respectively.

A point where a graph changes from concave up to concave down or vice versa is called a **point of inflection**. For example, the graph of $f(x) = x^3$ has a point of inflection at the origin. The tangents to a graph on one side of a point of inflection lie above the graph, and the tangents to the graph on the other side lie below the graph, as illustrated in Figure 54. The tangent at the point of inflection itself crosses the graph at that point. You can determine whether a point on the graph of a function is a point of inflection by checking whether its second derivative changes sign at that point.

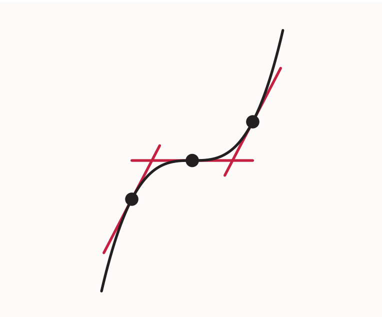


Figure 54 Tangents to a graph at and near a point of inflection

If the gradient of a graph at a point of inflection is zero, then the point is a **horizontal** point of inflection – this is a type of stationary point that you met earlier. Otherwise the point is a **slant** point of inflection. For example, the graph of $y = x^3$ and the graph in Figure 55(a) have horizontal points of inflection, and the graphs in Figures 55(b) and (c) have slant points of inflection.

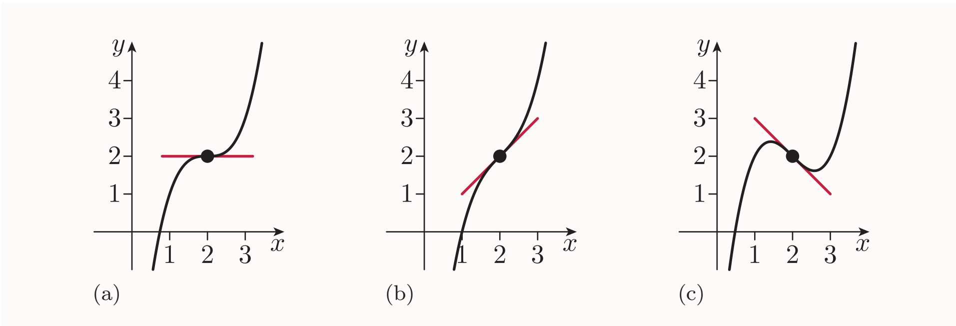


Figure 55 The graphs of (a) $y = x^3 - 6x^2 + 12x - 6$ (b) $y = x^3 - 6x^2 + 11x - 4$ (c) $y = x^3 - 6x^2 + 13x - 8$

If the graph of a function is concave up or concave down on an interval, then we also say that the function itself is concave up or concave down on the interval, respectively. (In general, for simplicity we often say that a function has a certain property if its graph has that property, and vice versa.) So the relationship between the sign of the second derivative of a function and the shape of the graph of the function can be summarised as follows.

Concave up/concave down criterion

If $f''(x)$ is positive for all x in an interval I, then f is concave up on I.

If $f''(x)$ is negative for all x in an interval I, then f is concave down on I.

Activity 26 *Using the concave up/concave down criterion*

Consider the function $f(x) = \frac{1}{6}x^4 - 2x^3 + 11x^2 - 18x$.

(a) Find the second derivative of this function.

(b) Hence show that this function is concave up on the whole of its domain (all the real numbers).

Hint for part (b): start by completing the square in your answer to part (a).

Activity 27 *Using the concave up/concave down criterion again*

Consider the function $f(x) = \frac{1}{12}x^4 - 2x^2$.

(a) Find the second derivative of this function.

(b) Hence show that this function is concave down on the interval $(-2, 2)$.

The connection between the sign of the second derivative of a function and the shape of the graph of the function gives a useful test, called the *second derivative test*, for determining the nature of some stationary points. This test can often, but not always, be used as an alternative to the first derivative test. It's based on the fact that the graph of a function is concave down at and around a local maximum, and concave up at and around a local minimum, as illustrated in Figure 56. In other words, by the concave up/concave down criterion, the second derivative of a function is negative at and around a local maximum, and positive at and around a local minimum.

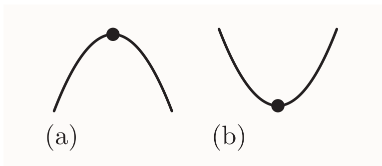


Figure 56 (a) A local maximum (b) a local minimum

Second derivative test (for determining the nature of a stationary point)

If, at a stationary point of a function, the value of the second derivative of the function is

- negative, then the stationary point is a local maximum
- positive, then the stationary point is a local minimum.

One way to remember the second derivative test is to keep in mind that a negative sign corresponds to a frown shape (a negative mood) and a positive sign corresponds to a smile shape (a positive mood). You might like to think of the following emoticons.

☹ ☺

Alternatively, you might prefer simply to remember that a local maximum corresponds to *decreasing gradient*, and hence a negative second derivative, whereas a local minimum corresponds to *increasing gradient*, and hence a positive second derivative. This is illustrated in Figure 57.

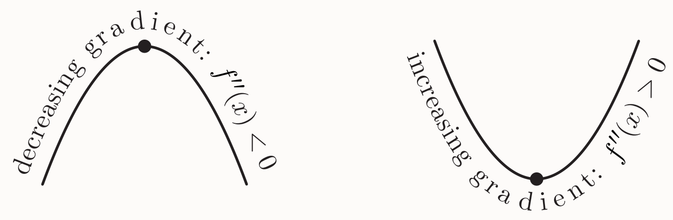


Figure 57 The change in the gradient at and around a local maximum and a local minimum

Example 12 *Determining the nature of stationary points using the second derivative test*

Consider the function

$$f(x) = x^3 - 3x^2 + 1.$$

(a) Find the stationary points of f.

(b) Use the second derivative test to determine the nature of each stationary point of f.

Solution

(a) We have

$$\begin{aligned} f'(x) &= 3x^2 - 6x \\ &= 3x(x-2). \end{aligned}$$

Solving the equation $f'(x) = 0$ gives

$$3x(x-2) = 0;$$

that is,

$$x = 0 \quad \text{or} \quad x = 2.$$

Hence the stationary points of f are 0 and 2.

(b) The second derivative is

$$f''(x) = 6x - 6.$$

For the stationary point 0,

$$f''(0) = 6 \times 0 - 6 = -6.$$

The second derivative is negative.

Hence, by the second derivative test, the stationary point 0 is a local maximum.

For the stationary point 2,

$$f''(2) = 6 \times 2 - 6 = 6.$$

The second derivative is positive.

Hence, by the second derivative test, the stationary point 2 is a local minimum.

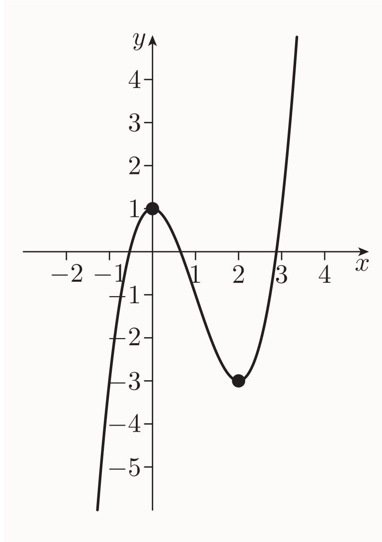


Figure 58 The graph of $f(x) = x^3 - 3x^2 + 1$

The graph of the function in Example 12 is shown in Figure 58. You can see that its stationary points are of the types determined in the example.

Activity 28 *Determining the nature of stationary points using the second derivative test*

Consider the function

$$f(x) = \tfrac{2}{3}x^3 - \tfrac{5}{2}x^2 + 2x.$$

(a) Find the stationary points of f, including the y-coordinates.

(b) Use the second derivative test to determine the nature of each stationary point.

(c) Using your answers to parts (a) and (b), sketch the graph of f.

Unfortunately, if the value of the second derivative of a function at a stationary point is *zero*, rather than either negative or positive, then you can't use the second derivative test to determine the nature of the stationary point. The stationary point might be a local maximum, a local minimum, a horizontal point of inflection, or none of these.

For example, Figure 59 shows the graphs of the functions $f(x) = x^4$, $g(x) = -x^4$, $h(x) = x^3$ and $k(x) = -x^3$. Each of these four functions has a stationary point at the origin, and at each of these stationary points the value of the second derivative is zero, as you might like to check. The stationary points on the four graphs are a local minimum, a local maximum and two horizontal points of inflection, respectively.

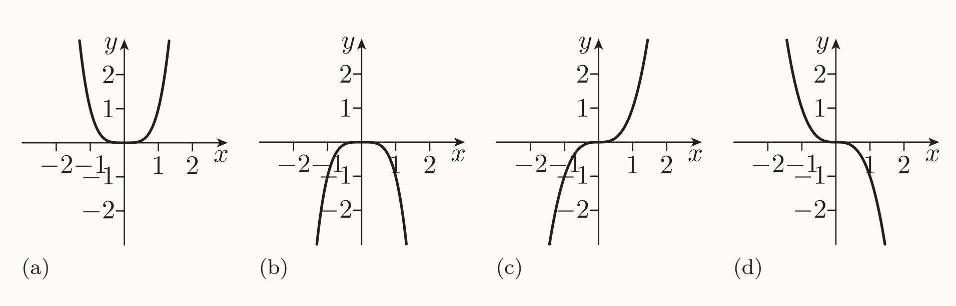


Figure 59 The graphs of (a) $f(x) = x^4$ (b) $g(x) = -x^4$ (c) $h(x) = x^3$ (d) $k(x) = -x^3$

So the second derivative test isn't as widely applicable as the first derivative test. If you're trying to use it to find the nature of a stationary point, and find that you can't because the value of the second derivative at the stationary point is zero, then try the first derivative test instead.

5.2 Rates of change of rates of change

Earlier in the unit you saw that if two variables x and y are related, then the derivative $\mathrm{d}y/\mathrm{d}x$ is the rate of change of y with respect to x. In the same way, the second derivative $\mathrm{d}^2y/\mathrm{d}x^2$ is the rate of change of $\mathrm{d}y/\mathrm{d}x$ with respect to x.

You saw that it's particularly useful to think of derivatives as rates of change when you're dealing with mathematical models of real-life situations.

For example, consider the graph in Figure 60, which is a repeat of Figure 32(b) on page 46. It's the displacement–time graph for a man walking along a straight path, and has equation

$$s = 6t - \tfrac{5}{8}t^2,$$

where t is the time in hours that the man has been walking, and s is his displacement in kilometres from his starting point.

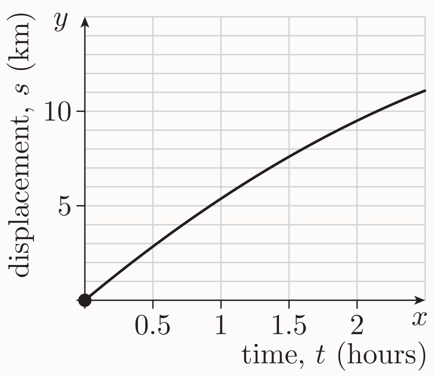


Figure 60 A displacement–time graph, with equation $s = 6t - \frac{5}{8}t^2$, for a man's walk along a straight path

As you know, the gradient of the displacement–time graph of an object is the object's velocity. So, as you saw in Example 5 on page 48, the man's velocity is given by the equation

$$v = \frac{\mathrm{d}s}{\mathrm{d}t};$$

that is,

$$v = 6 - \tfrac{5}{4}t.$$

The graph of this equation is shown in Figure 61. A graph like this, which shows an object's velocity plotted against time, is known as a **velocity–time graph**.

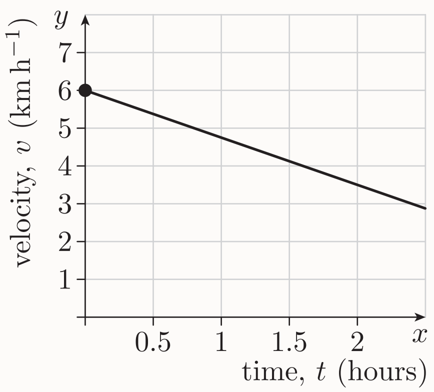


Figure 61 The velocity–time graph, with equation $v = 6 - \frac{5}{4}t$, corresponding to Figure 60

The velocity–time graph in Figure 61 shows that the man's velocity decreases as time goes on, as expected. For example, his velocity at the start of his walk is $6\,\mathrm{km\,h^{-1}}$, and his velocity one hour later is $4.75\,\mathrm{km\,h^{-1}}$.

The gradient of the velocity–time graph is the rate of change of the man's velocity with respect to time. The velocity–time graph is a straight line, so its gradient is the same at all points, and you can see from its equation, $v = 6 - \frac{5}{4}t$, that the numerical value of the gradient is $-\frac{5}{4} = -1.25$. Because the quantity on the horizontal axis (time) is measured in hours, and the quantity on the vertical axis (velocity) is measured in kilometres per hour, the gradient is measured in kilometres per hour per hour. So the gradient is -1.25 kilometres per hour per hour. This means that in each hour that passes, the man's velocity decreases by 1.25 kilometres per hour.

The units 'kilometres per hour per hour' are more usually called 'kilometres per hour squared', and are usually abbreviated as km/h^2 or $\text{km}\,\text{h}^{-2}$ (both read as 'kilometres per hour squared'). Similarly, 'metres per second per second' are known as 'metres per second squared' and abbreviated as m/s^2 or $\text{m}\,\text{s}^{-2}$, and so on.

The rate of change of velocity with respect to time is called **acceleration**, and is usually denoted by the variable a. Like displacement and velocity, acceleration can be constant, or it can take different values at different times. For example, you've just seen that the man on his walk has a constant acceleration of $-1.25\,\text{km}\,\text{h}^{-2}$. Also like displacement and velocity, the values that acceleration takes can be positive, negative or zero. The fact that the man's acceleration is *negative* at all times corresponds to the fact that his velocity is *decreasing* throughout his walk. Similarly, a positive acceleration would correspond to *increasing* velocity.

Because the gradient of the velocity–time graph of an object is the acceleration of the object, the important facts in the box on page 47 can be extended as follows.

Suppose that an object is moving in a straight line. If t is the time that has elapsed since some chosen point in time, and s, v and a are the displacement, velocity and acceleration of the object, respectively, then

$$v = \frac{\mathrm{d}s}{\mathrm{d}t}, \quad a = \frac{\mathrm{d}v}{\mathrm{d}t} \quad \text{and} \quad a = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}.$$

(Time, displacement, velocity and acceleration can be measured in any suitable units, as long as they are consistent.)

An example of a set of consistent units is seconds for time, metres for displacement, metres per second for velocity, and metres per second squared for acceleration.

In the discussion above, the value of the man's acceleration was found by using the fact that the equation relating his velocity to time is the equation of a straight line. An alternative way to find his acceleration is to use differentiation, as illustrated in the example below. The advantage of this method is, of course, that it can be used even when the acceleration isn't constant.

Example 13 *Using differentiation to find an acceleration*

Suppose that a man walks along a straight path, and his displacement s (in kilometres) from his starting point at time t (in hours) after he began his walk is given by the equation $s = 6t - \frac{5}{8}t^2$. Use differentiation to show that the man's acceleration is constant, and to find its value.

Solution

The man's displacement s (in km) at time t (in hours) is given by

$$s = 6t - \tfrac{5}{8}t^2.$$

Hence his velocity v (in $\mathrm{km\,h^{-1}}$) at time t (in hours) is given by

$$\begin{aligned} v &= \frac{\mathrm{d}s}{\mathrm{d}t} \\ &= 6 - \tfrac{5}{8} \times 2t \\ &= 6 - \tfrac{5}{4}t. \end{aligned}$$

It follows that his acceleration a (in $\mathrm{km\,h^{-2}}$) at time t (in hours) is given by

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -\tfrac{5}{4} = -1.25.$$

That is, the man's acceleration is constant, with value $-1.25\ \mathrm{km\,h^{-2}}$.

When you're working with displacement, velocity and acceleration along a straight line, it's important to keep in mind that *displacement* is different from *distance*, and *velocity* is different from *speed*. As you know, distance is the magnitude of displacement, and speed is the magnitude of velocity. For example, an object that has a displacement of -10 metres from some reference point has a distance of 10 metres from that point, and, similarly, an object that has a velocity of $-2\,\mathrm{m\,s^{-1}}$ has a speed of $2\,\mathrm{m\,s^{-1}}$. Unfortunately, the magnitude of acceleration is also known as acceleration, but this doesn't usually cause any confusion in practice.

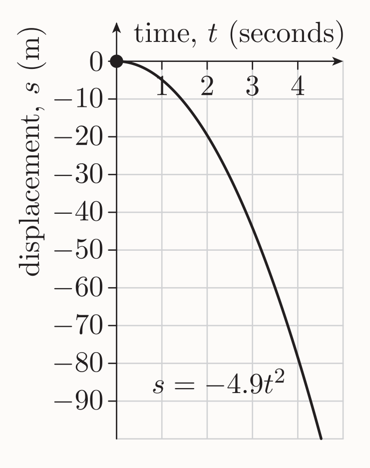


Figure 62 The displacement–time graph for a falling object

Consider the graph in Figure 62, which is the displacement–time graph for an object falling from rest, where the positive direction along the line of motion has been taken to be upwards and the reference point has been taken to be the point from which the object falls. Notice that the object's displacement from its starting point is decreasing, but its *distance* from its starting point is *increasing*. In general, if an object's displacement from a chosen reference point is always negative over some time interval, then saying that its displacement is decreasing over the time interval is the same as saying that its distance from the reference point is increasing over the time interval, and vice versa.

Similarly, if an object's velocity is always negative over some time interval, then saying that its velocity is decreasing over the time interval is the same as saying that its speed is increasing, and vice versa. For example, in Figure 62 the velocity (the gradient of the graph) is decreasing, but the speed of the object is increasing.

Activity 29 *Using differentiation to find an acceleration*

Suppose that the displacement of an object along a straight line is given by the equation

$$s = \tfrac{1}{3}t^3 - 5t^2 + 16t + 8,$$

where t is the time in seconds and s is the displacement in metres.

(a) Find an equation giving the object's velocity in terms of time, and an equation giving the object's acceleration in terms of time.

(b) Is the object's acceleration constant? Explain how you know.

(c) Find the object's velocity and acceleration at time 6 seconds.

(d) This part of the question is about the time interval from 5 seconds to 8 seconds (not including the endpoints of this time interval).

(i) Over this time interval, is the object's velocity increasing, decreasing or neither? Explain how you know.

(ii) Over the time interval, is the object's velocity always positive, always negative or neither?

(iii) Over this time interval, is the object's speed increasing, decreasing or neither? Explain how you know.

Activity 30 *Using differentiation to find another acceleration*

In Subsection 3.1 you saw that the displacement of an object falling from rest is given by the equation

$$s = -4.9t^2,$$

where t is the time in seconds that the object has been falling, and s is its displacement in metres from its starting point (where the positive direction is upwards). This is assuming that the effect of air resistance is negligible.

Show that the acceleration of the object is constant, and find its value.

In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for reelection.

(Rossi, H. (1996) 'Mathematics is an edifice, not a toolbox', *Notices of the American Mathematical Society*, vol. 43, no. 10.)

Activity 30 shows that the model for the motion of a falling object given in Subsection 3.1 is based on *constant acceleration*. It's a good model for the motion of any object falling from rest, provided that the effect of air resistance is negligible (which it is if the object is fairly compact and the fall is fairly short). The constant acceleration of a falling object in the absence of air resistance is known as the **acceleration due to gravity**.

If the falling object is near the surface of the Earth, then its acceleration due to gravity is approximately the value found in Activity 30. However, the value of the acceleration due to gravity varies slightly depending on the location of the object on the surface of the Earth. This is because of variations in the shape and density of the Earth. For example, the magnitude of the acceleration due to gravity in London is $9.812\,\mathrm{m\,s^{-2}}$, to three decimal places, whereas in Sydney it is $9.797\,\mathrm{m\,s^{-2}}$.

Solutions to activities

Solution to Activity 1

In the first half-hour the man walks about three kilometres, and in the final half-hour he walks about one and a half kilometres.

Solution to Activity 2

(a) Large negative gradient: graph E.

(b) Small negative gradient: graph C.

(c) Zero gradient: graph A.

(d) Small positive gradient: graph B.

(e) Large positive gradient: graph D.

Solution to Activity 3

The point on the graph with x-coordinate 0.9 has y-coordinate $0.9^2 = 0.81$. The gradient of the line through $(1, 1)$ and $(0.9, 0.81)$ is

$$\frac{\text{rise}}{\text{run}} = \frac{0.81 - 1}{0.9 - 1} = \frac{-0.19}{-0.1} = 1.9.$$

So another approximate value for the gradient of the graph at $(1, 1)$ is 1.9.

Solution to Activity 4

What you should have found in this activity is discussed in the text after the activity.

Solution to Activity 5

By formula (1), the gradient of the graph of $y = x^2$ at the point with x-coordinate 3 is

$$2 \times 3 = 6.$$

Similarly, the gradient at the point with x-coordinate -1.5 is

$$2 \times (-1.5) = -3.$$

Solution to Activity 6

(a) Multiplying out $(x+h)^4$ gives

$$\begin{aligned}
&(x+h)^4\\
&= (x+h)(x+h)^3\\
&= (x+h)(x^3 + 3x^2h + 3xh^2 + h^3)\\
&= x(x^3 + 3x^2h + 3xh^2 + h^3)\\
&\quad + h(x^3 + 3x^2h + 3xh^2 + h^3)\\
&= x^4 + 3x^3h + 3x^2h^2 + xh^3\\
&\qquad + x^3h + 3x^2h^2 + 3xh^3 + h^4\\
&= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4.
\end{aligned}$$

(b) The difference quotient for the function $f(x) = x^4$ at x is

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^4 - x^4}{h}.$$

By part (a),

$$\begin{aligned}
&\frac{f(x+h) - f(x)}{h}\\
&= \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}\\
&= \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}\\
&= 4x^3 + 6x^2h + 4xh^2 + h^3.
\end{aligned}$$

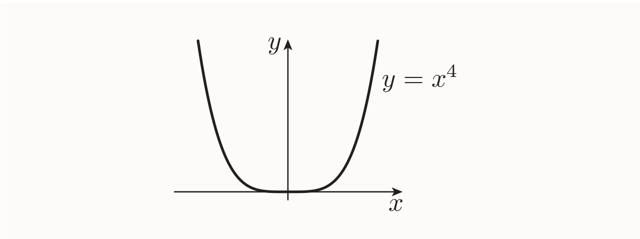
Each of the terms in the final expression above, except the first term, contains the factor h. Hence, as h gets closer and closer to zero, each of the terms except the first term gets closer and closer to zero. So the value of the whole expression gets closer and closer to the value of the first term, $4x^3$. That is, the formula for the derivative is

$$f'(x) = 4x^3.$$

(c) By the formula found in part (b), the gradient of the graph of the function $f(x) = x^4$ at the point with x-coordinate $\frac{1}{4}$ is

$$4 \times \left(\tfrac{1}{4}\right)^3 = 4 \times \tfrac{1}{64} = \tfrac{1}{16}.$$

(The graph of $f(x) = x^4$ is shown below.)



Solution to Activity 7

(a) $f(x) = x^8$, so

$$f'(x) = 8x^7.$$

(b) $f(x) = x^5$, so

$$f'(x) = 5x^4.$$

(c) $f(x) = \dfrac{1}{x^3} = x^{-3}$, so

$$f'(x) = -3x^{-4} = -\frac{3}{x^4}.$$

(d) $f(x) = x^{3/2}$, so
$$f'(x) = \tfrac{3}{2}x^{1/2} = \tfrac{3}{2}\sqrt{x}.$$
(e) $f(x) = \dfrac{1}{x} = x^{-1}$, so
$$f'(x) = -1 \times x^{-2} = -\frac{1}{x^2}.$$
(f) $f(x) = x^{5/2}$, so
$$f'(x) = \tfrac{5}{2}x^{3/2}.$$
(This derivative can also be written as $f'(x) = \tfrac{5}{2}\sqrt{x^3}$.)

(g) $f(x) = x^{4/3}$, so
$$f'(x) = \tfrac{4}{3}x^{1/3}.$$
(This form of the derivative is preferable to $f'(x) = \tfrac{4}{3}\sqrt[3]{x}$, which could be misread as $f'(x) = \tfrac{43}{3}\sqrt{x}$.)

(h) $f(x) = \dfrac{1}{x^8} = x^{-8}$, so
$$f'(x) = -8x^{-9} = -\frac{8}{x^9}.$$
(i) $y = \dfrac{1}{x^{1/4}} = x^{-1/4}$, so
$$\frac{dy}{dx} = -\frac{1}{4}x^{-5/4} = -\frac{1}{4} \times \frac{1}{x^{5/4}} = -\frac{1}{4x^{5/4}}.$$
(j) $y = x^{1/3}$, so
$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}.$$
(k) $y = \dfrac{1}{\sqrt{x}} = \dfrac{1}{x^{1/2}} = x^{-1/2}$, so
$$\frac{dy}{dx} = -\frac{1}{2}x^{-3/2} = -\frac{1}{2} \times \frac{1}{x^{3/2}} = -\frac{1}{2x^{3/2}}.$$
(l) $y = \sqrt[3]{x^5} = x^{5/3}$, so
$$\frac{dy}{dx} = \tfrac{5}{3}x^{2/3}.$$
(m) $y = x^{2/7}$, so
$$\frac{dy}{dx} = \frac{2}{7}x^{-5/7} = \frac{2}{7} \times \frac{1}{x^{5/7}} = \frac{2}{7x^{5/7}}.$$
(n) $f(x) = x^{-2}$, so
$$f'(x) = -2x^{-3} = -\frac{2}{x^3}.$$
(o) $f(x) = \dfrac{1}{x^{1/3}} = x^{-1/3}$, so
$$f'(x) = -\frac{1}{3}x^{-4/3} = -\frac{1}{3} \times \frac{1}{x^{4/3}} = -\frac{1}{3x^{4/3}}.$$
(p) $f(x) = \dfrac{1}{x^4} = x^{-4}$, so
$$f'(x) = -4x^{-5} = -\frac{4}{x^5}.$$

Solution to Activity 8

By the solution to Example 2(c), the derivative of $y = \sqrt{x}$ is
$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}.$$
So the gradient of the graph at the point with x-coordinate 4 is
$$\frac{1}{2\sqrt{4}} = \frac{1}{2 \times 2} = \frac{1}{4}.$$
(Check: the gradient of the line in Figure 22, in the margin beside this activity, appears to be about $\tfrac{1}{4}$.)

Solution to Activity 9

(a) $f(x) = 5x^3$, so
$$f'(x) = 5 \times 3x^2 = 15x^2.$$
(b) $f(x) = -x^7$, so
$$f'(x) = -7x^6.$$
(c) $f(x) = 2\sqrt{x} = 2x^{1/2}$, so
$$\begin{aligned} f'(x) &= 2 \times \frac{1}{2}x^{-1/2} \\ &= \frac{1}{x^{1/2}} \\ &= \frac{1}{\sqrt{x}}. \end{aligned}$$
(d) $f(x) = 6x$, so
$$f'(x) = 6 \times 1 = 6.$$
(e) $f(x) = \dfrac{x}{4} = \dfrac{1}{4}x$, so
$$f'(x) = \frac{1}{4} \times 1 = \frac{1}{4}.$$
(f) $f(x) = \dfrac{2}{x} = 2x^{-1}$, so
$$\begin{aligned} f'(x) &= 2 \times (-1)x^{-2} \\ &= -\frac{2}{x^2}. \end{aligned}$$
(g) $f(x) = -7x$, so
$$f'(x) = -7 \times 1 = -7.$$

(h) $y = \dfrac{\sqrt{x}}{3} = \dfrac{1}{3}x^{1/2}$, so

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{3} \times \frac{1}{2}x^{-1/2}\\ &= \frac{1}{6} \times \frac{1}{x^{1/2}}\\ &= \frac{1}{6\sqrt{x}}.\end{aligned}$$

(i) $y = \dfrac{8}{x^2} = 8x^{-2}$, so

$$\begin{aligned}\frac{dy}{dx} &= 8 \times (-2)x^{-3}\\ &= -\frac{16}{x^3}.\end{aligned}$$

(j) $y = -\dfrac{5}{x} = -5x^{-1}$, so

$$\begin{aligned}\frac{dy}{dx} &= -5 \times (-1)x^{-2}\\ &= \frac{5}{x^2}.\end{aligned}$$

(k) $y = \dfrac{4}{\sqrt{x}} = \dfrac{4}{x^{1/2}} = 4x^{-1/2}$, so

$$\begin{aligned}\frac{dy}{dx} &= 4 \times \left(-\frac{1}{2}\right) x^{-3/2}\\ &= -\frac{2}{x^{3/2}}.\end{aligned}$$

(l) $y = 4x^{3/2}$, so

$$\begin{aligned}\frac{dy}{dx} &= 4 \times \tfrac{3}{2}x^{1/2}\\ &= 6\sqrt{x}.\end{aligned}$$

(m) $y = \dfrac{1}{3\sqrt{x}} = \dfrac{1}{3x^{1/2}} = \dfrac{1}{3}x^{-1/2}$, so

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{3} \times \left(-\frac{1}{2}\right) x^{-3/2}\\ &= -\frac{1}{6} \times \frac{1}{x^{3/2}}\\ &= -\frac{1}{6x^{3/2}}.\end{aligned}$$

(n) $y = -12x^{1/3}$, so

$$\begin{aligned}\frac{dy}{dx} &= -12 \times \frac{1}{3}x^{-2/3}\\ &= -4 \times \frac{1}{x^{2/3}}\\ &= -\frac{4}{x^{2/3}}.\end{aligned}$$

Solution to Activity 10

The derivative of the function $f(x) = 3x^2$ is

$$f'(x) = 3 \times 2x = 6x.$$

So the gradient of this function f at the point with x-coordinate 2 is

$$6 \times 2 = 12.$$

Solution to Activity 11

(a) $f(x) = 6x^2 - 2x + 1$, so

$$f'(x) = 12x - 2.$$

(b) $f(x) = \frac{2}{3}x^3 + 2x^2 + x - \frac{1}{2}$, so

$$\begin{aligned}f'(x) &= \tfrac{2}{3} \times 3x^2 + 2 \times 2x + 1\\ &= 2x^2 + 4x + 1.\end{aligned}$$

(c) $f(x) = 5x + 1$, so

$$f'(x) = 5.$$

(Notice that in general the derivative of $f(x) = mx + c$ is $f'(x) = m$, as you'd expect.)

(d) $f(x) = \frac{1}{2}x + \sqrt{x} = \frac{1}{2}x + x^{1/2}$, so

$$\begin{aligned}f'(x) &= \frac{1}{2} + \frac{1}{2}x^{-1/2}\\ &= \frac{1}{2} + \frac{1}{2} \times \frac{1}{x^{1/2}}\\ &= \frac{1}{2} + \frac{1}{2} \times \frac{1}{\sqrt{x}}\\ &= \frac{1}{2}\left(1 + \frac{1}{\sqrt{x}}\right).\end{aligned}$$

(e)
$$\begin{aligned}f(x) &= (1 + x^2)(1 + 3x)\\ &= 1 + 3x + x^2 + 3x^3,\end{aligned}$$

so

$$f'(x) = 3 + 2x + 9x^2.$$

(f)
$$\begin{aligned}f(x) &= (x + 3)^2\\ &= x^2 + 6x + 9,\end{aligned}$$

so

$$f'(x) = 2x + 6.$$

(g) $f(x) = 30(x^{3/2} - x)$, so

$$\begin{aligned}f'(x) &= 30(\tfrac{3}{2}x^{1/2} - 1)\\ &= 30(\tfrac{3}{2}\sqrt{x} - 1)\\ &= 45\sqrt{x} - 30\\ &= 15(3\sqrt{x} - 2).\end{aligned}$$

(h) $f(x) = x(x^{3/2} - x) = x^{5/2} - x^2$, so

$$f'(x) = \tfrac{5}{2}x^{3/2} - 2x.$$

(Note that this function can't be differentiated in a similar way to the function in part (g), because the multiplier of the brackets, x, isn't a *constant*, and hence the constant multiple rule doesn't apply.)

(i)
$$\begin{aligned} y &= \frac{(x-2)(x+5)}{x} \\ &= \frac{x^2+3x-10}{x} \\ &= x + 3 - 10x^{-1}, \end{aligned}$$

so

$$\begin{aligned} \frac{dy}{dx} &= 1 - 10 \times (-1)x^{-2} \\ &= 1 + 10x^{-2} \\ &= 1 + \frac{10}{x^2}. \end{aligned}$$

(The final answer can also be expressed as

$$\frac{x^2+10}{x^2}.)$$

(j)
$$\begin{aligned} y &= \frac{x+\sqrt{x}}{x^2} \\ &= \frac{x + x^{1/2}}{x^2} \\ &= x^{-1} + x^{-3/2}, \end{aligned}$$

so

$$\begin{aligned} \frac{dy}{dx} &= -x^{-2} - \tfrac{3}{2}x^{-5/2} \\ &= -\frac{1}{x^2} - \frac{3}{2x^{5/2}}. \end{aligned}$$

(The final answer can be expressed in various ways. For example, you could combine the fractions and simplify the result like this:

$$\begin{aligned} -\frac{1}{x^2} - \frac{3}{2x^{5/2}} &= -\frac{2x}{2x^3} - \frac{3x^{1/2}}{2x^3} \\ &= -\frac{2x+3\sqrt{x}}{2x^3}.) \end{aligned}$$

(k)
$$\begin{aligned} y &= (x^{1/3}+1)(x^{1/3}+5x) \\ &= x^{2/3} + 5x^{4/3} + x^{1/3} + 5x, \end{aligned}$$

so

$$\begin{aligned} \frac{dy}{dx} &= \frac{2}{3}x^{-1/3} + \frac{20}{3}x^{1/3} + \frac{1}{3}x^{-2/3} + 5 \\ &= \frac{2}{3x^{1/3}} + \frac{20x^{1/3}}{3} + \frac{1}{3x^{2/3}} + 5. \end{aligned}$$

Solution to Activity 12

(a) The displacement s (in metres) of the object at time t (in seconds) is

$$s = -4.9t^2.$$

Hence the velocity v (in $\mathrm{m\,s^{-1}}$) of the object at time t (in seconds) is

$$\begin{aligned} v &= \frac{ds}{dt} \\ &= -9.8t. \end{aligned}$$

(b) When $t = 3$,

$$v = -9.8 \times 3 = -29.4.$$

So the velocity of the object 3 seconds into its fall is $-29.4\,\mathrm{m\,s^{-1}}$.

(c) When the object has a speed of $15\,\mathrm{m\,s^{-1}}$, it has a velocity of $-15\,\mathrm{m\,s^{-1}}$. So the time t at which the object is moving at this speed is given by

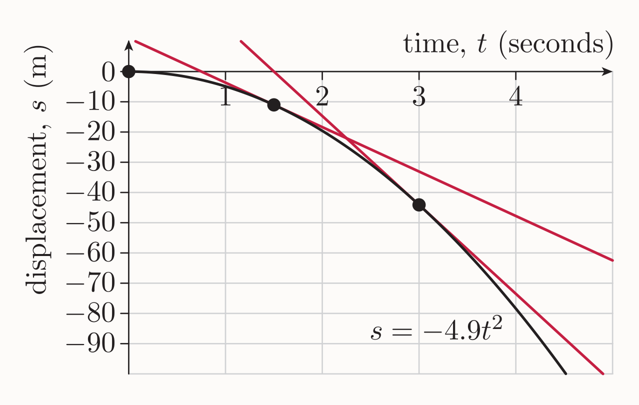
$$-15 = -9.8t.$$

This equation has solution

$$t = \frac{-15}{-9.8} = 1.5 \text{ (to 1 d.p.)}.$$

So the object takes about 1.5 seconds to reach a speed of $15\,\mathrm{m\,s^{-1}}$.

(As a check on parts (b) and (c), you can see that the gradients of the graph of the equation $s = -4.9t^2$, which is given in Figure 34 and repeated below, at the points with time-coordinates 1.5 and 3 seconds are about $-15\,\mathrm{m\,s^{-1}}$ and about $-30\,\mathrm{m\,s^{-1}}$, respectively. The tangents at these points are shown below.)



Solution to Activity 13

(a) The equation for c in terms of q is

$$c = 3000 + 4q + \tfrac{1}{125}q^2.$$

Hence

$$m = \frac{\mathrm{d}c}{\mathrm{d}q} = 4 + \tfrac{2}{125}q.$$

That is, an equation for m in terms of q is

$$m = 4 + \tfrac{2}{125}q.$$

(b) Substituting $q = 300$ into the equation found in part (a) gives

$$m = 4 + \tfrac{2}{125} \times 300 = 8.8.$$

So the marginal cost per kilogram of making extra chocolate when the amount of chocolate already being made is 300 kilograms is £8.80.

(c) The marginal cost is equal to the selling price when

$$m = 16.$$

By part (a), the value of q for which this happens is given by

$$4 + \tfrac{2}{125}q = 16.$$

Solving this equation gives

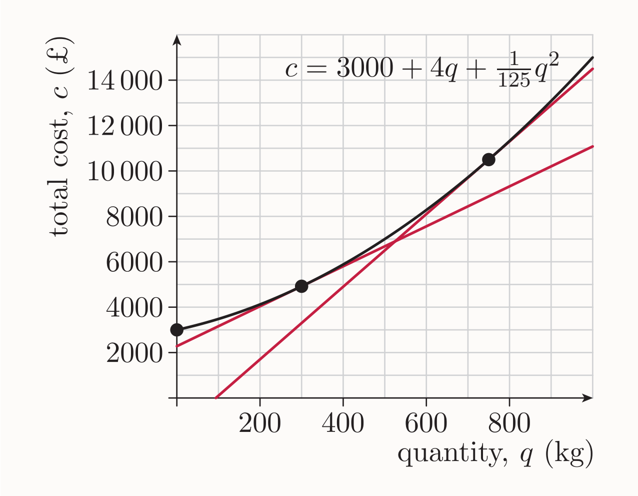
$$\tfrac{2}{125}q = 12,$$

and so

$$q = 750.$$

So the company should make 750 kg of chocolate per week.

(As a check on parts (b) and (c), you can see that the gradients of the graph in Figure 36 at the points with q-coordinates 300 and 750 kilograms are about £9 per kilogram and about £16 per kilogram, respectively. The tangents at these points are shown below.)



Solution to Activity 14

(a) $f(x) = \tfrac{2}{3}x^3 - 8x^2 + 30x - 36$, so

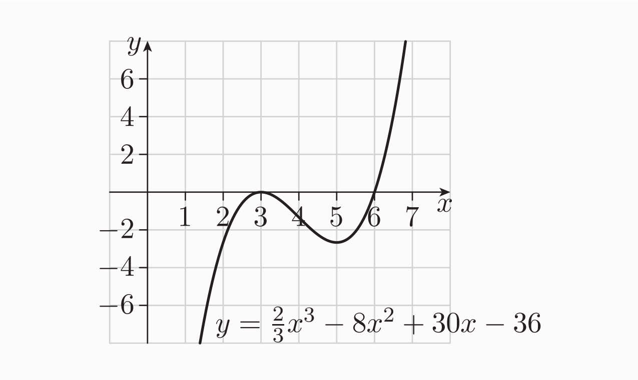
$$\begin{aligned} f'(x) &= 2x^2 - 16x + 30 \\ &= 2(x^2 - 8x + 15) \\ &= 2(x-3)(x-5). \end{aligned}$$

(b) When x is less than 3, the values of $x - 3$ and $x - 5$ are both negative, and hence the value of $f'(x) = 2(x-3)(x-5)$ is positive. Therefore, by the increasing/decreasing criterion, the function f is increasing on the interval $(-\infty, 3)$.

Similarly, when x is greater than 5, the values of $x - 3$ and $x - 5$ are both positive, and hence the value of $f'(x) = 2(x-3)(x-5)$ is also positive. Therefore, by the increasing/decreasing criterion, the function f is increasing on the interval $(5, \infty)$.

(c) When x is in the interval $(3, 5)$, the value of $x - 3$ is positive and the value of $x - 5$ is negative, and hence the value of $f'(x) = 2(x-3)(x-5)$ is negative. Therefore, by the increasing/decreasing criterion, the function f is decreasing on the interval $(3, 5)$.

(The graph of $f(x) = \frac{2}{3}x^3 - 8x^2 + 30x - 36$ is shown below.)



Solution to Activity 15

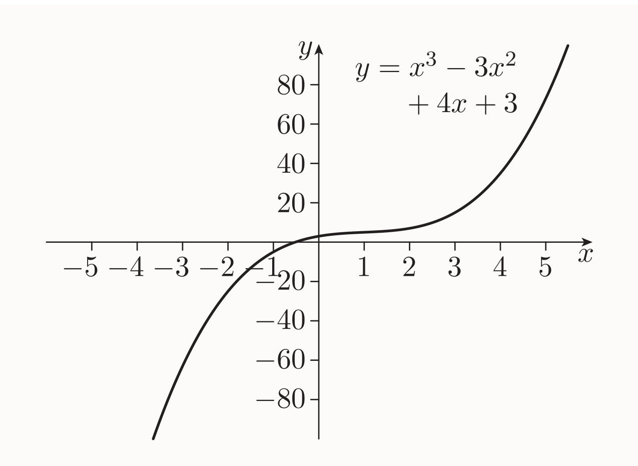
(a) $f(x) = x^3 - 3x^2 + 4x + 3$, so

$$\begin{aligned} f'(x) &= 3x^2 - 6x + 4 \\ &= 3(x^2 - 2x) + 4 \\ &= 3\left((x-1)^2 - 1\right) + 4 \\ &= 3(x-1)^2 + 1. \end{aligned}$$

(b) For every value of x, the expression $(x-1)^2$ is non-negative, and hence the expression $3(x-1)^2 + 1$ is positive.

That is, the derivative $f'(x)$ is positive on the interval $(-\infty, \infty)$. Therefore, by the increasing/decreasing criterion, the function f is increasing on the interval $(-\infty, \infty)$.

(The graph of $f(x) = x^3 - 3x^2 + 4x + 3$ is shown below.)



Solution to Activity 16

Here $f(x) = x^3 - x^2 - 2x$, so

$$f'(x) = 3x^2 - 2x - 2.$$

The expression for $f'(x)$ cannot be factorised easily, so we solve the equation $f'(x) = 0$ using the quadratic formula. This gives

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 3 \times (-2)}}{2 \times 3} \\ &= \frac{2 \pm \sqrt{28}}{6} \\ &= \frac{2 \pm 2\sqrt{7}}{6} \\ &= \frac{1 \pm \sqrt{7}}{3} \\ &= 1.22 \text{ or } -0.55 \text{ (to 2 d.p.).} \end{aligned}$$

That is, the stationary points are 1.22 and -0.55, to 2 decimal places.

Solution to Activity 17

(a) Here $f(x) = 3x^4 - 2x^3 - 9x^2 + 7$, so

$$\begin{aligned} f'(x) &= 12x^3 - 6x^2 - 18x \\ &= 6x(2x^2 - x - 3) \\ &= 6x(2x-3)(x+1). \end{aligned}$$

Solving the equation $f'(x) = 0$ gives

$$6x(2x-3)(x+1) = 0;$$

that is,

$$x = 0 \quad \text{or} \quad x = \tfrac{3}{2} \quad \text{or} \quad x = -1.$$

Hence the stationary points of f are 0, $\frac{3}{2}$ and -1.

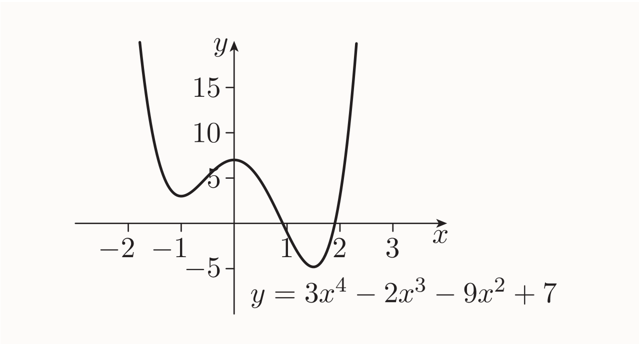
(b) A table of signs for $f'(x) = 6x(2x-3)(x+1)$ is given below. (The table is split to make it fit in a narrow text column.)

x	$(-\infty, -1)$	-1	$(-1, 0)$
$6x$	$-$	$-$	$-$
$2x-3$	$-$	$-$	$-$
$x+1$	$-$	0	$+$
$f'(x)$	$-$	0	$+$
slope of f	╲	—	╱

0	$(0, \frac{3}{2})$	$\frac{3}{2}$	$(\frac{3}{2}, \infty)$
0	$+$	$+$	$+$
$-$	$-$	0	$+$
$+$	$+$	$+$	$+$
0	$-$	0	$+$
—	↘	—	↗

The table shows that -1 is a local minimum, 0 is a local maximum, and $\frac{3}{2}$ is a local minimum.

(You can see these stationary points on the graph below.)



Solution to Activity 18

(a) $f(x) = x^4 - 2x^3$, so

$$f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3).$$

Solving the equation $f'(x) = 0$ gives

$$2x^2(2x - 3) = 0;$$

that is,

$$x = 0 \quad \text{or} \quad x = \tfrac{3}{2}.$$

Hence the stationary points of f are 0 and $\frac{3}{2}$.

(b) To determine the nature of the stationary points, apply the method described on page 62.

Consider the values -1, 1 and 2. The values -1 and 1 lie on each side of the stationary point 0, and the values 1 and 2 lie on each side of the stationary point $\frac{3}{2}$.

The function f is differentiable at all values of x (as is every polynomial function).

Also, there are no stationary points between -1 and 0 or between 0 and 1. Similarly, there are no stationary points between 1 and $\frac{3}{2}$ or between $\frac{3}{2}$ and 2.

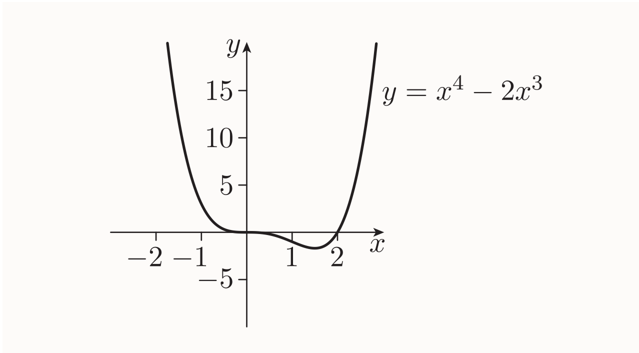
Since $f'(x) = 2x^2(2x - 3)$, we have

$$f'(-1) = 2(-1)^2(2(-1) - 3) = 2(-2 - 3) = -10$$
$$f'(1) = 2 \times 1^2(2 \times 1 - 3) = 2(2 - 3) = -2$$
$$f'(2) = 2 \times 2^2(2 \times 2 - 3) = 8(4 - 3) = 8.$$

Since the derivative is negative at both -1 and 1, the stationary point 0 is a horizontal point of inflection.

Since the derivative is negative at 1 and positive at 2, the stationary point $\frac{3}{2}$ is a local minimum.

(You can see these stationary points on the graph below.)



Solution to Activity 19

(a) Here

$$p = -0.006q^2 + 3.3q,$$

so

$$\frac{\mathrm{d}p}{\mathrm{d}q} = -0.012q + 3.3.$$

(b) The stationary point q is given by

$$\frac{\mathrm{d}p}{\mathrm{d}q} = 0;$$

that is,

$$-0.012q + 3.3 = 0,$$

so

$$q = \frac{3.3}{0.012} = 275.$$

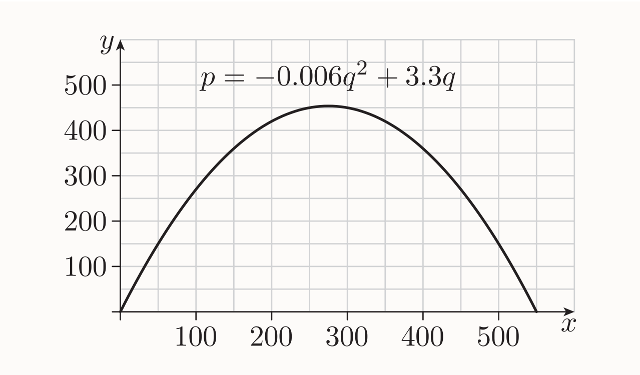
So if the bakery wants to earn the maximum profit, then it should make and sell 275 cakes per day.

(c) Substituting $q = 275$ into the formula for p gives

$$p = -0.006 \times 275^2 + 3.3 \times 275 = 453.75,$$

so the daily profit is £453.75.

(The graph of the equation $p = -0.006q^2 + 3.3q$ is shown below. You can see that the stationary point appears to be roughly $(275, 450)$, which accords with the results found above.)



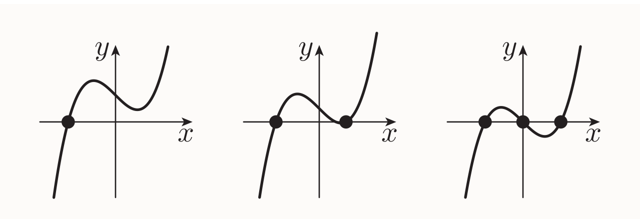
Solution to Activity 20

(a) If the coefficient of x^3 is positive, then the graph tends to infinity for large positive values of x, and tends to minus infinity for large negative values of x.

If the coefficient of x^3 is negative, then the graph tends to minus infinity for large positive values of x, and tends to infinity for large negative values of x.

(This is because for large positive and large negative values of x, the value of the term in x^3, the dominant term, overwhelms the values of the other terms.)

(b) A cubic function can have one, two or three x-intercepts, as illustrated below.



Solution to Activity 21

(a) The function is

$$f(x) = -\tfrac{1}{3}x^3 - 4x^2 - 12x - 4.$$

So the derivative is

$$\begin{aligned} f'(x) &= -x^2 - 8x - 12 \\ &= -(x^2 + 8x + 12) \\ &= -(x+6)(x+2). \end{aligned}$$

Solving the equation $f'(x) = 0$ gives

$$-(x+6)(x+2) = 0;$$

that is,

$$x = -6 \quad \text{or} \quad x = -2.$$

When $x = -6$,

$$\begin{aligned} y &= f(-6) \\ &= -\tfrac{1}{3}(-6)^3 - 4(-6)^2 - 12(-6) - 4 \\ &= 72 - 144 + 72 - 4 \\ &= -4. \end{aligned}$$

When $x = -2$,

$$\begin{aligned} y &= f(-2) \\ &= -\tfrac{1}{3}(-2)^3 - 4(-2)^2 - 12(-2) - 4 \\ &= \tfrac{8}{3} - 16 + 24 - 4 \\ &= \tfrac{20}{3}. \end{aligned}$$

So the stationary points are $(-6, -4)$ and $(-2, \frac{20}{3})$.

A table of signs for $f'(x) = -(x+6)(x+2)$ is given below. (The table is split to make it fit in a narrow text column.)

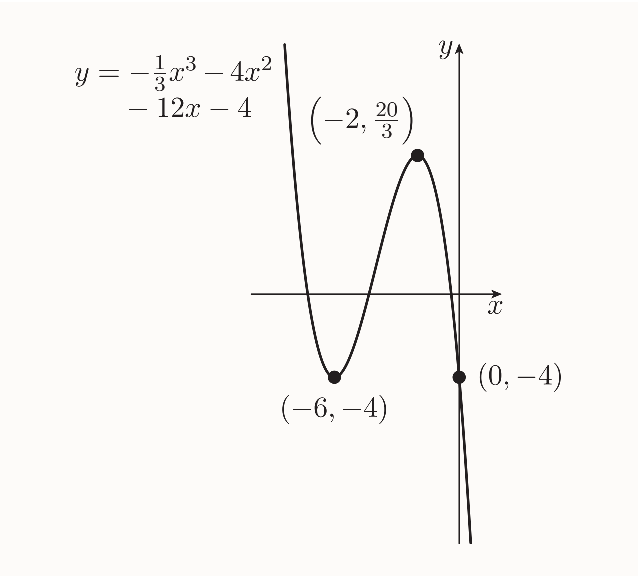
x	$(-\infty, -6)$	-6	$(-6, -2)$	-2	$(-2, \infty)$
-1	$-$	$-$	$-$	$-$	$-$
$x+6$	$-$	0	$+$	$+$	$+$
$x+2$	$-$	$-$	$-$	0	$+$
$f'(x)$	$-$	0	$+$	0	$-$
slope of f	$\searrow$	$-$	$\nearrow$	$-$	$\searrow$

By the first derivative test, $(-6, -4)$ is a local minimum, and $(-2, \frac{20}{3})$ is a local maximum.

When $x = 0$,

$$y = f(0) = -4.$$

A sketch of the graph is shown below.



(b) The function is

$$f(x) = \tfrac{1}{9}x^3 - x^2 + 3x.$$

So the derivative is

$$\begin{aligned} f'(x) &= \tfrac{1}{3}x^2 - 2x + 3 \\ &= \tfrac{1}{3}(x^2 - 6x + 9) \\ &= \tfrac{1}{3}(x-3)^2. \end{aligned}$$

Solving the equation $f'(x) = 0$ gives

$$\tfrac{1}{3}(x-3)^2 = 0;$$

that is,

$$x = 3.$$

When $x = 3$,

$$y = f(3) = \tfrac{1}{9} \times 3^3 - 3^2 + 3 \times 3 = 3.$$

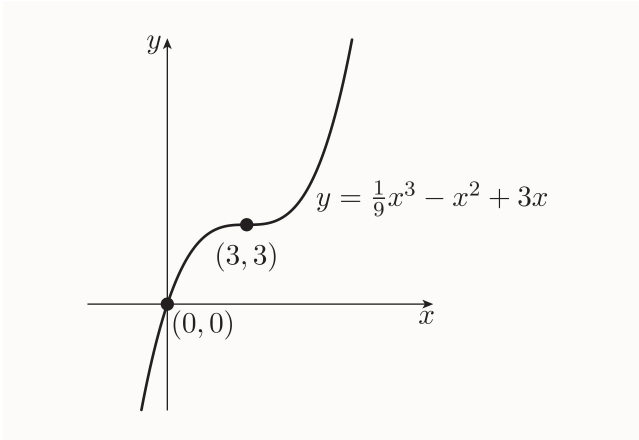
So there is just one stationary point, $(3, 3)$.

Since $f'(x)$ is positive everywhere except at $x = 3$, it follows, by the first derivative test, that $(3, 3)$ is a horizontal point of inflection and also, by the increasing/decreasing criterion, that f is increasing to the left and the right of the stationary point.

When $x = 0$,

$$y = f(0) = 0.$$

A sketch of the graph is shown below.



Solution to Activity 22

(a) Here

$$f(x) = \tfrac{1}{3}x^3 - x^2 - 8x + 1.$$

This gives

$$f'(x) = x^2 - 2x - 8 = (x-4)(x+2),$$

so the stationary points are -2 and 4.

Only the stationary point -2 is inside the interval, and the value of f at this point is

$$\begin{aligned} f(-2) &= \tfrac{1}{3}(-2)^3 - (-2)^2 - 8(-2) + 1 \\ &= -\tfrac{8}{3} - 4 + 16 + 1 \\ &= \tfrac{31}{3} = 10\tfrac{1}{3}. \end{aligned}$$

The values of f at the endpoints of the interval are

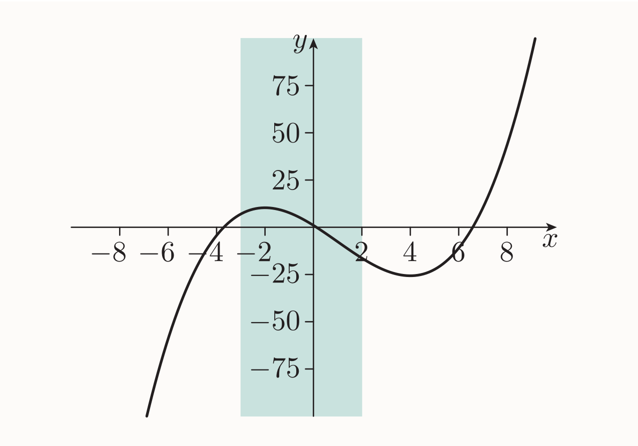
$$\begin{aligned} f(-3) &= \tfrac{1}{3}(-3)^3 - (-3)^2 - 8(-3) + 1 \\ &= -\tfrac{27}{3} - 9 + 24 + 1 \\ &= 7 \end{aligned}$$

and

$$\begin{aligned} f(2) &= \tfrac{1}{3} \times 2^3 - 2^2 - 8 \times 2 + 1 \\ &= \tfrac{8}{3} - 4 - 16 + 1 \\ &= -\tfrac{49}{3} = -16\tfrac{1}{3}. \end{aligned}$$

The greatest value of f on the interval $[-3, 2]$ is $10\frac{1}{3}$, and the least value is $-16\frac{1}{3}$.

(The graph of f is below.)



(b) Here

$$f(x) = 3x^2 - 2x + 5.$$

This gives

$$f'(x) = 6x - 2 = 2(3x - 1),$$

so the only stationary point is $\frac{1}{3}$.

This stationary point is inside the interval, and the value of f at this point is

$$\begin{aligned} f\left(\tfrac{1}{3}\right) &= 3\left(\tfrac{1}{3}\right)^2 - 2\left(\tfrac{1}{3}\right) + 5 \\ &= \tfrac{1}{3} - \tfrac{2}{3} + 5 \\ &= 4\tfrac{2}{3}. \end{aligned}$$

The values of f at the endpoints of the interval are

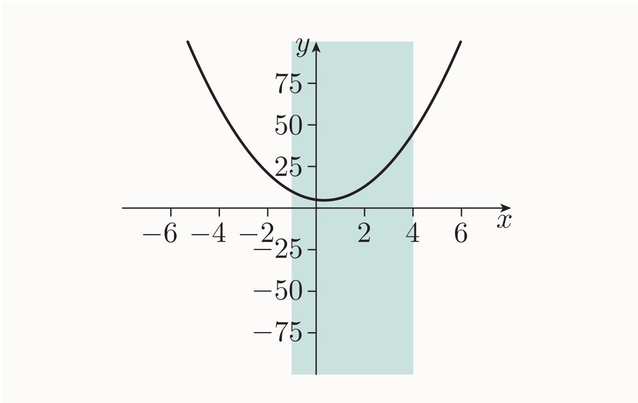
$$\begin{aligned} f(-1) &= 3(-1)^2 - 2(-1) + 5 \\ &= 3 + 2 + 5 \\ &= 10 \end{aligned}$$

and

$$\begin{aligned} f(4) &= 3 \times 4^2 - 2 \times 4 + 5 \\ &= 48 - 8 + 5 \\ &= 45. \end{aligned}$$

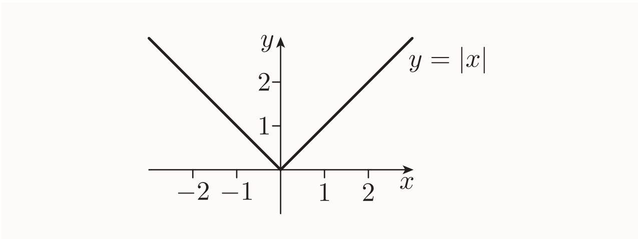
The greatest value of f on the interval $[-1, 4]$ is 45, and the least value is $4\frac{2}{3}$.

(The graph of f is below.)



Solution to Activity 23

An example is the function $f(x) = |x|$ and the interval $[-1, 1]$. This function is continuous on this interval. Its least value on the interval is 0, which occurs when $x = 0$, but 0 is neither a stationary point of the function nor an endpoint of the interval. The function is not differentiable when $x = 0$.



Solution to Activity 24

Here

$$f(x) = 2x^4 + 3x^2 + x,$$

so

$$f'(x) = 8x^3 + 6x + 1,$$

and

$$f''(x) = 24x^2 + 6.$$

Solution to Activity 25

The derivative of the function in graph A is shown in graph C.

The second derivative of the function in graph A is shown in graph D.

(There are various ways to work this out. For example, notice that the function in graph A has a stationary point when x is approximately 0. So the graph of its derivative must cross or touch the x-axis when x is approximately 0, which tells you that the derivative must be the function in graph C. Alternatively, notice that the function in graph A is decreasing for values of x just less than 0. So its derivative must be negative for these values of x, which again tells you that the derivative must be the function in graph C. You can use similar arguments to work out that the derivative of the function in graph C is the function in graph D.)

(Graph B shows the third derivative of the function in graph A.)

Solution to Activity 26

(a) We have

$$f(x) = \tfrac{1}{6}x^4 - 2x^3 + 11x^2 - 18x,$$

so

$$f'(x) = \tfrac{2}{3}x^3 - 6x^2 + 22x - 18,$$

and

$$f''(x) = 2x^2 - 12x + 22.$$

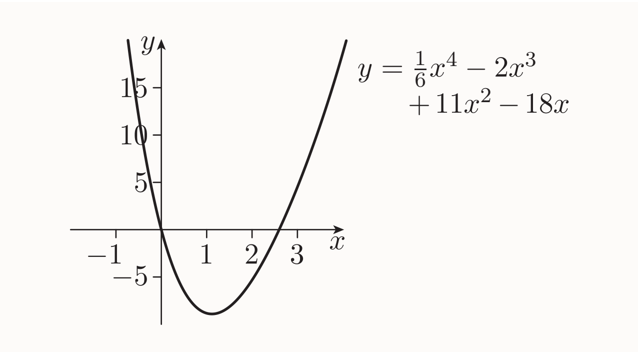
(b) To show that f is concave up on all of its domain, which is all the real numbers, we show that f'' is positive on the interval $(-\infty, \infty)$. One way to do that is by completing the square, as follows.

$$\begin{aligned} f''(x) &= 2x^2 - 12x + 22 \\ &= 2(x^2 - 6x) + 22 \\ &= 2\left((x-3)^2 - 9\right) + 22 \\ &= 2(x-3)^2 - 18 + 22 \\ &= 2(x-3)^2 + 4. \end{aligned}$$

The first term in this expression for $f''(x)$ is greater than or equal to zero for every value of x, and the second term is positive, so $f''(x)$ is positive for every value of x.

It follows, by the concave up/concave down criterion, that f is concave up on $(-\infty, \infty)$.

(The graph of f is shown below.)



Solution to Activity 27

(a) We have

$$f(x) = \tfrac{1}{12}x^4 - 2x^2,$$

so

$$f'(x) = \tfrac{1}{3}x^3 - 4x,$$

and

$$f''(x) = x^2 - 4.$$

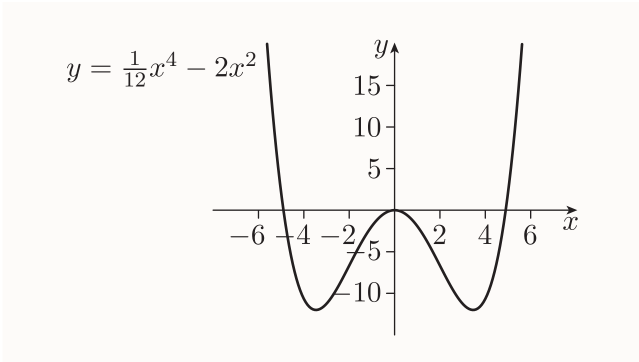
(b) Factorising the expression for $f''(x)$ found in part (a) gives

$$f''(x) = (x+2)(x-2).$$

When x is in the interval $(-2, 2)$, the value of $x + 2$ is positive and the value of $x - 2$ is negative, so the value of $f''(x)$ is negative.

It follows, by the concave up/concave down criterion, that f is concave down on $(-2, 2)$.

(The graph of f is shown below.)



Solution to Activity 28

(a) We have

$$f(x) = \tfrac{2}{3}x^3 - \tfrac{5}{2}x^2 + 2x,$$

so

$$\begin{aligned} f'(x) &= 2x^2 - 5x + 2 \\ &= (2x-1)(x-2). \end{aligned}$$

Hence the stationary points of f are $\frac{1}{2}$ and 2.

The y-coordinate of the stationary point $\frac{1}{2}$ is given by

$$\begin{aligned} y &= \tfrac{2}{3}(\tfrac{1}{2})^3 - \tfrac{5}{2}(\tfrac{1}{2})^2 + 2 \times \tfrac{1}{2} \\ &= \tfrac{2}{24} - \tfrac{5}{8} + 1 \\ &= \tfrac{11}{24}. \end{aligned}$$

The y-coordinate of the stationary point 2 is given by

$$y = \tfrac{2}{3} \times 2^3 - \tfrac{5}{2} \times 2^2 + 2 \times 2$$
$$= \tfrac{16}{3} - 10 + 4$$
$$= -\tfrac{2}{3}.$$

Hence the stationary points are $(\frac{1}{2}, \frac{11}{24})$ and $(2, -\frac{2}{3})$.

(b) We have

$$f''(x) = 4x - 5,$$

so

$$f''(\tfrac{1}{2}) = 4 \times \tfrac{1}{2} - 5 = 2 - 5 = -3$$

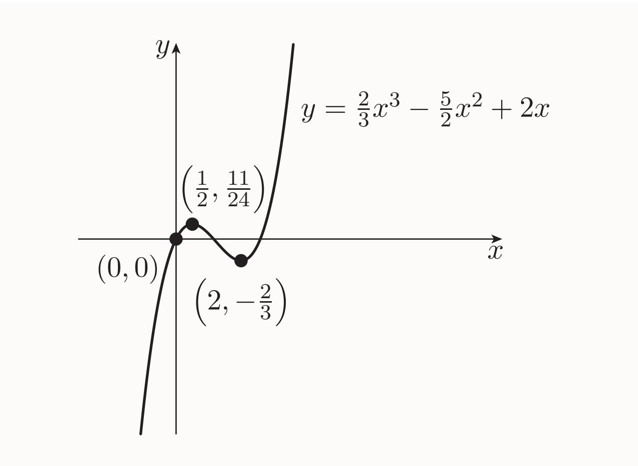
and

$$f''(2) = 4 \times 2 - 5 = 8 - 5 = 3.$$

Hence, by the second derivative test, $(\frac{1}{2}, \frac{11}{24})$ is a local maximum and $(2, -\frac{2}{3})$ is a local minimum.

(c) The y-intercept of f is 0.

A sketch of the graph of f is shown below.



Solution to Activity 29

(a) The displacement s metres of the object at time t seconds is given by

$$s = \tfrac{1}{3}t^3 - 5t^2 + 16t + 8.$$

Hence the velocity $v\,\mathrm{m\,s^{-1}}$ of the object at time t seconds is given by

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = t^2 - 10t + 16,$$

and the acceleration $a\,\mathrm{m\,s^{-2}}$ of the object at time t seconds is given by

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = 2t - 10.$$

(b) The object's acceleration is not constant, because the expression that gives the object's acceleration in terms of time is $2t - 10$, which is not a constant (it contains the variable t).

(c) When $t = 6$,

$$v = 6^2 - 10 \times 6 + 16 = -8,$$

and

$$a = 2 \times 6 - 10 = 2.$$

So at time 6 seconds, the velocity of the object is $-8\,\mathrm{m\,s^{-1}}$ and its acceleration is $2\,\mathrm{m\,s^{-2}}$.

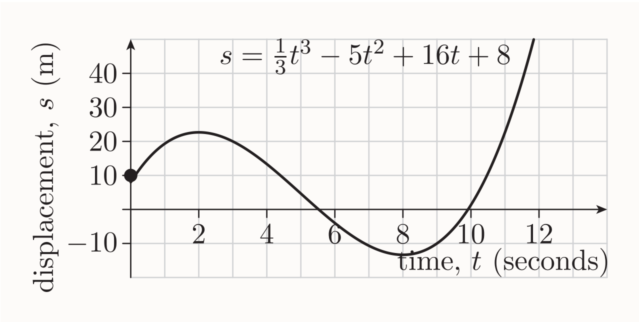
(d) (i) When t is between 5 and 8 (exclusive), the value of $a = 2t - 10$ is always positive. Hence the acceleration of the object is always positive in the given time interval, so its velocity is increasing over this time interval.

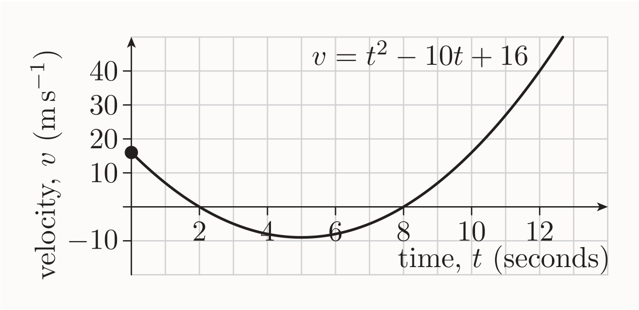
(ii) We have $v = t^2 - 10t + 16 = (t - 2)(t - 8)$. When t is between 5 and 8 (exclusive), the value of $t - 2$ is always positive and the value of $t - 8$ is always negative, so the value of $v = (t - 2)(t - 8)$ is always negative. That is, the velocity of the object is always negative over the given time interval.

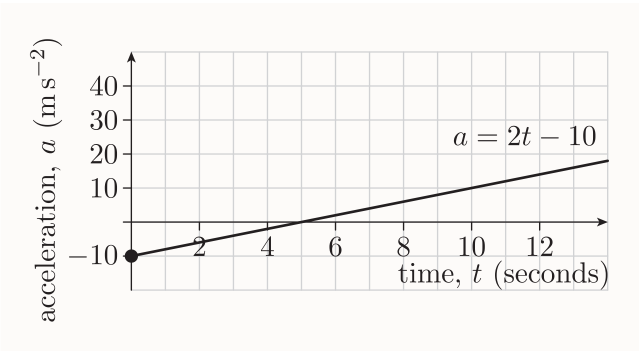
(iii) By parts (b)(i) and (b)(ii), over the given time interval the object's velocity is always negative, and always increasing. So its speed is decreasing over this time interval.

(The object's displacement, velocity and acceleration are shown in the graphs below. Notice in particular that over the time interval from 5 seconds to 8 seconds (excluding the endpoints), the object's velocity is negative and increasing. So the magnitude of its velocity is decreasing;

that is, its speed is decreasing.)







Solution to Activity 30

The displacement s metres of the falling object at time t seconds is given by

$$s = -4.9t^2.$$

Hence the velocity $v\,\mathrm{m\,s^{-1}}$ of the object at time t seconds is given by

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = -9.8t,$$

and the acceleration $a\,\mathrm{m\,s^{-2}}$ of the object at time t seconds is given by

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = -9.8.$$

So the acceleration of the object is constant, with value $-9.8\,\mathrm{m\,s^{-2}}$.

(The object's displacement, velocity and acceleration are shown in the graphs below. Notice in particular that although the velocity of the object is decreasing, its speed is increasing, because its velocity is negative.)