

MST125

Essential mathematics 2

Differential equations

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Introduction

This unit introduces differential equations. A *differential equation* is an equation that includes derivatives – here are two examples:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y \qquad \text{and} \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6x\frac{\mathrm{d}y}{\mathrm{d}x} + 9x^2y = \sin(2x).$$

In both these equations y is being differentiated with respect to x, so y is a function of x. The process of finding y as a function of x is called 'solving the differential equation'.

Differential equations arise in many situations in mathematics and the mathematical sciences, such as physics, chemistry, engineering, biology and even economics and finance. This is because the rates of change of the quantities involved (which are derivatives) often behave in much simpler ways than the quantities themselves. Some important applications of differential equations are mentioned at the end of this introduction.

This unit concentrates on three types of differential equations, the simplest one of which is a special case of each of the other two. After studying the unit, you should be able to recognise differential equations of each of the three types, and apply an appropriate method to solve each type.

Section 1 deals with differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$$

Such equations can be solved by *direct integration*.

Section 2 shows how differential equations that can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$$

can be solved by using the method of *separation of variables*.

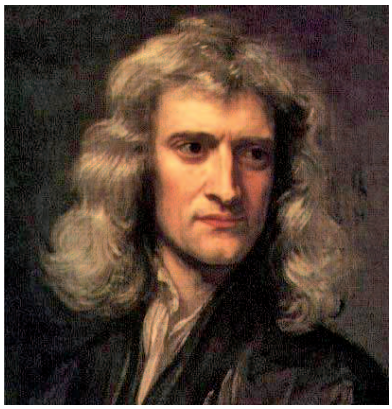
Section 3 investigates a number of applications of differential equations to real-world problems. It differs from the other sections in this unit, in that here you will see how differential equations can arise in applications. The section goes on to solve these differential equations and explore some properties of their solutions.

Section 4 deals with differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = h(x).$$

These equations are called *linear* differential equations, and are solved using a 'trick' that turns them into a form that can be solved by direct integration.

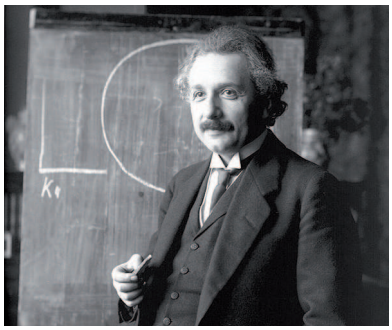
Section 5 summarises and revises all the methods for solving differential equations that are covered in the unit. Finally, in Section 6 you'll see how differential equations can be solved on a computer.



Isaac Newton (1642–1727)



James Clerk Maxwell (1831–1879)



Albert Einstein (1879–1955)



Erwin Schrödinger (1887–1961)

Differential equations in science

In physics, differential equations often arise from fundamental physical laws. Here are four important examples.

- *Newton's second law of motion* is essentially a differential equation that describes how objects move under the influence of forces.
- *Maxwell's equations* are differential equations that describe the behaviour of electricity and magnetism.
- *Einstein's equations of general relativity* are differential equations that describe the motion of extremely massive objects interacting via gravity, such as stars, galaxies and black holes.
- *Schrödinger's equation* is a differential equation that describes the motion of particles on the atomic and subatomic scale.

Other fundamental physical laws are also described by differential equations, and together these differential equations accurately describe the physics and chemistry of the vast majority of the Universe as we know it.

Differential equations also arise in subjects such as biology, engineering and economics, and also in physics and chemistry, to describe the behaviour of very large numbers of individual objects in aggregate, when the complexity of describing every individual in the group is too great.

For example, the motion of water could be described in principle by describing the motion of each individual water molecule. However, since there are approximately 3×10^{22} molecules in a single gram of water, the resulting system of differential equations would be extremely complicated. Instead, to a very good approximation, the motion can be described by a small number of differential equations, together called the *Navier–Stokes equations*.

Other phenomena that can be described in this way include: the weather, the spread of disease, population changes, electrical circuits and the economies of countries.

1 Differential equations and direct integration

This section introduces differential equations and the idea of solving a differential equation. It then illustrates these concepts by looking at the simplest type of differential equation, namely those that can be solved by *direct integration*.

1.1 Differential equations

In general, a **differential equation** is an equation involving an unknown function and its derivatives. In the introduction to this unit you saw the following two examples:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y, \tag{1}$$

and

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6x\frac{\mathrm{d}y}{\mathrm{d}x} + 9x^2y = \sin(2x). \tag{2}$$

In each of these differential equations the variable y is a function of x; we sometimes denote this fact by writing $y = y(x)$. Since y is a function of x in these equations, y is the **dependent variable** and x is the **independent variable**. Note that the dependent variable y is differentiated with respect to the independent variable x.

Letters other than x and y are sometimes used for the independent and dependent variables in a differential equation, especially t for the independent variable to represent time. For example,

$$\frac{\mathrm{d}q}{\mathrm{d}t} = t^2 \quad \text{is the same differential equation as} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = x^2,$$

except that the independent variable has changed from x to t and the dependent variable from y to q.

The nth derivative of a function is said to be the derivative of **order** n, and the **order of a differential equation** is the order of the highest derivative that appears in the equation. For example, differential equation (1) is of first order since the highest derivative that appears is $\mathrm{d}y/\mathrm{d}x$, and differential equation (2) is of second order since the highest derivative that appears is $\mathrm{d}^2y/\mathrm{d}x^2$. In this unit you will meet only first-order differential equations. More specifically, you will meet only first-order differential equations that can be expressed in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y),$$

where $f(x, y)$ denotes an expression that depends on the variables x and y. For example, the differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + y^2,$$

are both of this form.

A **solution** of a differential equation is a function $y = y(x)$ that satisfies the differential equation when substituted into that equation. For example, the function $y = e^{2x}$ is a solution of the first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y, \tag{3}$$

because with $y = e^{2x}$ the left-hand side of the differential equation equals

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{2x}\right) = 2e^{2x} = 2y,$$

which is the same as the right-hand side.

Usually a differential equation has more than one solution. For example, two other solutions of differential equation (3) are the functions $y = 3e^{2x}$ and $y = -5e^{2x}$, as you can easily check. In fact, any function of the form $y = Ae^{2x}$, where A is a constant, is a solution of equation (3).

To **solve a differential equation** you usually try to find the complete family of solutions, expressed in the form of a **general solution** involving one or more arbitrary constants. For example, as you will see later in the unit, the general solution of equation (3) is $y = Ae^{2x}$, where A is an arbitrary constant.

If you choose a particular value for each arbitrary constant in the general solution of a differential equation, then you obtain a **particular solution** of the differential equation. For example, $y = e^{2x}$, $y = 3e^{2x}$ and $y = -5e^{2x}$ are all particular solutions of equation (3).

1.2 Direct integration

The simplest type of differential equation is one of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$$

Here the right-hand side is an expression in the independent variable x and contains no terms involving the dependent variable y. We call differential equations of this type **directly integrable**.

For example, the differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = x^2, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \sin x, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

are all of this type, but

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = x - y$$

are not.

Activity 1 *Identifying directly integrable differential equations*

Which of the following differential equations are directly integrable?

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\cos x$ (b) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x\cos x$ (c) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x\cos t$

You can solve a directly integrable differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \tag{4}$$

simply by integrating both sides with respect to the independent variable, provided that you *can* integrate $f(x)$.

Since integration reverses differentiation, this gives

$$y = \int f(x)\,\mathrm{d}x,$$

so the solution of equation (4) is just the indefinite integral of the function f. This method is called **direct integration**.

Let's apply direct integration to a simple example.

Example 1 *Solving a differential equation by direct integration*

Use direct integration to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2.$$

Solution

The differential equation has the form $\mathrm{d}y/\mathrm{d}x = f(x)$, so it can be solved by direct integration.

Integrating both sides of the differential equation with respect to x gives

$$y = \int x^2\,\mathrm{d}x.$$

Doing the integration, we obtain the general solution

$$y = \tfrac{1}{3}x^3 + c,$$

where c is an arbitrary constant.

When you have solved a differential equation, it's often a good idea to check that your solution does satisfy the original differential equation.

With $y = \frac{1}{3}x^3 + c$ in Example 1, the left-hand side of the differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{1}{3}x^3 + c\right) = x^2,$$

which is the same as the right-hand side, so the differential equation is satisfied.

Here's another example.

Example 2 *Solving another differential equation by direct integration*

(a) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 - 5.$$

(b) Check that your solution satisfies the differential equation.

Solution

(a) The differential equation has the form $\mathrm{d}y/\mathrm{d}x = f(x)$, so it can be solved by direct integration.

Integrating both sides gives

$$y = \int (x^3 - 5)\,\mathrm{d}x = \tfrac{1}{4}x^4 - 5x + c,$$

where c is an arbitrary constant.

(b) With $y = \frac{1}{4}x^4 - 5x + c$ the left-hand side of the differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{1}{4}x^4 - 5x + c\right) = x^3 - 5,$$

which is the same as the right-hand side, so the differential equation is satisfied.

Here's an example of direct integration for you to try.

Activity 2 *Solving a differential equation by direct integration*

(a) Use direct integration to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 3.$$

(b) Check that your solution satisfies the differential equation.

General solutions obtained by using direct integration always contain an arbitrary constant. For example, in Example 1 you saw that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 \tag{5}$$

has the general solution

$$y = \tfrac{1}{3}x^3 + c, \tag{6}$$

where c is an arbitrary constant. If you set the arbitrary constant in a general solution to be a particular value, then you obtain a particular

solution. For example, if you set $c = 5$ in equation (6), then you obtain the function $y = \frac{1}{3}x^3 + 5$, which is a particular solution of differential equation (5).

Here is an example where you are asked to identify particular solutions.

Example 3 *Finding a general solution and particular solutions*

(a) Find the general solution of the differential equation

$$\frac{dy}{dx} = e^{2x}.$$

(b) Which of the following functions are particular solutions of this differential equation?

(i) $y = \frac{1}{2}e^{2x} + 7$ (ii) $y = \frac{1}{2}e^{2x}$ (iii) $y = \frac{1}{2}e^{2x} + 3x$

(iv) $y = \frac{1}{2}(e^{2x} + 3)$

Solution

(a) The differential equation has the form $dy/dx = f(x)$, so it can be solved by direct integration.

Integrating both sides of $dy/dx = e^{2x}$ gives the general solution

$$y = \int e^{2x}\,dx; \quad \text{that is,} \quad y = \tfrac{1}{2}e^{2x} + c,$$

where c is an arbitrary constant.

Check: differentiating $y = \frac{1}{2}e^{2x} + c$ gives $dy/dx = e^{2x}$, as required.

(b) The functions in (i), (ii) and (iv) are particular solutions obtained by setting the constant c equal to 7, 0 and $\frac{3}{2}$, respectively. The function in (iii) is not a particular solution as it is not of the form $y = \frac{1}{2}e^{2x} + c$, where c is a constant.

Here is a similar example, in which t is the independent variable.

Activity 3 *Finding a general solution and particular solutions*

(a) Find the general solution of the differential equation $\dfrac{dx}{dt} = \sin t$.

(b) Which of the following functions are particular solutions of this differential equation?

(i) $x = 2\cos t + 1$ (ii) $x = \cos t - 3$ (iii) $x = 1 - \cos t$

(iv) $x = -\cos t + 2t$

When using differential equations, you often need to find a particular solution that has a given additional property. To illustrate the ideas, consider once again the differential equation from Example 1,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2,$$

which has the general solution

$$y = \tfrac{1}{3}x^3 + c,$$

where c is an arbitrary constant. Different values for c give different particular solutions, and graphs of the particular solutions corresponding to $c = \frac{1}{3}, \frac{2}{3}, 1$ and $\frac{4}{3}$ are shown in Figure 1.

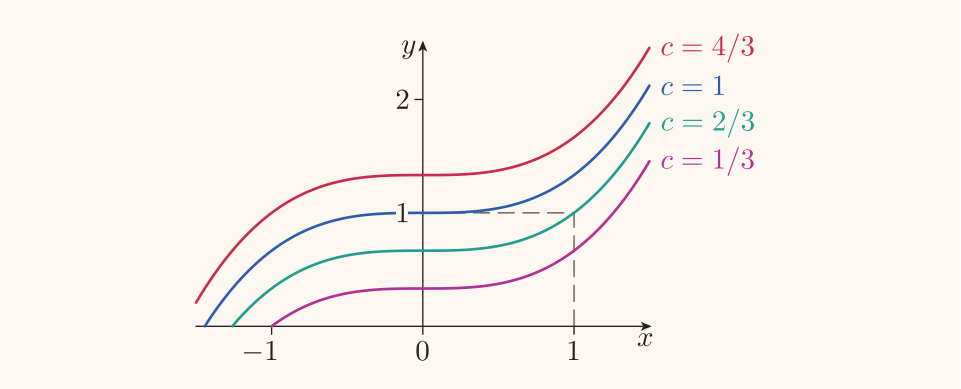


Figure 1 The graphs of $y = \frac{1}{3}x^3 + c$, for $c = \frac{1}{3}$, $\frac{2}{3}$, 1 and $\frac{4}{3}$

Now, you might want to choose the particular solution whose graph passes through the point $(1, 1)$. As shown in Figure 1, the particular solution with this property is

$$y = \tfrac{1}{3}x^3 + \tfrac{2}{3}.$$

Notice that if you choose any point in the plane, then only one of the graphs of the form $y = \frac{1}{3}x^3 + c$ passes through this point. So choosing a point through which the graph of the solution must pass is equivalent to picking a particular solution.

Requiring that the solution to a differential equation passes through a given point in the plane is a very common way of specifying a particular solution. If a particular solution is specified in this way, by requiring that its graph passes through a given point, then this requirement is called an **initial condition**. In the example above, the particular solution $y = \frac{1}{3}x^3 + \frac{2}{3}$ satisfies the initial condition $y = 1$ when $x = 1$.

Here is an example of a problem that uses an initial condition to specify a particular solution.

Example 4 *Applying an initial condition*

In Example 2 you saw that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 - 5$$

has the general solution

$$y = \tfrac{1}{4}x^4 - 5x + c,$$

where c is an arbitrary constant.

Find the particular solution of this differential equation that satisfies the initial condition $y = -1$ when $x = 2$.

Solution

We need to find the specific value of c such that $y = -1$ when $x = 2$.

We substitute the values $x = 2$ and $y = -1$ (simultaneously) into the equation $y = \frac{1}{4}x^4 - 5x + c$. This gives

$$-1 = \tfrac{1}{4} \times 2^4 - 5 \times 2 + c = -6 + c,$$

so $c = 5$. The required particular solution is therefore

$$y = \tfrac{1}{4}x^4 - 5x + 5.$$

An initial condition can be written using the $y(x)$ notation for the unknown function. For example, you can write the initial condition

$$y = -1 \text{ when } x = 2$$

more simply as

$$y(2) = -1.$$

Here is a similar problem for you to try.

Activity 4 *Applying an initial condition*

The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5}{2\sqrt{x}}$$

has general solution

$$y = 5\sqrt{x} + c,$$

where c is an arbitrary constant. Find the particular solution of this differential equation that satisfies the initial condition $y = 7$ when $x = 1$; that is, $y(1) = 7$.

Here is a summary of the method of solution by direct integration.

Direct integration

The general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

is the indefinite integral

$$y = \int f(x)\,\mathrm{d}x,$$

which can be written as

$$y = F(x) + c,$$

where $F(x)$ is any antiderivative of $f(x)$ and c is an arbitrary constant.

An initial condition of the form

$$y = b \text{ when } x = a, \quad \text{that is,} \quad y(a) = b,$$

enables you to find a value for the arbitrary constant c. The corresponding particular solution satisfies both the differential equation and the initial condition.

This method can always be applied to solve a differential equation of the form $\mathrm{d}y/\mathrm{d}x = f(x)$, as long as you can find the indefinite integral involved.

A problem in which you have to find the particular solution of a differential equation that satisfies a given initial condition is called an **initial value problem**. Generally there are two stages in solving an initial value problem.

Solving an initial value problem

To solve an initial value problem:

1. Find the general solution of the differential equation.
2. Use the general solution to find the particular solution that satisfies the initial condition.

Here is an example of solving an initial value problem. In this example the differential equation has a restriction $t > 0$. This restriction is given since the expression $1/t^2$ is not defined when $t = 0$, so we look for a solution restricted to an interval that contains the value of t in the initial condition. This is because when we are solving a differential equation we are usually interested only in solutions that are *continuous* functions.

Example 5 *Solving an initial value problem*

Solve the initial value problem

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{1}{t^2} \quad (t > 0), \quad \text{where } q = 1 \text{ when } t = 2.$$

Solution

First find the general solution. The differential equation has the form $\mathrm{d}q/\mathrm{d}t = f(t)$, so it can be solved by direct integration.

Integrating both sides of the differential equation with respect to t gives the general solution

$$q = \int \frac{1}{t^2}\,\mathrm{d}t = \int t^{-2}\,\mathrm{d}t = -t^{-1} + c \quad (t > 0),$$

where c is an arbitrary constant.

Substitute the initial condition to find c.

Since $q = 1$ when $t = 2$, we obtain

$$1 = -\frac{1}{2} + c,$$

which gives $c = \frac{3}{2}$. Hence the particular solution that satisfies the initial condition is

$$q = -\frac{1}{t} + \frac{3}{2} \quad (t > 0).$$

Here are some similar examples for you to try.

Activity 5 *Solving initial value problems*

Solve each of the following initial value problems.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x}$, where $y = 5$ when $x = 4$

(b) $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \cos(2x)$, where $u = -2$ when $x = \frac{1}{4}\pi$

Sometimes a whole family of similar differential equations can be solved at the same time by including a letter that denotes a constant in the equation; such a letter is often called a **parameter**. Here's an example of a differential equation with a parameter for you to try to solve.

Activity 6 *Solving an initial value problem with a parameter*

Solve the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{k}{t} \quad (t > 0), \quad \text{where } k \text{ is a positive constant, and } x = 3 \text{ when } t = 1.$$

The final activity in this section involves a possible application of differential equations. In this application the derivative $\mathrm{d}m/\mathrm{d}t$ represents the rate of change of the mass, m, of a rocket with respect to time, t. The mass of the rocket is decreasing with time as the fuel is used up, and hence its rate of change is always negative.

Activity 7 *An application: finding the mass of a burning rocket*

A toy rocket consists of a casing and fuel. Initially the total mass of the rocket is 200 grams. The fuel is ignited at time $t = 0$ and then burns in such a way that the rocket's total mass m (in grams) at time t (in seconds) satisfies the differential equation

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -8t \quad (t \geq 0).$$

(a) Find the general solution of the given differential equation.

(b) Find the particular solution of the differential equation that satisfies the initial condition $m = 200$ when $t = 0$.

(c) When the fuel is exhausted, the residual mass of the rocket casing is 100 grams. Find the time at which the fuel is exhausted; that is, find the time t (in seconds) when $m = 100$ for the particular solution obtained in part (b).

2 Separable differential equations

In this section we consider a second type of first-order differential equation, called a *separable* differential equation, and we show how to solve separable differential equations to find the general and particular solutions.

2.1 Separation of variables

Direct integration applies only to the very simplest type of differential equation, those of the form $\mathrm{d}y/\mathrm{d}x = f(x)$. In this subsection, we consider how to solve first-order differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y), \tag{7}$$

where the right-hand side is the *product* of a function of x and a function of y. We can't solve such equations by direct integration, because the right-hand side depends on both x and the unknown function $y = y(x)$.

Differential equations of the form in equation (7) are called **separable**. For example,

$$\frac{dy}{dx} = xy^2, \qquad \frac{dy}{dx} = \sin x \cos y, \qquad \frac{dy}{dx} = \frac{x}{y^2}, \qquad \frac{dy}{dx} = \frac{\sin y}{\cos x},$$

are all separable, but

$$\frac{dy}{dx} = x + y, \qquad \frac{dy}{dx} = xy^2 + 1, \qquad \frac{dy}{dx} = \ln(xy),$$

are not separable because their right-hand sides cannot be written as a function of x multiplied by a function of y.

Activity 8 *Identifying separable equations*

Identify which of the following differential equations are separable; that is, which of them are of the form

$$\frac{\text{d(dependent variable)}}{\text{d(independent variable)}} = f(\text{independent variable})\, g(\text{dependent variable}).$$

For each equation that is separable, identify the corresponding functions f and g.

(a) $\dfrac{dy}{dx} = x \cos y$ (b) $\dfrac{dx}{dt} = t \cos t$ (c) $\dfrac{dp}{dt} = p \cos p$

(d) $\dfrac{dv}{dy} = v + y$ (e) $\dfrac{dv}{dy} = v + vy$ (f) $\dfrac{dy}{dx} = \sin(xy)$

To illustrate the method for solving a separable differential equation, let's start with a fairly simple example. Consider the differential equation

$$\frac{dy}{dx} = \frac{x^2}{y} \qquad (y > 0). \tag{8}$$

This is of the form in equation (7), with $f(x) = x^2$ and $g(y) = 1/y$, so it is a separable differential equation. The condition $y > 0$ is included because the right-hand side of the differential equation is not defined when $y = 0$.

To solve differential equation (8), first rearrange it so that the dependent variable and its derivative appear only on the left-hand side and the independent variable appears only on the right-hand side. Multiplying both sides of the equation by y gives

$$y\frac{dy}{dx} = x^2.$$

Now integrate both sides with respect to x:

$$\int y \frac{dy}{dx}\, dx = \int x^2\, dx. \tag{9}$$

Next recall the rule for integration by substitution from MST124, revised in Section 1 of Unit 7, which tells you that any integral of the form

$$\int f(u)\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x \tag{10}$$

is equal to the simpler integral

$$\int f(u)\,\mathrm{d}u.$$

(It's easy to remember this rule because it looks as if we simply cancel a '$\mathrm{d}x$' in the denominator with a '$\mathrm{d}x$' in the numerator, though of course that's not what's actually happening.)

By applying this rule to the left-hand side of equation (9), which is of the same form as expression (10) with y instead of u, we can simplify equation (9) to

$$\int y\,\mathrm{d}y = \int x^2\,\mathrm{d}x. \tag{11}$$

Now you can see that the variables x and y have been 'separated', so you can integrate both sides of this equation to give an equation relating y to x.

Doing the integrations on both sides gives

$$\tfrac{1}{2}y^2 = \tfrac{1}{3}x^3 + c,$$

where c is an arbitrary constant.

(You could have introduced an arbitrary constant on both sides to obtain

$$\tfrac{1}{2}y^2 + c_1 = \tfrac{1}{3}x^3 + c_2,$$

but the constants c_1 and c_2 could then be combined to give $\frac{1}{2}y^2 = \frac{1}{3}x^3 + c$, where $c = c_2 - c_1$.)

Hence

$$y = \pm\sqrt{\tfrac{2}{3}\,x^3 + C}, \tag{12}$$

where $C = 2c$ is another arbitrary constant.

Now remember the condition $y > 0$ that applied to the original differential equation. This tells you that the solution $y = y(x)$ must take only positive values. Hence the general solution of the differential equation, with this condition, is

$$y = \sqrt{\tfrac{2}{3}\,x^3 + C}. \tag{13}$$

At this point you could check this solution by differentiating it and substituting it back into the original differential equation, as in Subsection 1.2. This would be slightly laborious but perfectly doable. However, later in this section you will learn a technique for checking your answer that is usually more straightforward.

Notice that the general solution found above contains an arbitrary constant, as expected. However, unlike the general solution of a directly integrable differential equation, this general solution is not of the form $y = F(x) + c$, since the arbitrary constant in it cannot be taken outside the square root.

You can find particular solutions of separable differential equations in exactly the same way as for directly integrable differential equations. Suppose, for example, that you want to find the particular solution of equation (8) that satisfies the initial condition

$$y = 2 \text{ when } x = 3.$$

To do this you simply substitute $x = 3$ and $y = 2$ into the general solution, which gives

$$2 = \sqrt{\tfrac{2}{3} \times 3^3 + C}.$$

Squaring both sides gives

$$2^2 = \tfrac{2}{3} \times 3^3 + C,$$

so $C = -14$. Hence the required particular solution is

$$y = \sqrt{\tfrac{2}{3}x^3 - 14}.$$

Notice that this particular solution is a function that is not defined for all values of x. We shall return to this point later in the section.

Before you see some further examples of separating variables, here is a shortcut that makes the process easier. In the discussion above, we used integration by substitution to show that if

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y},$$

then

$$\int y \,\mathrm{d}y = \int x^2 \,\mathrm{d}x.$$

A convenient way to remember this step is to *imagine* that the derivative $\mathrm{d}y/\mathrm{d}x$ is a fraction, at least temporarily. Then you separate the variables by taking everything involving y to the left-hand side of the differential equation and everything involving x to the right-hand side, and finally putting integral signs on both sides.

This shortcut works for any differential equation of the form in equation (7) and is summarised in the following box.



A shortcut for separating variables

To get from the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\,g(y)$$

to

$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x,$$

imagine that $\mathrm{d}y/\mathrm{d}x$ is a fraction, and take everything involving y to the left and everything involving x to the right, and finally put integral signs on both sides.

You should remember however, that the derivative $\mathrm{d}y/\mathrm{d}x$ is not really a fraction, but the clever notation $\mathrm{d}y/\mathrm{d}x$ makes this manipulation appear natural.

Once you have separated variables in this way, various possibilities can occur when you try to proceed.

- You may be able to find both integrals and rearrange the resulting equation to express y explicitly in terms of x. The resulting equation is a solution in **explicit form**.
- You may be able to find both of the integrals, but the resulting equation relating y and x may be so complicated that you cannot rearrange it to express y explicitly in terms of x. In this case, the equation relating y and x is said to be a solution in **implicit form**.
- One or both of the two integrals may be difficult or impossible to find. In this case the method has failed.

Here is an example that demonstrates the use of the shortcut for separating variables and also illustrates the second possibility above.

Example 6 *Finding a general solution by separation of variables*

Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\sin x}{y+1} \qquad (y > 0). \tag{14}$$

Solution

This differential equation is of the form $\mathrm{d}y/\mathrm{d}x = f(x)g(y)$, with $f(x) = \sin x$ and $g(y) = y/(y+1)$. So take everything involving y to the left and everything involving x to the right, and then integrate both sides.

Separating the variables in equation (14) gives

$$\int \frac{y+1}{y}\,\mathrm{d}y = \int \sin x\,\mathrm{d}x;$$

that is,

$$\int \left(1 + \frac{1}{y}\right) \mathrm{d}y = \int \sin x \, \mathrm{d}x.$$

Doing both integrations gives

$$y + \ln|y| = -\cos x + c,$$

where c is an arbitrary constant. Since we know that $y > 0$, we have $|y| = y$, so this solution can be written as

$$y + \ln y = -\cos x + c. \tag{15}$$

The equation $y + \ln y = -\cos x + c$ cannot be rearranged to give a general solution in explicit form, so equation (15) is the general solution in implicit form.

Figure 2 shows the curves in the x, y-plane given by the equation $y + \ln y = -\cos x + c$, for $c = -1$, 0 and 1. Each of these curves seems to be the graph of a function since each value of x appears to correspond to just one value of y. This is in fact the case, but we shall not investigate this further in this unit.

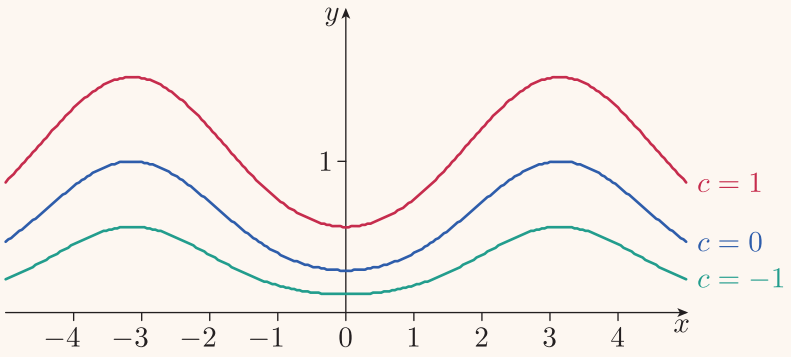


Figure 2 The graphs of $y + \ln y = -\cos x + c$, for $c = -1$, 0 and 1

In seeking to solve a differential equation, you should aim to find a solution in explicit form wherever possible. However, if this is impossible, as in Example 6, then you must settle for a solution in implicit form.

If you have found the general solution only in implicit form, then you can still find a particular solution (in implicit form) from a given initial condition. Here's an example.

Example 7 *Finding a particular solution in implicit form*

In Example 6 you saw that the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y \sin x}{y+1}$$

has general solution

$$y + \ln y = -\cos x + c,$$

in implicit form, where c is an arbitrary constant.

Find the particular solution that satisfies the initial condition

$$y = 1 \text{ when } x = 0.$$

Solution

Substituting the initial condition $y = 1$ when $x = 0$ into the general solution $y + \ln y = -\cos x + c$, we obtain

$$1 + \ln 1 = -1 + c,$$

so $c = 2 + \ln 1 = 2$. Hence, the particular solution in implicit form that satisfies the given initial condition is

$$y + \ln y = -\cos x + 2.$$

The method for solving differential equations that you have met in this section is known as **separation of variables** and is summarised below.

Separation of variables

This method applies to differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\, g(y). \tag{16}$$

1. Separate the variables and put integral signs on both sides to obtain
$$\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x.$$
2. Carry out the two integrations, introducing *one* arbitrary constant, to obtain the general solution in implicit form.
3. If possible, manipulate the resulting equation to make y the subject, thus expressing the general solution in explicit form.

In some cases it is not possible to carry out step 2 or step 3.

Once a general solution has been obtained, in implicit form or explicit form, it can be checked by substituting back into the differential equation. You will see how to do this for solutions in implicit form in the next subsection.

Note that if $g(y) = 1$ in equation (16), then the separable differential equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$$

In this case the method of separation of variables reduces to simple direct integration, the method covered in Subsection 1.2. Hence direct integration is just a special case of separation of variables.

Here are some problems involving separation of variables for you to try.

Activity 9 *Finding a solution in explicit form*

(a) Find the general solution in explicit form of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} \quad (y > 0).$$

(b) Find the particular solution of this differential equation that satisfies the initial condition $y = 1$ when $x = 0$.

Activity 10 *Finding a solution in implicit form*

(a) Find the general solution in implicit form of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + e^x}{1 + \cos y} \quad (-\pi < y < \pi).$$

(b) Find the particular solution of this differential equation that satisfies the initial condition $y(0) = 0$.

2.2 Verifying solutions in implicit form

In Subsection 1.2 you saw how to verify that an *explicit* solution obtained by direct integration does satisfy the original differential equation. You will now see how to check that a solution you have found in *implicit* form satisfies the differential equation from which it was obtained.

To do this you'll need to use the chain rule for differentiation, which you revised in Unit 1.

Chain rule

If y is a function of u and u is a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

The chain rule is needed for checking solutions to differential equations when you have found a solution in implicit form involving the independent variable x and the dependent variable y, and you want to check that this solution satisfies the original differential equation, which involves dy/dx. In that situation, you may need to differentiate a function of y such as

$$G(y) = y^2, \quad \text{with respect to } x,$$

so

$$G \text{ is a function of } y \qquad \text{and} \qquad y \text{ is a function of } x.$$

In this case the chain rule gives

$$\frac{dG}{dx} = \frac{dG}{dy}\frac{dy}{dx} = 2y\frac{dy}{dx}$$

or, equivalently,

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dy}\left(y^2\right)\frac{dy}{dx} = 2y\frac{dy}{dx}.$$

Here is an example.



Example 8 *Verifying a solution given in implicit form*

Show that

$$y^3 + y = x^2 + 9$$

is a solution in implicit form of the differential equation

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 1}.$$

Solution

Differentiating both sides of the equation $y^3 + y = x^2 + 9$ with respect to x gives

$$\frac{d}{dx}\left(y^3 + y\right) = \frac{d}{dx}\left(x^2 + 9\right).$$

Doing the differentiation on the right-hand side of this equation is straightforward. For the left-hand side, use the chain rule to differentiate y^3 and y with respect to x.

By the chain rule, the left-hand side is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y^3+y\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right)+\frac{\mathrm{d}}{\mathrm{d}x}\left(y\right)$$
$$= 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}x}.$$

The right-hand side is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+9\right) = 2x.$$

So

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}x} = 2x.$$

Rearranging gives

$$\left(3y^2+1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x;$$

that is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{3y^2+1},$$

which is the required differential equation.

Here are two similar problems for you to try; recall that in Activity 10, you were asked to solve the differential equation in Activity 11.

Activity 11 *Verifying a solution given in implicit form*

Show that

$$y+\sin y = x+e^x-1,$$

is a solution in implicit form of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+e^x}{1+\cos y} \quad (-\pi < y < \pi).$$

Activity 12 *Verifying another solution given in implicit form*

Show that

$$y+\ln y = (1-2x)e^{3x}$$

is a solution in implicit form of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(1-6x)e^{3x}}{y+1} \quad (y>0).$$

A note of caution

Some care is required when applying separation of variables. First, you can divide both sides of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\,g(y)$$

by $g(y)$ only if $g(y) \neq 0$, since division by zero is undefined. Therefore, when separating variables, it is necessary to exclude from consideration any value of y for which $g(y) = 0$. This can be done by imposing a suitable condition on y from the outset. For example, in Activity 9 we applied the condition $y > 0$ to the differential equation $\mathrm{d}y/\mathrm{d}x = -x/y$, and obtained the positive general solution $y = \sqrt{b - x^2}$. As you saw earlier, conditions like this often arise from the context of the problem being considered.

In addition to restrictions on the dependent variable, sometimes solutions may be valid only for a certain range of values of the independent variable. For example, in Activity 9(b) you found the particular solution $y = \sqrt{1 - x^2}$. For this function to be defined we require $1 - x^2 \geq 0$, which is equivalent to $-1 \leq x \leq 1$. Since $x = 1$ and $x = -1$ both give $y = 0$, which is excluded by the restriction $y > 0$ in the question, the domain of this particular solution is $-1 < x < 1$. This example shows that consideration of the appropriate domain for a solution can sometimes be a bit complicated.

Activity 13 *Finding a solution in explicit form of an initial value problem*

Solve each of the following initial value problems, giving your answers in explicit form.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -y^3 \quad (y > 0), \quad$ where $y = \frac{1}{2}$ when $x = 0$.

(b) $\dfrac{u^2}{x^4}\dfrac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad (u > 0,\ x > 0), \quad$ where $u = 1$ when $x = 1$.

Activity 14 *Finding a solution in implicit form of an initial value problem*

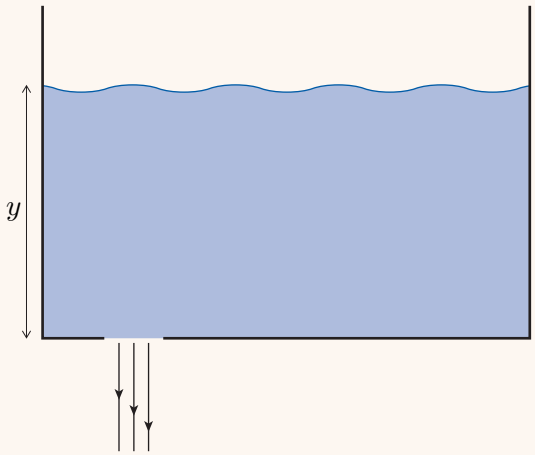
Solve the following initial value problem, giving your answer in implicit form.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 \cos x}{y + 1} \quad (y > 0), \quad \text{where } y = 2 \text{ when } x = \frac{\pi}{2}.$$

In the final activity of this section you will use separation of variables to solve a practical problem concerning water flow, in which the differential equation is derived from a physical law called *Torricelli's theorem*. In this activity the differential equation involves a parameter k and the initial condition contains a parameter y_0. Remember that k and y_0 are *constants*. The notation y_0 is used to denote the value of the variable y at the initial time $t = 0$.

Activity 15 *An application: modelling water flow*

This activity concerns a mathematical model for the flow of water from a tank, of constant horizontal cross-section, through a small hole at its base. This is shown in the figure below.



According to the model, the height y of the water surface above the hole is related to the time t that has elapsed since the hole was uncovered. This relationship is described by the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky^{1/2} \quad (y > 0),$$

where k is a positive constant.

(The condition $y > 0$ is natural in the context of the problem and the value of k depends on the area of the hole and the horizontal cross-sectional area of the tank.)

(a) Find the general solution in implicit form of this differential equation.

(b) Find the particular solution for which $y = y_0$ at $t = 0$. (Here y_0 is the height of the water column above the hole at the moment when the hole is uncovered.)

(c) Show that, according to this model, the tank becomes half-empty after time $\dfrac{(2-\sqrt{2})\sqrt{y_0}}{k}$.

3 Applications of differential equations

In this section we will investigate some applications of differential equations to real-world problems. You will see how differential equations arise in a variety of contexts, and go on to solve these differential equations and explore some properties of their solutions.

We will concentrate on real-world problems that can be modelled by first-order differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = Ky + C,$$

where K and C are constants. A differential equation of this type can be solved using separation of variables, and the resulting behaviour is that of exponential growth or decay, as you will see.

The variety of applications of differential equations that you will see indicates how ubiquitous they are throughout the mathematical sciences.

3.1 Continuous model for population change

In this subsection you will see how to use a differential equation to model the variation in the size of a population over time. The population could be the people in a country or in some other geographical region, or it could be the bacteria in a Petri dish, or any other population of individual organisms.

Since a population is made up of individuals, the size of a population can take only integer values, so the graph of the population size against time will not be continuous. However, if the population size is large, then its behaviour over time can be approximated by a function with a continuous graph – in other words, a continuous function.

If you approximate a population size over time by a continuous function, then you can consider the rate of change of the population size at any particular point in time. This rate of change is the difference between the rate at which new organisms are created, and the rate at which older organisms die. (For some populations, there might also be contributions from immigration and emigration.)

To illustrate the ideas behind the way that we will model the change in a population size over time, let's consider a population of bacteria in a Petri dish. Suppose that at a particular point in time the population size is 100 000 bacteria, and the rate of change of the population size is 30 000 bacteria per hour. Now consider a few hours later, and suppose that the population size has doubled, to 200 000 bacteria. Since there are now twice as many bacteria available to create more bacteria and to die, you would expect the rate of change of the population size to be twice what it was – that is, you would expect it to be 60 000 bacteria per hour. In general, if the population size is multiplied by some number, then you would expect its rate of change to be multiplied by the same number.

In fact you would expect this property to hold not only for increasing population sizes, but also for decreasing ones. For example, consider a population of wild birds whose size is decreasing over time. If the population size decreases to half of what it was, then you would expect the rate of change of the population size to also decrease to half of what it was, since there are only half as many birds available to die and to create new birds. In general, if the population size is multiplied by some number (which can be less than 1), then you would expect its rate of change to be multiplied by the same number.

If two quantities are related in such a way that whenever one of the quantities is multiplied by a number the other quantity is multiplied by the same number, then we say that the quantities are **proportional** (or **directly proportional**).

So, from the discussion above, you would expect the following modelling assumption to hold for a population size that changes over time.

Modelling assumption for a population size

The rate of change of the population size is proportional to the population size.

This modelling assumption does indeed work well for many populations, particularly over periods of time that are not too long, and for population sizes that are not too large. If a population size becomes too large, then resources such as food may start to be in short supply, so the modelling assumption above might not apply.

If a quantity is proportional to another quantity, then the relationship between the two quantities is described by an equation of the form

$$\text{first quantity} = K \times (\text{second quantity}),$$

where K is a constant.

So the modelling assumption above tells you that the relationship between the population size and the rate of change of the population size is given by an equation of the form

$$\text{rate of change of the population size} = K \times (\text{population size}), \qquad (17)$$

for some constant K.

The constant K is called the **proportionate growth rate** of the population size. This terminology is used because, by equation (17), the constant K is equal to

$$\frac{\text{rate of change of the population size}}{\text{population size}},$$

so it is a measure of how fast the population size is increasing or decreasing relative to its size. For example, if the proportionate growth rate is $K = 0.05$ per year, then, by equation (17), at any point in time the

population size is growing at the rate of 0.05 of its current size per year. In other words, it is growing at the rate of 5% per year. Notice that the units for a proportionate growth rate are 'per year' or 'per hour', or similarly for any other measure of time. These units can be written as year^{-1}, or hour^{-1}, and so on, as appropriate.

The larger the value of the proportionate growth rate K for a population size is, the more rapidly the population size will increase. A negative proportionate growth rate corresponds to a decreasing population size, and the larger its magnitude is, the more rapidly the population size will decrease. A proportionate growth rate of zero corresponds to a population size that does not change.

Let's write equation (17) in symbols. Let P be the population size at time t. Then the rate of change of P with respect to t is

$$\frac{\mathrm{d}P}{\mathrm{d}t}.$$

So equation (17) can be written as

$$\frac{\mathrm{d}P}{\mathrm{d}t} = KP, \tag{18}$$

where K is the proportionate growth rate for the population. This equation is a differential equation that relates the time t and the population size P.

We can solve this differential equation to find the relationship between t and P. To find the precise relationship, with no arbitrary constant, we need an initial condition: we need to know the population size at a particular point in time. Let's assume that when $t = 0$ the population size is P_0. When solving the differential equation, we can assume that $P > 0$, since P is a population size. So we have the following initial value problem:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = KP \quad (P > 0), \quad \text{where } P = P_0 \text{ when } t = 0.$$

In the next activity you are asked to use the methods that you learned in Section 2 to solve this initial value problem.

Activity 16 *Solving the population size initial value problem*

Consider the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = KP \quad (P > 0),$$

where K is a constant.

(a) Use separation of variables to show that the general solution of this differential equation is

$$P = Ae^{Kt},$$

where A is an arbitrary positive constant.

(b) Show that the particular solution that satisfies the initial condition

$$P = P_0 \text{ when } t = 0$$

is given by

$$P = P_0 e^{Kt}.$$

Figure 3 shows some typical graphs of the solution $P = P_0 e^{Kt}$ found in Activity 16. The case $K > 0$ corresponds to population growth, $K < 0$ corresponds to population decline and $K = 0$ gives a static population size.

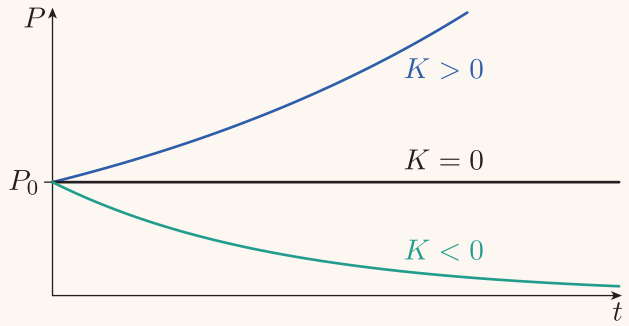


Figure 3 The graph of $P = P_0 e^{Kt}$ for $K > 0$, $K = 0$ and $K < 0$

Figure 4 illustrates that the larger the magnitude of the proportionate growth rate K in the equation $P = P_0 e^{Kt}$, the faster the population size P grows or decays.

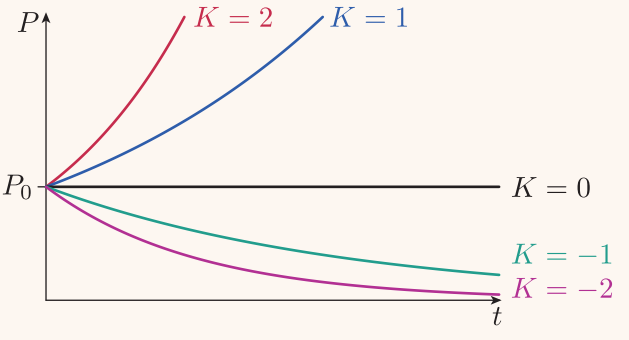


Figure 4 The graph of $P = P_0 e^{Kt}$ for different values of K

In fact, Activity 16 shows that, with the modelling assumption that you have seen, the variation in the size of a population over time is given by an *exponential growth* or *decay function* (or a constant function, in the case $K = 0$). Remember from MST124 Unit 3 that an **exponential growth** or **decay function** is a function of the form $y = ae^{kx}$, where a and k are non-zero constants; a positive constant k gives an exponential growth function, and a negative constant k gives an exponential decay function. You saw in MST124 Unit 3 that population sizes can often be

modelled by functions of this form, and the discussion above explains why this is; it arises from the modelling assumption discussed in this subsection.

This model for population change is called the **exponential model**, or sometimes the **continuous exponential model**, for population change. It illustrates how an equation that relates two quantities may in fact be a consequence of a simpler *differential* equation relating the quantities.

Here is a summary of the model.

Continuous exponential model for population change

If a population size changes with constant proportionate growth rate K, and the population size at time $t = 0$ is P_0, then the population size P at time t is given by

$$P = P_0 e^{Kt}.$$

The next activity, which is similar to some problems in MST124 Unit 3, shows how the equation $P = P_0 e^{Kt}$ can be put to use. Remember that the notation year^{-1} means 'per year'.

Activity 17 *Working with the exponential model for population growth*

The population size of a country is currently 5.1 million, and its proportionate growth rate is 0.032 year^{-1}.

(a) Use the equation $P = P_0 e^{Kt}$ to predict what the population size will be after each of the following time intervals. Give your answers to two significant figures.

(i) 10 years (ii) 100 years

(b) After how long will the population size reach 50 million? Give your answer to the nearest year.

A population size whose change over time is given by the exponential model, with a particular proportionate growth rate, has the following property.

> The factor by which the population size increases or decreases over a time interval depends only on the length of the time interval (and not on the time at the start of the time interval or the population size at the start of the time interval).

For example, if the population size increases by a factor of 1.05 (that is, grows by 5%) over some time interval of length one year, say, then it will increase by the factor 1.05 over *every* time interval of length one year. Similarly, if the population size decreases by a factor of 0.95 (that is, declines by 5%) over some time interval of length one year, say, then it will decrease by the factor 0.95 over *every* time interval of length one year. You

met this property of exponential growth and decay functions in MST124 Unit 3.

To see why this property holds, suppose that the population size is given by the equation $P = P_0 e^{Kt}$, where the symbols here have the same meanings as earlier, and consider the time interval of length T from time t to time $t + T$, say. The population sizes at the start and end of the time interval are

$$P_0 e^{Kt} \quad \text{and} \quad P_0 e^{K(t+T)},$$

respectively. So the factor by which the population size changes over the time interval is

$$\frac{P_0 e^{K(t+T)}}{P_0 e^{Kt}} = \frac{P_0 e^{Kt} e^{KT}}{P_0 e^{Kt}} = e^{KT}.$$

As you can see, this value depends only on the length T of the time interval. (The proportionate growth rate K is a constant.)

Example 9 *Calculating a growth factor*

Suppose that a population size increases according to the exponential model, with proportionate growth rate 0.04 year^{-1}. By what factor will the population size increase over five years? Give your answer to three significant figures.

Solution

Suppose that the initial population size, at time $t = 0$, is P_0. The proportionate growth rate is $K = 0.04$. Hence the population size P at time t is given by

$$P = P_0 e^{0.04t}.$$

Consider a five-year time interval from time t to time $t + 5$. The population sizes at the start and end of this time interval are

$$P_0 e^{0.04t} \quad \text{and} \quad P_0 e^{0.04(t+5)},$$

respectively.

So the factor by which the population size will increase over the time interval is

$$\frac{P_0 e^{0.04(t+5)}}{P_0 e^{0.04t}} = \frac{e^{0.04t} e^{0.04\times 5}}{e^{0.04t}} = e^{0.04\times 5} = e^{0.2} = 1.221\ldots.$$

Hence the population size will increase by a factor of 1.22 (to 3 s.f.) over five years.

So the population size in Example 9 will increase by about 22% over *any* time interval of five years.

Note that the working in Example 9 can be made a little shorter by using the result that the factor by which the population size changes over a time interval of length T is e^{KT}, where K is the proportionate growth rate. This formula was mentioned immediately before the example. However, to save having to remember this formula, you can just use the equation $P = P_0 e^{Kt}$, as was done in the solution to Example 9.

Activity 18 *Calculating a proportion remaining of a decreasing population size*

Suppose that a population size decreases according to the exponential model, with proportionate growth rate -0.03 year^{-1}. What proportion of the population size will remain after a decade? Give your answer to two significant figures.

You have seen that if a population size changes according to the exponential model, with proportionate growth rate K, then over any time interval of length T, the population size is multiplied by the factor e^{KT}.

It follows that if the population size is increasing, then the time T that it takes to double is fixed and given by the equation

$$e^{KT} = 2.$$

This time T is known as the **doubling time** of the population size.

Similarly, it follows that if the population size is decreasing, then the time T that it takes to halve is fixed and given by the equation

$$e^{KT} = \tfrac{1}{2}.$$

This time T is known as the **halving time** or **half-life** of the population size.

You can rearrange each of these equations to express T in terms of K, by first taking the natural logarithm of each side. This gives the results below, which you met for exponential growth and decay functions in general in MST124 Unit 3.

Doubling and halving times

Suppose that a population size changes with proportionate growth rate K.

- If the population size is increasing (so $K > 0$), then its doubling time is $\dfrac{\ln 2}{K}$.
- If the population size is decreasing (so $K < 0$), then its halving time is $-\dfrac{\ln 2}{K}$.

Activity 19 *Finding doubling and halving times*

Find, to three significant figures, the doubling time for the population described in Example 9, and the halving time for the population described in Activity 18.

3.2 Radioactive decay

Radioactive substances are composed of atoms that 'decay' spontaneously; that is, they are transformed into other types of atoms, releasing energy and subatomic particles (both often called radiation) in the process. The rate at which this occurs varies according to the radioactive substance concerned.

For example, the radioactive element ruthenium-106 slowly decays into another element, rhodium-106. Each atom of ruthenium-106 has a known probability of decay in any given length of time, which is a constant throughout time. In fact each atom of ruthenium-106 has almost exactly a 50% chance of decay in a year. So if we have 10 000 atoms of ruthenium-106, about 5000 will decay in a year, and if we have 100 000, about 50 000 will decay in a year.

Because the number of atoms present in even one gram of ruthenium-106 is enormous (about 5.7×10^{21} atoms), it is sensible to use a continuous model to describe this situation, measuring the amount in grams, rather than in numbers of atoms.

With such a model, we can consider the rate of change of the amount of ruthenium-106 at any particular moment in time. This rate of change will be proportional to the amount of ruthenium-106 present. This is because, for example, if the amount of ruthenium-106 is halved, then there are only half as many atoms to decay (each with the same probability in any given length of time), so the rate of change of the amount of ruthenium-106 will also halve.

Other radioactive atoms have different probabilities of decay in a given time interval, but these are also a constant throughout time and hence a similar model will hold. In general, the rate of decay of any radioactive substance can be modelled by the following relationship.

Radioactive decay

The rate of change of the amount of a radioactive substance is proportional to the amount of the substance present.

So we have the following equation, where K is a constant:

$$\left.\begin{array}{l}\text{rate of change of}\\ \text{the amount of the}\\ \text{radioactive substance}\end{array}\right) = K \times \left.\begin{array}{l}\text{amount of}\\ \text{the radioactive}\\ \text{substance}\end{array}\right).$$

Since the amount of the radioactive substance is decreasing, the constant K in the equation above is negative, and we usually write it as $-k$, where k is a positive constant, called the **decay constant**. Like a proportionate growth rate, a decay constant is measured in units such as year^{-1}, hour^{-1}, and so on.

Let's write the equation above in symbols. Let m be the amount (mass) of the radioactive substance at time t. Then the rate of change of the amount of the substance at time t is $\mathrm{d}m/\mathrm{d}t$.

So the equation can be written as

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -km,$$

where k is the decay constant.

To find a particular solution of this equation we require an initial condition. If the amount of the radioactive substance present at time $t = 0$ is m_0, then the initial condition is

$$m = m_0 \text{ when } t = 0.$$

So we have the following initial value problem:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -km \quad (m > 0), \quad \text{where } m = m_0 \text{ when } t = 0.$$

The condition $m > 0$ holds because the amount of the substance is always positive.

This initial value problem has the same mathematical form as the initial value problem for the population change model (with $-k$ replacing K). Hence its solution can be written down immediately, by comparison with the exponential model for population change. The solution is therefore

$$m = m_0 e^{-kt}.$$

So we have the following result.

Exponential model for radioactive decay

If a radioactive substance decays with decay constant k, and the original amount of the substance is m_0, then the amount m of the substance at time t is given by

$$m = m_0 e^{-kt}.$$

The graph of the solution $m = m_0 e^{-kt}$, shown in Figure 5, illustrates that this solution has the characteristics we would expect of radioactive decay. The amount of the radioactive substance decreases, initially quite quickly

and then more slowly, until eventually there is practically none of the original substance left.

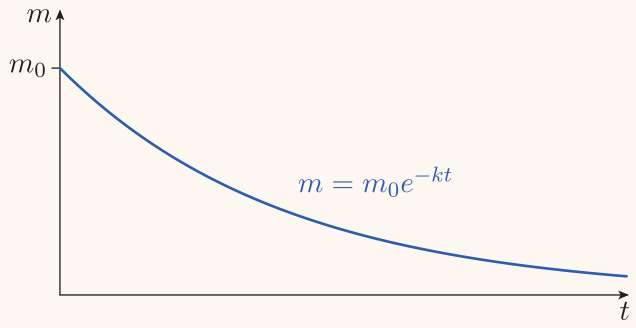


Figure 5 The graph of $m = m_0e^{-kt}$ for $k > 0$

Different radioactive substances have different values for the decay constant k. The larger the value of the decay constant k, the faster the substance decays. This is illustrated in Figure 6.

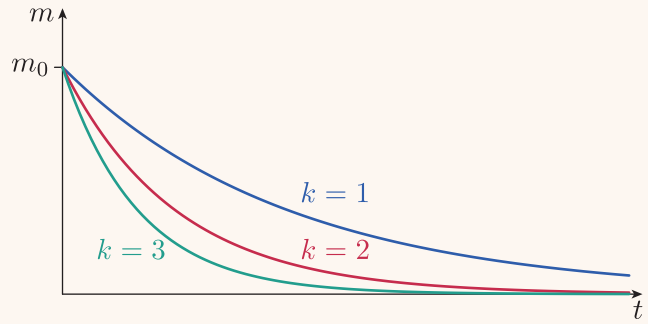


Figure 6 The graph of $m = m_0e^{-kt}$ for several values of k

Because the solution $m = m_0e^{-kt}$ is an exponential decay function, the properties of exponential decay functions discussed in the last subsection hold for radioactive decay. In particular, every radioactive substance has a **half-life**, which is the time that it takes for the amount of the substance to halve, and the relationship between the decay constant and the half-life is as follows.

Half-life of a radioactive substance

If a radioactive substance decays with decay constant k, then its half-life T is given by

$$T = \frac{\ln 2}{k}.$$

Although the speed of decay of a radioactive substance can be described in terms of the decay constant k, it is more usual to use the half-life T.

Different radioactive substances have a very wide range of half-lives, from billions of years down to fractions of a second.

Activity 20 *Using a half-life*

Silicon-31 has a half-life of 2.62 hours.

(a) Find the value of the decay constant k for silicon-31, to three significant figures.

(b) Hence calculate, to two significant figures, the proportion of a sample of silicon-31 that will still be present after 6 hours.

Ernest Rutherford was a New Zealand-born British physicist of great distinction. He discovered the concept of radioactive half-life and was the first to show that radioactivity was caused by the decay of heavy atoms into lighter atoms, for which he was awarded the 1908 Nobel Prize in Chemistry.

Carbon dating

One useful application of radioactivity is for dating archaeological finds of a biological origin. Carbon-14 is a radioactive element present in the atmosphere that decays (to nitrogen) with a half-life of about 5570 years. This decay is balanced by the production of new carbon-14 in the atmosphere by cosmic rays, so that the proportion of carbon-14 in the atmosphere is approximately constant. It is absorbed by living organisms when they are alive, but is no longer replenished after the organism dies, so the amount of carbon-14 in the remains of the organism declines as the carbon-14 decays. By measuring the proportion of carbon-14 in a dead organism and comparing it to the proportion in a living one, the number of years since death can be estimated. This technique is known as **carbon dating** or **radiocarbon dating**.

Activity 21 *Using carbon dating*

Suppose that the bones of an animal are found in an archaeological dig and analysis produces the estimate that 85% of the original amount of carbon-14 is still present in the bones. Taking the half-life of carbon-14 to be 5570 years, find the approximate age of the bones to the nearest 100 years.

3.3 Newton's law of cooling

When you make a hot cup of tea and leave it on your desk, the tea cools down, eventually reaching the ambient temperature of the room. Furthermore the colder the room is, the faster the tea cools down. This observation is the foundation of the following law.

Newton's law of cooling

The rate at which a body cools is proportional to the temperature difference between the body and the surrounding medium.

In our example the body is the cup of tea and the surrounding medium is the air in the room.

Empirical observation has shown that Newton's law of cooling holds, to a good approximation, in a wide variety of situations where the body and the surrounding medium are both good conductors of heat.

Let's write Newton's law of cooling in symbols. Let T be the temperature of the body (such as the cup of tea) at time t, and let T_a denote the ambient temperature of the surrounding medium (such as the air), where the subscript 'a' is short for 'ambient'. Then the rate at which the body cools is $\mathrm{d}T/\mathrm{d}t$, and the temperature difference between the body and the surrounding medium is $T - T_\mathrm{a}$.

Hence, according to Newton's law of cooling, we have

$$\frac{\mathrm{d}T}{\mathrm{d}t} = K(T - T_\mathrm{a}),$$

where K is a constant. Since the temperature of the body is decreasing, the value of $\mathrm{d}T/\mathrm{d}t$ is always negative. Also, the ambient temperature is always lower than the temperature of the body, so $T - T_\mathrm{a}$ is always positive. Therefore K is a negative constant, and hence we can write $K = -\lambda$, where λ is a positive constant. The equation above then becomes

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\lambda(T - T_\mathrm{a}).$$

The positive constant λ is dependent on both the body and the surrounding medium.

If we assume that the surrounding medium is large in comparison to the hot body (as in the case of the air surrounding the cup of tea), then it will hardly heat up as the body is cooling down, so we can assume that T_a is a constant. Under these assumptions we can solve the differential equation above using separation of variables. In doing so we can assume that $T - T_\mathrm{a} > 0$, a fact mentioned above.

To find a particular solution of the differential equation, we need an initial condition. If the initial temperature of the hot body is T_0, then the initial condition is

$$T = T_0 \text{ when } t = 0.$$

In the next activity you are asked to solve the differential equation that arises from Newton's law of cooling, with the initial condition above.

Activity 22 *Solving Newton's law of cooling*

Consider the differential equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\lambda(T - T_\mathrm{a}) \qquad (T - T_\mathrm{a} > 0),$$

which arises from Newton's law of cooling.

(a) Use separation of variables to show that the general solution of this differential equation is

$$T = T_\mathrm{a} + Ae^{-\lambda t},$$

where A is an arbitrary positive constant.

(b) Show that the particular solution that satisfies the initial condition

$$T = T_0 \text{ when } t = 0$$

is

$$T = T_\mathrm{a} + (T_0 - T_\mathrm{a})e^{-\lambda t}.$$

(c) Suppose that a body cools according to Newton's law of cooling. Use the particular solution found in part (b) to find the eventual temperature of the body after a long time.

Here is a summary of the solution determined in Activity 22.

Solution of Newton's law of cooling

If a body cools in a surrounding medium with constant temperature T_a, and the temperature of the body at time $t = 0$ is T_0, then the temperature T of the body at time t is given by

$$T = T_\mathrm{a} + (T_0 - T_\mathrm{a})e^{-\lambda t},$$

where λ is a positive constant.

Activity 23 *Determining a time of death*

The temperature of a healthy human being is 37 °C. When a person dies, the body starts to cool, and its rate of cooling depends on the surrounding ambient air temperature.

Suppose that a dead body is found and a pathologist is called in to determine the time of death. The pathologist finds that the room in which the body is located is kept at a constant temperature of 20 °C.

She takes the temperature of the body when she first arrives and finds it to be 30 °C. She takes another temperature reading an hour later and finds it to be 28.5 °C.

(a) By finding the values of the constants in the solution of Newton's law of cooling that apply to this situation, show that the temperature T of the body as a function of time t, taking $t = 0$ to be the time at which the pathologist arrived, is

$$T = 20 + 10 \times 0.85^t.$$

(b) How long before the pathologist arrived did the person die? Give your answer to the nearest quarter of an hour.

4 Solving linear differential equations

This section presents a third method of finding solutions to first-order differential equations. The method, called the *integrating factor* method, applies only to a particular form of equation known as a *linear* differential equation.

4.1 Linear differential equations

A first-order differential equation involving the independent variable x and the dependent variable y is **linear** if it can be expressed in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = h(x), \tag{19}$$

where $g(x)$ and $h(x)$ are given functions.

The linear first-order differential equation (19) is **homogeneous** if $h(x) = 0$ for all x, and **inhomogeneous** or **non-homogeneous** otherwise.

The term *linear* here refers to the dependent variable y. Although any functions $g(x)$ and $h(x)$ of the independent variable x are allowed in the differential equation, the only occurrences of y allowed are those in the terms $\mathrm{d}y/\mathrm{d}x$ and $g(x)y$. This means that terms involving y^2, y^3, $\sin y$, and so on, are not allowed.

So, for example, the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - x^2 y = x^3$$

is linear, with $g(x) = -x^2$ and $h(x) = x^3$, whereas the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2$$

is not linear, due to the presence of the term y^2.

Activity 24 *Identifying linear differential equations*

State which of the following differential equations are linear, identifying the functions $g(x)$ and $h(x)$ in each case that is linear.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + x^3 y = x^5$ (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x \sin x$ (c) $\dfrac{\mathrm{d}z}{\mathrm{d}t} = -3z^{1/2}$

(d) $\dfrac{\mathrm{d}y}{\mathrm{d}t} + y^2 = t$ (e) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = y^2$ (f) $(1 + x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 3x^2$

Note that if $g(x) = 0$, then equation (19) reduces to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = h(x),$$

which is the directly integrable type considered in Subsection 1.2. So, directly integrable differential equations are also linear differential equations. Since we have already considered this case in some detail, for the remainder of this section we will concentrate on the case where $g(x) \neq 0$.

4.2 Constant-coefficient linear differential equations

One of the simplest examples of a linear differential equation is the case where $h(x) = 0$ (so the equation is homogeneous) and $g(x) = A$, where A is a constant. In this case, equation (19) becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} + Ay = 0.$$

We dealt with this differential equation in some detail in the previous section (in the equivalent form $\mathrm{d}P/\mathrm{d}t = KP$). It has solution $y = y_0 e^{-Ax}$, where y_0 is a constant.

Now consider the case where $h(x) \neq 0$ (that is, the differential equation is inhomogeneous) and $g(x) = A$, where A is a constant. This gives the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + Ay = h(x), \tag{20}$$

which is called a **constant-coefficient** first-order linear differential equation. This is because in the context of the linear differential equation (19), the function $g(x)$ is often called the coefficient of y.

We can use a trick to solve differential equations of this form. Here's the idea. First multiply both sides by e^{Ax}:

$$e^{Ax}\frac{\mathrm{d}y}{\mathrm{d}x} + Ae^{Ax}y = e^{Ax}h(x). \tag{21}$$

Now, by the product rule for differentiation,

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{Ax}y) = e^{Ax}\frac{\mathrm{d}y}{\mathrm{d}x} + Ae^{Ax}y.$$

So the left-hand side of equation (21) is equal to the derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{Ax}y)$$

and hence equation (21) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{Ax}y) = e^{Ax}h(x).$$

This equation can be solved by integrating both sides with respect to x, which gives

$$e^{Ax}y = \int e^{Ax}h(x)\,\mathrm{d}x.$$

Hence the general solution of differential equation (20) is

$$y = e^{-Ax}\left(\int e^{Ax}h(x)\,\mathrm{d}x\right). \tag{22}$$

So, if we can find the integral on the right-hand side, then we can solve differential equation (20). Here is an example where the integral can be found by using integration by parts.

Example 10 *Solving a simple linear differential equation*

(a) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x.$$

(b) Check that your solution satisfies the differential equation.

Solution

(a) The differential equation has the form of equation (20), with $A = 2$ and $h(x) = x$. So the solution is given by equation (22).

By equation (22), the general solution is

$$y = e^{-2x}\left(\int xe^{2x}\,\mathrm{d}x\right).$$

The integrand is a product of two expressions, x and e^{2x}. You can integrate the second expression, e^{2x}, and differentiating the first expression, x, makes it simpler. So try integration by parts.

Integrating by parts gives

$$\begin{aligned} y &= e^{-2x}\left(\tfrac{1}{2}xe^{2x} - \tfrac{1}{2}\int e^{2x}\,\mathrm{d}x\right) \\ &= \tfrac{1}{2}e^{-2x}\left(xe^{2x} - \int e^{2x}\,\mathrm{d}x\right). \end{aligned}$$

Do the integration inside the brackets, remembering to include the constant of integration inside the brackets.

So, the general solution is

$$y = \tfrac{1}{2}e^{-2x}\left(xe^{2x} - \tfrac{1}{2}e^{2x} + c\right)$$
$$= \tfrac{1}{2}\left(x - \tfrac{1}{2} + ce^{-2x}\right),$$

where c is an arbitrary constant.

(b) Differentiating the general solution we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{1}{2}\left(1 - 2c\,e^{-2x}\right).$$

Substituting this and the general solution into the left-hand side of the differential equation gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \tfrac{1}{2}\left(1 - 2c\,e^{-2x}\right) + \tfrac{2}{2}\left(x - \tfrac{1}{2} + ce^{-2x}\right)$$
$$= \tfrac{1}{2} - c\,e^{-2x} + x - \tfrac{1}{2} + c\,e^{-2x}$$
$$= x,$$

which is equal to the right-hand side. So $y = \frac{1}{2}\left(x - \frac{1}{2} + ce^{-2x}\right)$ satisfies the differential equation.

Here is a similar example for you to try.

Activity 25 *Solving a simple linear differential equation*

(a) Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^{2x}.$$

(b) Check that your solution satisfies the differential equation.

In the next subsection you'll see that equation (22) can be adapted to solve the case where the coefficient $g(x)$ is not constant.

4.3 The integrating factor method

The key to solving the constant-coefficient linear differential equation was to multiply both sides by e^{Ax}. This allowed us to integrate the left-hand side of the equation. For this reason e^{Ax} is called an **integrating factor** for the differential equation. We now generalise this trick to solve the general linear first-order differential equation (19).

Integrating factor method

To find the general solution of a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = h(x), \tag{23}$$

determine the functions $g(x)$ and $h(x)$, and then perform the following steps.

1. Write down the integrating factor

$$p(x) = \exp\left(\int g(x)\,\mathrm{d}x\right). \tag{24}$$

   Find the integral, if possible, but don't include a constant of integration.

2. Write down the general solution, given by the formula

$$y = \frac{1}{p(x)}\left(\int p(x)h(x)\,\mathrm{d}x\right). \tag{25}$$

   Find the integral, if possible, and include the constant of integration inside the brackets.

Remember that $\exp(x)$ is just another way to write e^x, so equation (24) can be written as

$$p(x) = e^{\int g(x)\,\mathrm{d}x}.$$

You will see why the integrating factor method works later in the section. Notice, however, that it is a generalisation of the method that was used for the constant-coefficient linear case in Subsection 4.2. In that case $g(x) = A$, so the integrating factor given by equation (24) is

$$p(x) = \exp\left(\int A\,\mathrm{d}x\right) = \exp(Ax) = e^{Ax}.$$

This is exactly the integrating factor that was used in Subsection 4.2. The general solution given by equation (25) is therefore

$$y = e^{-Ax}\left(\int e^{Ax}h(x)\,\mathrm{d}x\right),$$

exactly the same as equation (22).

As with the separation of variables method, when you use the integrating factor method it is not always possible to find the necessary integrals. However, this *is* always possible in the examples and activities in this unit.



Example 11 *Applying the integrating factor method*

Use the integrating factor method to find the general solution of each of the following differential equations.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = e^x \sin x$ (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y-1}{x}$ $(x > 0)$

Solution

(a) Verify that the given differential equation is linear.

The given differential equation has the form of equation (23) with

$$g(x) = -1 \quad \text{and} \quad h(x) = e^x \sin x,$$

so we use the integrating factor method.

First calculate an integrating factor, without including a constant of integration.

By equation (24), the integrating factor is

$$p(x) = \exp\left(\int (-1)\,\mathrm{d}x\right) = \exp(-x) = e^{-x}.$$

Calculate the general solution, including the constant of integration inside the brackets.

The general solution given by equation (25) is

$$y = \frac{1}{e^{-x}}\left(\int e^{-x}(e^x \sin x)\,\mathrm{d}x\right) = e^x\left(\int \sin x\,\mathrm{d}x\right).$$

Doing the integration, we obtain

$$y = e^x(-\cos x + c),$$

where c is an arbitrary constant. This is the general solution.

(b) Verify that the given differential equation is linear.

We rearrange the differential equation so it is in the form of equation (23):

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x}y = -\frac{1}{x}. \tag{26}$$

This now has the form of equation (23), with

$$g(x) = -\frac{1}{x} \quad \text{and} \quad h(x) = -\frac{1}{x},$$

so we use the integrating factor method.

First calculate the integrating factor, without including a constant of integration.

By equation (24), the integrating factor is

$$\begin{aligned} p(x) &= \exp\left(\int\left(-\frac{1}{x}\right)\,\mathrm{d}x\right) \\ &= \exp\left(-\ln|x|\right) \\ &= \exp\left(-\ln x\right) \qquad (\text{since } x > 0) \\ &= \exp\left(\ln\left(x^{-1}\right)\right) \\ &= \frac{1}{x}. \end{aligned}$$

Calculate the general solution, including the constant of integration inside the brackets.

The general solution, given by equation (25), is

$$\begin{aligned} y &= \frac{1}{1/x}\left(\int\left(\frac{1}{x}\right)\left(-\frac{1}{x}\right)\,\mathrm{d}x\right) \\ &= x\left(-\int\frac{1}{x^2}\,\mathrm{d}x\right) \\ &= x\left(\frac{1}{x}+c\right) \\ &= 1+cx, \end{aligned}$$

where c is an arbitrary constant. This is the general solution.

Note that the differential equation in part (b) of Example 11 can also be solved by the method of separation of variables, if you assume that $y \neq 1$ (see 'A note of caution' on page 26). You might like to check that you get the same answer if you solve it this way (except that one particular solution is omitted because of the condition $y \neq 1$).

Here are some differential equations that you can solve by the integrating factor method.

Activity 26 *Using the integrating factor method to find general solutions*

Find the general solution of each of the following differential equations.

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y + x$ (b) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} - 3y = x \quad (x > 0)$

Activity 27 *Using the integrating factor method to find particular solutions*

Solve each of the following initial value problems.

(a) $\dfrac{\mathrm{d}u}{\mathrm{d}x} = xu$, where $u(0) = 2$

(b) $t\dfrac{\mathrm{d}y}{\mathrm{d}t} + 2y = t^2 \quad (t > 0)$, where $y(1) = 1$

Activity 28 *Newton's law of cooling: change of ambient temperature*

In Subsection 3.3 you met Newton's law of cooling, given by the differential equation

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\lambda(T - T_\mathrm{a}),$$

where λ is a positive constant.

The solution of this differential equation describes the temperature T of a body at time t, when it is placed in a surrounding medium of ambient temperature T_a. In that subsection we assumed that the ambient temperature T_a was a constant. However, if T_a varies reasonably slowly with time, the equation remains valid and it can be rearranged to give

$$\frac{\mathrm{d}T}{\mathrm{d}t} + \lambda T = \lambda T_\mathrm{a}(t), \tag{27}$$

which is a first-order linear differential equation.

(a) Suppose that the ambient temperature T_a is a function of time that decays exponentially with time; that is,

$$T_\mathrm{a}(t) = T_{\mathrm{a}0}\, e^{-kt},$$

where k and $T_{\mathrm{a}0}$ are positive constants, and $0 < k < \lambda$. This equation models a surrounding medium that has ambient temperature $T_{\mathrm{a}0}$ at time $t = 0$, and then cools to temperature 0 after a very long time. Then equation (27) becomes

$$\frac{\mathrm{d}T}{\mathrm{d}t} + \lambda T = \lambda T_{\mathrm{a}0}\, e^{-kt}.$$

Show that the general solution of this equation is

$$T = \frac{\lambda T_{\mathrm{a}0}\, e^{-kt}}{\lambda - k} + A\, e^{-\lambda t},$$

where A is an arbitrary constant.

(b) Check that when $k = 0$ the function $T_\mathrm{a}(t)$ is a constant and the general solution reduces to that obtained in Activity 22.

As always, when you have solved a differential equation it is a good idea to check that the solution you obtain satisfies the original differential equation. As before, you can do this by substituting your solution and its derivative into the differential equation.

Example 12 *Verifying a solution*

Verify that the solution to Example 11(a) satisfies the differential equation in that example.

Solution

The differential equation was

$$\frac{dy}{dx} - y = e^x \sin x$$

and its general solution was calculated to be

$$y = e^x(-\cos x + c).$$

Differentiating this general solution using the product rule gives

$$\begin{aligned}\frac{dy}{dx} &= e^x \frac{d}{dx}(-\cos x + c) + \frac{d}{dx}(e^x)(-\cos x + c) \\ &= e^x \sin x + e^x(-\cos x + c).\end{aligned}$$

Substituting for y and dy/dx in the left-hand side of the differential equation gives

$$\begin{aligned}\frac{dy}{dx} - y &= e^x \sin x + e^x(-\cos x + c) - e^x(-\cos x + c) \\ &= e^x \sin x,\end{aligned}$$

which is equal to the right-hand side, as required.

Here's an example for you to try.

Activity 29 *Verifying another solution*

Verify that the solution to Example 11(b) satisfies the differential equation in that example.

4.4 Why the integrating factor method works

If you would like to know why the integrating factor method works, then you can read the following justification. Otherwise, skip this material and go on to the next section.

The general first-order linear differential equation is of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = h(x), \tag{28}$$

and we show that its general solution is

$$y = \frac{1}{p(x)}\left(\int p(x)h(x)\,\mathrm{d}x\right), \tag{29}$$

where

$$p(x) = \exp\left(\int g(x)\,\mathrm{d}x\right).$$

Here $p(x)$ is the integrating factor.

The integrating factor $p(x)$ is chosen to be of this form because, when it is differentiated with respect to x, we obtain the following equation:

$$\frac{\mathrm{d}p}{\mathrm{d}x} = p(x)g(x). \tag{30}$$

To see why this equation holds, we set $p(x) = e^u$, where $u = \int g(x)\,\mathrm{d}x$, and then use the chain rule for differentiation to give

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}u}(e^u)\frac{\mathrm{d}u}{\mathrm{d}x} = e^u g(x) = p(x)g(x),$$

as required.

By the product rule and equation (30),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(p(x)y) &= p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}p}{\mathrm{d}x}y \\ &= p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)g(x)y.\end{aligned} \tag{31}$$

Now, multiplying both sides of equation (28) by $p(x)$ gives

$$p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)g(x)y = p(x)h(x).$$

The left-hand side of this equation is the same as the right-hand side of equation (31), so

$$\frac{\mathrm{d}}{\mathrm{d}x}(p(x)y) = p(x)h(x).$$

It follows that

$$p(x)y = \int p(x)h(x)\,\mathrm{d}x.$$

Dividing both sides by $p(x)$ gives the general solution as

$$y = \frac{1}{p(x)}\left(\int p(x)h(x)\,\mathrm{d}x\right),$$

which is equation (29), as required.

Finally, you may have been wondering whether we would obtain further solutions to this differential equation by including a constant of integration when we find the integrating factor. But if we did this, then the integrating factor would be of the form

$$p(x) = \exp\left(\int g(x)\,\mathrm{d}x + C\right) = e^C \exp\left(\int g(x)\,\mathrm{d}x\right),$$

where C is a constant, so the new integrating factor would simply be a constant multiple of the original one, and in equation (29) this constant multiple would be cancelled out.

5 Summary and consolidation

It's time to summarise and review the main results of this unit so far. You have seen three types of first-order differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x), \quad \frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y), \quad \frac{\mathrm{d}y}{\mathrm{d}x} + g(x)y = h(x).$$

The first type is *directly integrable* and is solved by direct integration, the second type is *separable* and is solved by separation of variables, and the third type is *linear* and is solved by the integrating factor method. Also, remember that the first type is a special case of the other two.

You need to be able to recognise each type of differential equation and know its method of solution. It is important to realise that often variables other than x and y are used, and that a differential equation may need to be rearranged in order to put it into one of these forms.

Here is an activity to check your ability to recognise each of these three types of differential equations, followed by several activities that are intended to strengthen your ability to solve them.

Activity 30 *Identifying differential equations*

Which method would you use to try to solve each of the following differential equations? If more than one method can be used, you should list all the methods. (You are *not* asked to solve these equations.)

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + x^3 y = x^5$ (b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x \sin x$

(c) $\dfrac{\mathrm{d}v}{\mathrm{d}u} + 5v = 0$ (d) $(1 + x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 1 - x^2$

(e) $\dfrac{\mathrm{d}y}{\mathrm{d}t} - \dfrac{2y}{t} = y + t$ (f) $\dfrac{\mathrm{d}u}{\mathrm{d}x} = xu^2 + 2(u^2 - x) - 4$

Activity 31 *An application: a melting snowball*

The rate of change of the radius, r, of a melting snowball with respect to time, t, can be modelled by the differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -k \quad (r \geq 0),$$

where k is a positive constant. It is assumed that the snowball is spherical at all times and that the temperature of the air around it is a constant (above 0 °C). The radius r of the snowball is measured in centimetres and the time t is measured in days since melting started.

(a) Find the general solution of this differential equation.

(b) The snowball originally has a radius of 100 cm. Find the particular solution that describes its radius as a function of time.

(c) The snowball is found to have lost half its initial volume after 2 days. Show that $k = 50\left(1 - 2^{-1/3}\right)$. (Remember that the volume V of a sphere is given in terms of its radius r by $V = \frac{4}{3}\pi r^3$.)

(d) How many days does it take for the snowball to disappear completely? Give your answer to the nearest day.

Activity 32 *Finding a general solution and a particular solution*

(a) Find the general solution of the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t\cos t.$$

(b) Find the particular solution that satisfies the initial condition $x = 2$ when $t = 0$.

Activity 33 *Solving an initial value problem*

(a) Solve, in explicit form, the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = tx \quad (x > 0), \quad \text{where } x = 2 \text{ when } t = 0.$$

Activity 34 *Solving an initial value problem and checking the solution*

(a) Solve, in implicit form, the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t}{\cos x + x} \quad (x \geq 0), \quad \text{where } x(1) = 0.$$

(b) Verify that the particular solution that you have found satisfies the differential equation.

Activity 35 *Solving an initial value problem*

Solve the initial value problem

$$e^{3t}\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - e^{3t}y, \quad \text{where } y(0) = 3.$$

Activity 36 *Finding a general solution and a particular solution*

(a) Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x - \frac{2xy}{x^2 + 1}.$$

(b) Find the particular solution of the differential equation that satisfies $y(0) = \frac{1}{4}$.

6 Other methods of solving differential equations

This final section describes various other approaches to finding solutions of first-order differential equations and to understanding the behaviour of these solutions.

Consider the general first-order differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y),$$

where $f(x, y)$ is some given function of x and y. As you have seen, for certain special types of function $f(x, y)$, you can solve the differential equation, obtaining a general solution for y in either explicit or implicit form, and then obtain particular solutions by using initial conditions.

If such a solution to a differential equation can be obtained, then the differential equation is said be **solvable analytically** and the solution is said to be an **analytic solution**.

There are more elaborate techniques, which are beyond the scope of this module, for finding analytic solutions when $f(x, y)$ has certain more complicated forms. Indeed, the module computer algebra system can find analytic solutions to a much wider variety of first-order differential equations than you have seen here. Sometimes such solutions are expressed in terms of functions that are familiar, like sin, cos, exp, ln, and so on, and in other cases the solutions are in terms of more obscure functions.

However, there are many functions $f(x, y)$ for which the differential equation $\mathrm{d}y/\mathrm{d}x = f(x, y)$ has *no* analytic solutions. For example, the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(xy),$$

has no analytic solutions in explicit or implicit form. Such differential equations occur frequently in the mathematical sciences, and are probably more common than differential equations with analytic solutions!

Mathematicians and scientists have developed a number of techniques to explore the solutions of such differential equations. These include:

- finding approximate analytic solutions
- plotting *direction fields*
- calculating *numerical solutions*.

The first of these techniques consists of a wide range of methods for constructing approximate analytic solutions to differential equations, which are close to exact analytic solutions. This is a vast area of applied mathematics, well beyond the scope of this module, so we will say no more about it.

The second technique, plotting direction fields, is described in Subsection 6.1. The third technique, calculating numerical solutions, is discussed briefly in Subsection 6.2. Then, in Subsection 6.3, the use of the computer for solving differential equations is covered.

6.1 Direction fields

A direction field is a graphical representation of a differential equation.

To appreciate the idea, consider the simple differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2,$$

from Subsection 1.2, which has general solution

$$y = \tfrac{1}{3}x^3 + c,$$

where c is an arbitrary constant. This general solution consists of an infinite family of functions, some of whose graphs are shown in Figure 7.

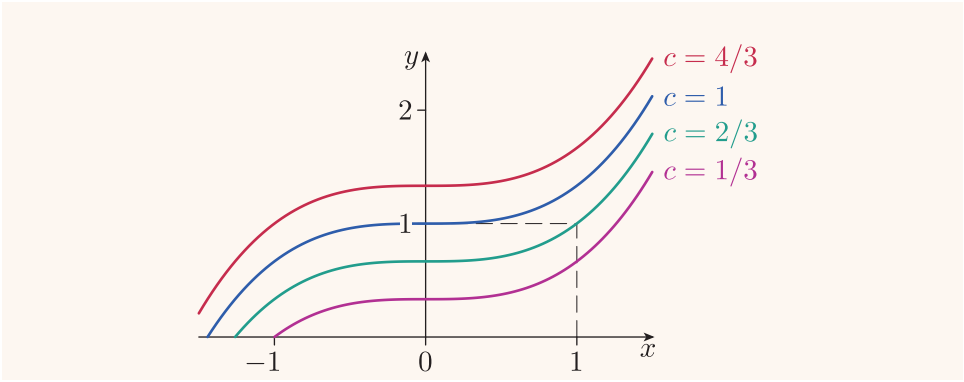


Figure 7 The graphs of $y = \frac{1}{3}x^3 + c$, for $c = \frac{1}{3}$, $\frac{2}{3}$, 1 and $\frac{4}{3}$

At each point (x, y) on one of these graphs, the equation $\mathrm{d}y/\mathrm{d}x = x^2$ must hold; that is, the gradient of the graph is x^2. For example, at the point $(x, y) = (1, 1)$ the gradient is $x^2 = 1^2 = 1$.

This property is also true for a general differential equation of the form $\mathrm{d}y/\mathrm{d}x = f(x, y)$. That is, each solution of the differential equation is a function, $y = y(x)$, with the property that at each point (x, y) on the graph of the function the gradient is $f(x, y)$. When these gradients are depicted graphically at many points of the plane, we obtain what is called a direction field.

This is done for the simple case of $f(x, y) = x^2$ in Figure 8. Here a grid of points has been chosen and at each point (x, y) of the grid a short line segment has been drawn centred on (x, y), with the gradient x^2. The idea is that if the spacing between the grid points is not too large, then a good picture is obtained of the shape of the graphs of the solutions to the differential equation.

In this figure, the graph of one of the particular solutions, $y = \frac{1}{3}x^3 + \frac{2}{3}$, has also been plotted. As you can see, the short line segments near this graph are close to being tangent to it.

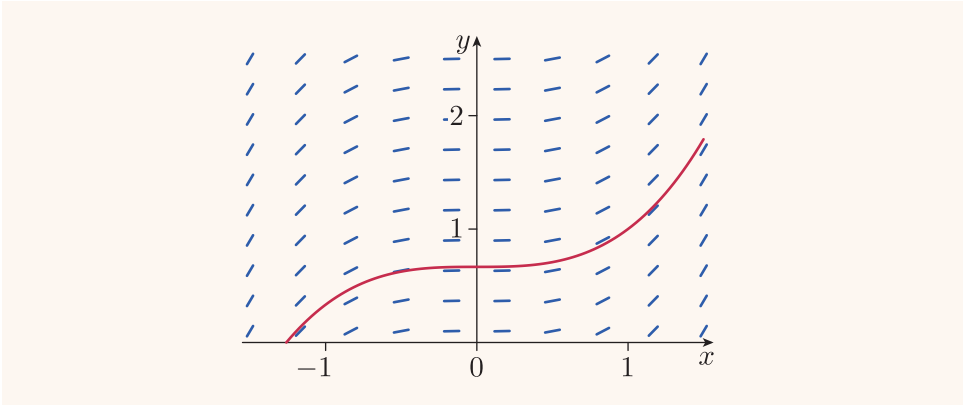


Figure 8 A direction field for $\mathrm{d}y/\mathrm{d}x = x^2$, together with the particular solution $y = \frac{1}{3}x^3 + \frac{2}{3}$

The distribution over the plane of such short line segments with the appropriate gradients is called a **direction field** for the differential equation. Even when you can't find the solution to a differential equation $\mathrm{d}y/\mathrm{d}x = f(x, y)$ directly, you can *always* plot a direction field for the differential equation.

Figure 9 shows a direction field for the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(xy),$$

which can't be solved analytically. You can see that the gradients of the line segments plotted do correspond to the numerical values of $f(x, y) = \sin(xy)$. For example, these gradients are 0 when $x = 0$ (that is, along the y-axis) and also when $y = 0$ (that is, along the x-axis), and the gradient at the point $(x, y) = (1, 1)$ is $\sin 1 = 0.841\ldots$.

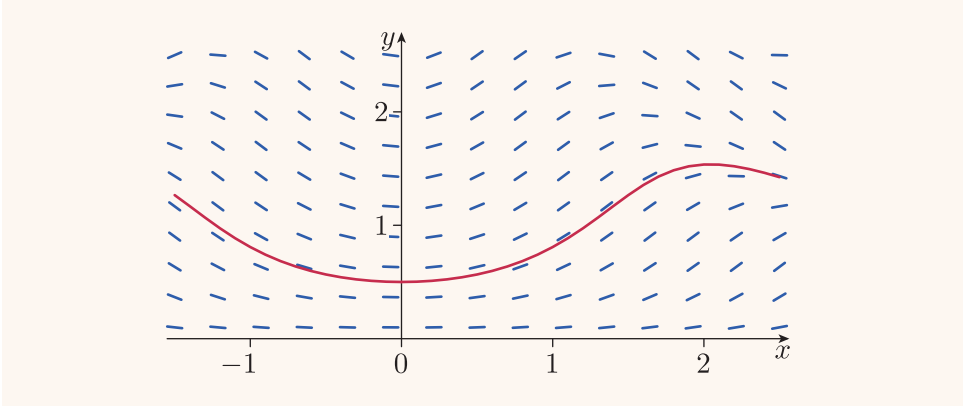


Figure 9 A direction field for $\mathrm{d}y/\mathrm{d}x = \sin(xy)$, and the particular solution satisfying the initial condition $y(0) = 0.5$

The graph of any particular solution to the differential equation must have a gradient at each point that matches the gradient provided by the direction field. You can see that the gradients at points on the graph of the particular solution plotted in Figure 9 match the gradients of the nearby line segments very closely. Thus the direction field gives a good intuitive impression of how solutions of this differential equation behave. If the short line segments were plotted more closely together, then this impression would be even clearer.

So, a direction field is a graphical representation of a differential equation, which gives you an idea of what the solutions are like. Direction fields are most often used for differential equations that can't be solved analytically, but they are also useful even if an analytic solution can be obtained, giving a qualitative understanding of the behaviour of the general solution. Such graphical representations are used extensively in more advanced studies of differential equations.

In principle, direction fields can be plotted by hand, although it is a tedious task. In Activity 37, you will see how the module computer algebra system can be used to plot direction fields.

6.2 Numerical solutions of differential equations

You have seen that the direction fields introduced in Subsection 6.1 give useful qualitative information about the solutions of a differential equation. This geometric approach can be developed further in order to use a computer to produce *quantitative* information about the solutions of the differential equation, in the form of *numerical solutions*.

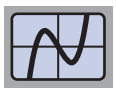
A **numerical solution** of the differential equation $\mathrm{d}y/\mathrm{d}x = f(x, y)$ consists of a sequence of points in the x, y-plane. These points are chosen in such a way that taken together the points form an approximation to the graph of a particular solution of the differential equation. Roughly speaking, they are obtained by 'joining up' short line segments in a direction field for the differential equation. There are various ways of calculating such a sequence of points, the details of which are beyond the scope of this module, but the idea is as follows.

We start with a given value of x and the corresponding value of y, given by the initial condition. The value of x is then increased by a small amount to a new value and an estimate is obtained for the corresponding new value of y by using the expression $f(x, y)$ for the derivative $\mathrm{d}y/\mathrm{d}x$. This step is then repeated many times until an approximate numerical solution has been obtained for the required range of values of x. The more points there are in the sequence, and hence the closer together adjacent points are, the more accurate the numerical solution is likely to be.

6.3 Solving differential equations using a computer

In the following activity, you can learn how to use the computer algebra system to find analytic and numerical solutions of differential equations and to investigate direction fields.

Activity 37 *Solving differential equations with a computer*



To complete your study of Unit 8, work through Section 7 of the *Computer algebra guide*, where you will learn how to obtain analytic and numerical solutions of differential equations, and how to plot direction fields.

Solutions to activities

Solution to Activity 1

Directly integrable differential equations have the form

$$\frac{\text{d(dependent variable)}}{\text{d(independent variable)}} = f(\text{independent variable}).$$

Equation (a) is of this form. Equations (b) and (c) are not.

Solution to Activity 2

(a) Integrating both sides of the differential equation with respect to x gives

$$y = \int (2x + 3)\,\mathrm{d}x$$
$$= x^2 + 3x + c,$$

where c is an arbitrary constant.

(b) With $y = x^2 + 3x + c$ the left-hand side of the differential equation gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 3x + c\right) = 2x + 3,$$

which is the same as the right-hand side, so the differential equation is satisfied.

Solution to Activity 3

(a) The general solution of $\mathrm{d}x/\mathrm{d}t = \sin t$ is

$$x = \int \sin t\,\mathrm{d}t$$
$$= -\cos t + c,$$

where c is an arbitrary constant.

(Check: differentiating $x = -\cos t + c$ gives $\mathrm{d}x/\mathrm{d}t = \sin t$, as required.)

(b) Only (iii) is a particular solution of the differential equation. It corresponds to $c = 1$.

Solution to Activity 4

Substitute the values $x = 1$ and $y = 7$ into the equation $y = 5\sqrt{x} + c$. This gives

$$7 = 5\sqrt{1} + c = 5 + c,$$

so $c = 2$. The required particular solution is therefore

$$y = 5\sqrt{x} + 2.$$

Solution to Activity 5

(a) The general solution of $\mathrm{d}y/\mathrm{d}x = \sqrt{x} = x^{1/2}$ is

$$y = \int x^{1/2}\,\mathrm{d}x$$
$$= \tfrac{2}{3}x^{3/2} + c,$$

where c is an arbitrary constant.

The initial condition is $y = 5$ when $x = 4$. Substituting $x = 4$ and $y = 5$ into the general solution gives

$$5 = \tfrac{2}{3} \times 4^{3/2} + c = \tfrac{16}{3} + c,$$

so $c = -\frac{1}{3}$. Hence the particular solution that satisfies the initial condition is

$$y = \tfrac{2}{3}x^{3/2} - \tfrac{1}{3}.$$

(b) The general solution of $\mathrm{d}u/\mathrm{d}x = \cos(2x)$ is

$$u = \int \cos(2x)\,\mathrm{d}x$$
$$= \tfrac{1}{2}\sin(2x) + c,$$

where c is an arbitrary constant. Substituting $u = -2$ and $x = \frac{1}{4}\pi$ into the general solution gives

$$-2 = \tfrac{1}{2}\sin(\tfrac{1}{2}\pi) + c = \tfrac{1}{2} + c,$$

so $c = -\frac{5}{2}$. Hence the particular solution that satisfies the initial condition is

$$u = \tfrac{1}{2}\sin(2x) - \tfrac{5}{2}.$$

Solution to Activity 6

The general solution of $\mathrm{d}x/\mathrm{d}t = k/t$ $(t > 0)$ is

$$x = \int \frac{k}{t}\,\mathrm{d}t$$
$$= k\ln|t| + c,$$

where c is an arbitrary constant. However, we have the restriction $t > 0$, so the general solution is

$$x = k\ln t + c.$$

Using the initial condition, $x = 3$ when $t = 1$, we obtain

$$3 = k\ln 1 + c = c.$$

Hence $c = 3$, and the solution of the initial value problem is

$$x = k\ln t + 3.$$

Solution to Activity 7

(a) The general solution of $\mathrm{d}m/\mathrm{d}t = -8t$ is

$$m = \int (-8t)\,\mathrm{d}t$$

$$= -4t^2 + c \quad (t \geq 0),$$

where c is an arbitrary constant. (This general solution can be checked by differentiation.)

(b) The initial condition is $m = 200$ when $t = 0$. Substituting $t = 0$ and $m = 200$ into the general solution gives

$$200 = 0 + c,$$

so $c = 200$. Hence the required particular solution is

$$m = 200 - 4t^2 \quad (t \geq 0).$$

(c) When $m = 100$, the corresponding value of t is given by

$$100 = 200 - 4t^2; \quad \text{that is,} \quad t^2 = 25.$$

Hence $t = 5$ when $m = 100$, so the rocket's fuel is exhausted after 5 seconds.

Solution to Activity 8

Equations (a), (b), (c) and (e) are of the appropriate form, but equations (d) and (f) are not.

(a) $f(x) = x, \quad g(y) = \cos y$

(b) $f(t) = t\cos t, \quad g(x) = 1$

(c) $f(t) = 1, \quad g(p) = p\cos p$

(e) $f(y) = 1 + y, \quad g(v) = v$

These are the most likely choices for f and g, but they are not the only possible answers. For equation (a), for example, you could have chosen $f(x) = 2x$ and $g(y) = \frac{1}{2}\cos y$.

(Equation (b) can also be solved using direct integration; this is the most straightforward method of solution.)

Solution to Activity 9

(a) The differential equation $\mathrm{d}y/\mathrm{d}x = -x/y$ has a right-hand side of the form $f(x)g(y)$, where $f(x) = -x$ and $g(y) = 1/y$, so it is separable. By separating the variables, we obtain

$$\int y\,\mathrm{d}y = \int (-x)\,\mathrm{d}x.$$

Carrying out the two integrations gives

$$\tfrac{1}{2}y^2 = -\tfrac{1}{2}x^2 + c,$$

where c is an arbitrary constant. Since the left-hand side is positive when $y > 0$, the constant c must be positive.

This implicit form of the general solution can be manipulated to make y the subject, giving

$$y = \pm\sqrt{2c - x^2}.$$

However, we have a restriction of $y > 0$ in the question, so

$$y = \sqrt{2c - x^2},$$

or, equivalently,

$$y = \sqrt{b - x^2},$$

where $b = 2c$ is another arbitrary positive constant.

(The implicit form of the general solution can be written in the form $x^2 + y^2 = b$, which is the equation of a circle. The condition $y > 0$ led to the explicit solution $y = \sqrt{b - x^2}$, whose graph is the upper half of this circle.)

(b) The initial condition is $y = 1$ when $x = 0$. Substituting $x = 0$ and $y = 1$ into the general solution gives

$$1 = \sqrt{b - 0^2} = \sqrt{b},$$

so $b = 1$. Hence the particular solution that satisfies the initial condition is

$$y = \sqrt{1 - x^2}.$$

Solution to Activity 10

(a) The differential equation has a right-hand side of the form $f(x)g(y)$, where $f(x) = 1 + e^x$ and $g(y) = 1/(1 + \cos y)$, so it is separable. Separating the variables we obtain

$$\int (1 + \cos y)\,\mathrm{d}y = \int (1 + e^x)\,\mathrm{d}x.$$

Doing the integrations gives

$$y + \sin y = x + e^x + c,$$

where c is an arbitrary constant. This is the general solution in implicit form; it cannot be rearranged into the form $y = y(x)$.

(b) The initial condition is $y(0) = 0$; that is, $y = 0$ when $x = 0$. Substituting $x = 0$ and $y = 0$ into the general solution, we obtain

$$0 + \sin 0 = 0 + e^0 + c,$$

which gives

$$0 = 1 + c,$$

so $c = -1$. Hence the required particular solution in implicit form is

$$y + \sin y = x + e^x - 1.$$

Solution to Activity 11

The given equation is $y + \sin y = x + e^x - 1$.
Differentiating both sides with respect to x gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(y + \sin y) = \frac{\mathrm{d}}{\mathrm{d}x}(x + e^x - 1).$$

By the chain rule, the left-hand side is

$$\frac{\mathrm{d}}{\mathrm{d}y}(y + \sin y)\frac{\mathrm{d}y}{\mathrm{d}x} = (1 + \cos y)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

The right-hand side is

$$\frac{\mathrm{d}}{\mathrm{d}x}(x + e^x - 1) = 1 + e^x.$$

Hence we obtain

$$(1 + \cos y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + e^x.$$

Since $-\pi < y < \pi$, we have $1 + \cos y > 0$, so we can divide through by $1 + \cos y$ to give

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + e^x}{1 + \cos y},$$

which is the required differential equation.

Solution to Activity 12

The given equation is $y + \ln y = (1 - 2x)e^{3x}$.
Differentiating both sides with respect to x gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(y + \ln y) = \frac{\mathrm{d}}{\mathrm{d}x}\left((1 - 2x)e^{3x}\right).$$

By the chain rule, the left-hand side is

$$\frac{\mathrm{d}}{\mathrm{d}y}(y + \ln y)\frac{\mathrm{d}y}{\mathrm{d}x} = \left(1 + \frac{1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

The right-hand side is, by the product rule for differentiation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left((1 - 2x)e^{3x}\right) &= (1 - 2x)\left(3e^{3x}\right) + (-2)\left(e^{3x}\right)\\ &= (1 - 6x)e^{3x}.\end{aligned}$$

Hence we obtain

$$\left(1 + \frac{1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = (1 - 6x)e^{3x}.$$

Multiplying both sides by $y/(y + 1)$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y(1 - 6x)e^{3x}}{y + 1},$$

as required.

Solution to Activity 13

(a) The differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -y^3 \quad (y > 0).$$

So the differential equation is separable and, after separating the variables, we obtain

$$\int \left(-\frac{1}{y^3}\right)\mathrm{d}y = \int 1\,\mathrm{d}x;$$

that is,

$$\frac{1}{2y^2} = x + c,$$

where c is an arbitrary constant. To make y the subject of this equation, first take the reciprocals of both sides and then divide through by 2, to obtain

$$y^2 = \frac{1}{2(x + c)}.$$

Since $y > 0$, the general solution in explicit form is therefore

$$y = \frac{1}{\sqrt{2(x + c)}}.$$

The initial condition is $y = \frac{1}{2}$ when $x = 0$. Substituting $x = 0$ and $y = \frac{1}{2}$ in the general solution, we obtain

$$\frac{1}{2} = \frac{1}{\sqrt{2c}},$$

so $c = 2$. Hence the required particular solution is

$$y = \frac{1}{\sqrt{2(x + 2)}}.$$

(b) We can rearrange the differential equation as follows:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x^4}{u^2}.$$

So the equation is separable and, after separating the variables, we obtain

$$\int u^2\,\mathrm{d}u = \int x^4\,\mathrm{d}x;$$

that is,

$$\tfrac{1}{3}u^3 = \tfrac{1}{5}x^5 + c,$$

where c is an arbitrary constant.

Making u the subject of this equation, we obtain the explicit form of the general solution as

$$u = \left(\tfrac{3}{5}x^5 + b\right)^{1/3},$$

where $b = 3c$ is another arbitrary constant.

The initial condition is $u = 1$ when $x = 1$. Substituting $x = 1$ and $u = 1$ in the general solution, we obtain

$$1 = \left(\tfrac{3}{5} + b\right)^{1/3},$$

so $b = \frac{2}{5}$. Hence the required particular solution is

$$u = \left(\tfrac{3}{5}x^5 + \tfrac{2}{5}\right)^{1/3}.$$

Solution to Activity 14

The differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 \cos x}{y+1} \quad (y > 0).$$

So the equation is separable and, after separating the variables, we obtain

$$\int \frac{y+1}{y^2}\,\mathrm{d}y = \int \cos x\,\mathrm{d}x;$$

that is,

$$\int \left(\frac{1}{y} + \frac{1}{y^2}\right)\mathrm{d}y = \int \cos x\,\mathrm{d}x.$$

Doing the integrations, we obtain

$$\ln|y| - \frac{1}{y} = \sin x + c,$$

where c is an arbitrary constant. Since $y > 0$, this gives

$$\ln y - \frac{1}{y} = \sin x + c,$$

where c is an arbitrary constant. This is the general solution in implicit form, and it cannot be rearranged to make it explicit.

To find the particular solution we substitute the initial condition, $y = 2$ when $x = \pi/2$, into the general solution. This gives

$$\ln 2 - \tfrac{1}{2} = \sin(\pi/2) + c = 1 + c,$$

so $c = -\frac{3}{2} + \ln 2$. Hence the particular solution, in implicit form, is

$$\ln y - \frac{1}{y} = \sin x - \tfrac{3}{2} + \ln 2.$$

Solution to Activity 15

(a) The differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -ky^{1/2} \quad (y > 0).$$

Separating the variables, we obtain

$$\int y^{-1/2}\,\mathrm{d}y = \int (-k)\,\mathrm{d}t;$$

that is,

$$2y^{1/2} = -kt + c.$$

This is the general solution in implicit form, where c is an arbitrary constant.

(b) The initial condition is $y = y_0$ when $t = 0$. Substituting $t = 0$ and $y = y_0$ into the general solution, we obtain

$$2y_0^{1/2} = c.$$

Hence the required particular solution is

$$2y^{1/2} = -kt + 2y_0^{1/2},$$

which can be written as

$$kt = 2(y_0^{1/2} - y^{1/2}).$$

(c) The tank becomes half-empty when $y = \frac{1}{2}y_0$. This occurs when

$$kt = 2\left(y_0^{1/2} - \left(\tfrac{1}{2}y_0\right)^{1/2}\right).$$

This gives

$$kt = 2y_0^{1/2} - \frac{2}{2^{1/2}}\,y_0^{1/2};$$

that is,

$$t = \frac{(2-\sqrt{2})\sqrt{y_0}}{k},$$

as required.

Solution to Activity 16

(a) The differential equation is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = KP.$$

Separating the variables gives

$$\int \frac{1}{P}\,\mathrm{d}P = \int K\,\mathrm{d}t.$$

Doing the integrations, and using the fact that $P > 0$, gives

$$\ln P = Kt + c,$$

where c is an arbitrary constant. This is the general solution in implicit form.

To obtain the general solution in explicit form, we can apply the exponential function (the inverse of the natural logarithm function) to each side. This gives

$$e^{\ln P} = e^{Kt+c};$$

that is,

$$P = e^{Kt+c}.$$

By an index law, we obtain

$$P = e^c e^{Kt}.$$

Since c is an arbitrary constant, e^c is also an arbitrary constant, and we can denote it by A. The value of A must be positive since $e^c > 0$ for each real number c.

Hence the general solution is

$$P = Ae^{Kt},$$

where A is an arbitrary positive constant.

(b) The initial condition is $P = P_0$ when $t = 0$. Putting $t = 0$ and $P = P_0$ into the general solution, we have

$$P_0 = Ae^0 = A.$$

Hence $A = P_0$, and the required particular solution is

$$P = P_0 e^{Kt}.$$

Solution to Activity 17

(a) The initial population size is $P_0 = 5.1 \times 10^6$, and the proportionate growth rate is $K = 0.032$ year^{-1}. The population size P after time t years is given by

$$P = P_0 e^{Kt};$$

that is,

$$P = 5.1 \times 10^6 e^{0.032t}.$$

(i) After 10 years the population size will be

$$\begin{aligned} P &= 5.1 \times 10^6 e^{0.032 \times 10} \\ &= 5.1 \times 10^6 e^{0.32} \\ &= 7.0 \times 10^6 \text{ (to 2 s.f.).} \end{aligned}$$

That is, it will be 7.0 million, to two significant figures.

(ii) After 100 years the population size will be

$$\begin{aligned} P &= 5.1 \times 10^6 e^{0.032 \times 100} \\ &= 5.1 \times 10^6 e^{3.2} \\ &= 1.3 \times 10^8 \text{ (to 2 s.f.).} \end{aligned}$$

That is, it will be 130 million, to two significant figures.

(b) The population size will reach 50×10^6 at the time t years given by

$$5.1 \times 10^6 e^{0.032t} = 50 \times 10^6;$$

that is,

$$e^{0.032t} = \frac{50}{5.1}.$$

Taking the natural logarithm of each side, we obtain

$$0.032t = \ln\left(\frac{50}{5.1}\right);$$

that is,

$$t = \frac{1}{0.032} \ln\left(\frac{50}{5.1}\right) = 71.336\ldots.$$

Hence it will take 71 years (to the nearest year) for the population size to reach 50 million.

Solution to Activity 18

Suppose that the initial population size, at time $t = 0$, is P_0. The proportionate growth rate is $K = -0.03$. Hence the population size P at time t is given by

$$P = P_0 e^{-0.03t}.$$

Consider a ten-year time interval from time t to time $t + 10$. The population sizes at the start and end of this time interval are

$$P_0 e^{-0.03t} \quad \text{and} \quad P_0 e^{-0.03(t+10)},$$

respectively.

Hence the proportion of the population size that will remain at the end of the time interval is

$$\begin{aligned} \frac{P_0 e^{-0.03(t+10)}}{P_0 e^{-0.03t}} &= \frac{e^{-0.03t} e^{-0.03 \times 10}}{e^{-0.03t}} \\ &= e^{-0.03 \times 10} \\ &= e^{-0.3} \\ &= 0.7408\ldots. \end{aligned}$$

So a proportion of 0.74, that is, 74% (to 2 s.f.), of the population will remain at the end of the decade.

Solution to Activity 19

For the population described in Example 9, the proportionate growth rate is $K = 0.04$ year^{-1}. Hence the doubling time is

$$T = \frac{\ln 2}{0.04} = 17.3 \text{ years (to 3 s.f.)}.$$

For the population described in Activity 18, the proportionate growth rate is $K = -0.03$ year^{-1}. Hence the halving time is

$$T = -\frac{\ln 2}{-0.03} = \frac{\ln 2}{0.03} = 23.1 \text{ years (to 3 s.f.)}.$$

Solution to Activity 20

(a) By the equation in the box above the activity, the decay constant k corresponding to the half-life $T = 2.62$ hours is

$$k = \frac{\ln 2}{T} = \frac{\ln 2}{2.62} = 0.265 \text{ hour}^{-1} \text{ (to 3 s.f.)}.$$

(b) The amount m of a sample of silicon-31 that will remain after time t is given by

$$m = m_0 e^{-kt},$$

where k is the decay constant found in part (a), and m_0 is the amount of silicon-31 at the start time $t = 0$.

Hence the amount that will remain after 6 hours is

$$m_0 e^{-k\times 6} = m_0 e^{-6k}.$$

It follows that the proportion that will remain after 6 hours is

$$\frac{m_0 e^{-6k}}{m_0} = e^{-6k} = e^{-6\times 0.2645\ldots} = 0.2044\ldots.$$

So, to two significant figures, 20% of the original amount of silicon-31 remains after 6 hours.

Solution to Activity 21

The decay constant k corresponding to the half-life $T = 5570$ years is

$$k = \frac{\ln 2}{T} = \frac{\ln 2}{5570} = 1.2444\ldots \times 10^{-4} = 1.24 \times 10^{-4} \text{ year}^{-1} \text{ (to 3 s.f.)}.$$

Let us assume that the animal died at time $t = 0$. Then the amount m of carbon-14 that will remain after time t is given by

$$m = m_0 e^{-kt},$$

where k is the decay constant found above, and m_0 is the amount of carbon-14 at the time of the animal's death.

So the proportion of carbon-14 that will remain after time t is

$$\frac{m_0 e^{-kt}}{m_0} = e^{-kt}.$$

Hence the time t at which the proportion of carbon-14 remaining is 85%, that is, 0.85, is given by

$$e^{-kt} = 0.85.$$

Taking the natural logarithm of both sides gives

$$-kt = \ln 0.85;$$

that is,

$$t = -\frac{\ln 0.85}{k} = -\frac{\ln 0.85}{1.2444\ldots \times 10^{-4}} = 1305.971\ldots.$$

So the animal died approximately 1300 years ago.

Solution to Activity 22

(a) The differential equation is

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\lambda(T - T_\mathrm{a}).$$

Separating the variables gives

$$\int \frac{1}{T - T_\mathrm{a}}\,\mathrm{d}T = -\int \lambda\,\mathrm{d}t.$$

Doing the integrations, and using the fact that $T - T_\mathrm{a} > 0$, we obtain

$$\ln(T - T_\mathrm{a}) = -\lambda t + c,$$

where c is an arbitrary constant. This is the general solution in implicit form.

To obtain the general solution in explicit form, we can apply the exponential function to each side. This gives

$$e^{\ln(T-T_\mathrm{a})} = e^{-\lambda t + c}.$$

From this we obtain

$$T - T_\mathrm{a} = e^c e^{-\lambda t};$$

that is,

$$T = T_\mathrm{a} + e^c e^{-\lambda t}.$$

Since c is an arbitrary constant, e^c is also an arbitrary constant, and we can denote it by A. The value of A must be positive since $e^c > 0$ for each real number c.

Hence the general solution is

$$T = T_a + Ae^{-\lambda t},$$

where A is an arbitrary positive constant.

(b) The initial condition is $T = T_0$ when $t = 0$. Putting $t = 0$ and $T = T_0$ into the general solution gives

$$\begin{aligned} T_0 &= T_a + Ae^0 \\ &= T_a + A. \end{aligned}$$

Hence $A = T_0 - T_a$, and the required particular solution is

$$T = T_a + (T_0 - T_a)e^{-\lambda t}.$$

(c) If t is very large, then $e^{-\lambda t}$ is very small (remember that λ is positive), and we can approximate it by 0. Hence we have

$$\begin{aligned} T &\approx T_a + (T_0 - T_a) \times 0 \\ &= T_a. \end{aligned}$$

Hence in the long term the temperature of the body is equal to the ambient temperature of the surrounding medium.

(This agrees with our experience.)

Solution to Activity 23

(a) The solution to Newton's law of cooling is

$$T = T_a + (T_0 - T_a)e^{-\lambda t},$$

where

- T is the temperature of the cooling body at time t
- T_0 is the temperature of the body at time $t = 0$
- T_a is the ambient temperature
- λ is a positive constant.

We know that $T_a = 20$. Let $t = 0$ be the time at which the pathologist arrived. Then $T_0 = 30$. So the solution to Newton's law of cooling becomes

$$T = 20 + (30 - 20)e^{-\lambda t};$$

that is

$$T = 20 + 10e^{-\lambda t}.$$

We now determine the value of λ.

At time $t = 1$ (that is, one hour after the pathologist arrived), the temperature of the body was $T = 28.5$. So

$$28.5 = 20 + 10e^{-\lambda \times 1};$$

that is,

$$8.5 = 10e^{-\lambda}.$$

Hence

$$0.85 = e^{-\lambda}.$$

Taking the natural logarithm of both sides gives

$$\ln 0.85 = \ln(e^{-\lambda});$$

that is,

$$\ln 0.85 = -\lambda.$$

Hence we obtain

$$\lambda = -\ln 0.85.$$

So the equation for the temperature of the body as a function of time is

$$\begin{aligned} T &= 20 + 10e^{(\ln 0.85)t} \\ &= 20 + 10(e^{\ln 0.85})^t \\ &= 20 + 10 \times 0.85^t, \end{aligned}$$

as required.

(b) To find the time at which the person died, we need to find the value of t that gives $T = 37$.

This value of t is given by

$$37 = 20 + 10 \times 0.85^t.$$

Rearranging, we obtain

$$0.85^t = 1.7.$$

Taking the natural logarithm of both sides gives

$$t \ln 0.85 = \ln 1.7,$$

so

$$t = \frac{\ln 1.7}{\ln 0.85} = -3.265\ldots.$$

So the person died three and a quarter hours before the pathologist arrived, to the nearest quarter of an hour.

Solution to Activity 24

(a) The equation $\mathrm{d}y/\mathrm{d}x + x^3y = x^5$ is linear, with $g(x) = x^3$ and $h(x) = x^5$.

(b) The equation $\mathrm{d}y/\mathrm{d}x = x \sin x$ is linear, with $g(x) = 0$ and $h(x) = x \sin x$.

(c) The equation $\mathrm{d}z/\mathrm{d}t = -3z^{1/2}$ is not linear, because of the $z^{1/2}$ term.

(d) The equation $\mathrm{d}y/\mathrm{d}t + y^2 = t$ is not linear, because of the y^2 term.

(e) The equation $x\,(\mathrm{d}y/\mathrm{d}x) + y = y^2$ is not linear, because of the y^2 term.

(f) The equation $(1+x^2)(\mathrm{d}y/\mathrm{d}x) + 2xy = 3x^2$ is linear, since we can divide through by $1+x^2$ to obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{2x}{1+x^2}\right) y = \frac{3x^2}{1+x^2},$$

which is of the required form with

$$g(x) = \frac{2x}{1+x^2} \quad \text{and} \quad h(x) = \frac{3x^2}{1+x^2}.$$

Solution to Activity 25

(a) The differential equation has the same form as equation (20), with $A = 1$ and $h(x) = e^{2x}$. So the general solution is given by equation (22):

$$\begin{aligned} y &= e^{-x}\left(\int e^x e^{2x}\mathrm{d}x\right) \\ &= e^{-x}\left(\int e^{3x}\mathrm{d}x\right) \\ &= e^{-x}\left(\tfrac{1}{3}e^{3x} + c\right) \\ &= \tfrac{1}{3}e^{2x} + ce^{-x}, \end{aligned}$$

where c is an arbitrary constant.

(b) Differentiating the general solution we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{2}{3}e^{2x} - ce^{-x}.$$

Substituting this and the general solution into the left-hand side of the differential equation gives

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + y &= \left(\tfrac{2}{3}e^{2x} - ce^{-x}\right) + \left(\tfrac{1}{3}e^{2x} + ce^{-x}\right) \\ &= e^{2x}, \end{aligned}$$

which is equal to the right-hand side. So $y = \frac{1}{3}e^{2x} + ce^{-x}$ satisfies the differential equation.

Solution to Activity 26

(a) The given differential equation can be rearranged into the form of equation (23) as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x} - y = x,$$

so

$$g(x) = -1 \quad \text{and} \quad h(x) = x.$$

The integrating factor is

$$p(x) = \exp\left(\int (-1)\,\mathrm{d}x\right) = e^{-x}.$$

Hence the general solution, equation (25), is

$$y = e^x\left(\int xe^{-x}\,\mathrm{d}x\right).$$

Integration by parts gives

$$\begin{aligned} y &= e^x\left(-xe^{-x} + \int e^{-x}\,\mathrm{d}x\right) \\ &= e^x\left(-xe^{-x} - e^{-x} + c\right) \\ &= ce^x - x - 1, \end{aligned}$$

where c is an arbitrary constant.

(b) After division by x (where $x > 0$), the differential equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{3}{x}y = 1,$$

so

$$g(x) = -\frac{3}{x} \quad \text{and} \quad h(x) = 1.$$

The integrating factor is

$$\begin{aligned} p(x) &= \exp\left(\int\left(-\frac{3}{x}\right)\mathrm{d}x\right) \\ &= \exp(-3\ln x) \quad (\text{since } x > 0) \\ &= \exp(\ln(x^{-3})) \\ &= x^{-3}. \end{aligned}$$

Thus the general solution is

$$\begin{aligned} y &= x^3\left(\int x^{-3}\,\mathrm{d}x\right) \\ &= x^3\left(-\tfrac{1}{2}x^{-2} + c\right) \\ &= cx^3 - \tfrac{1}{2}x, \end{aligned}$$

where c is an arbitrary constant.

Solution to Activity 27

(a) The differential equation can be rearranged into the form of equation (23) as follows:

$$\frac{\mathrm{d}u}{\mathrm{d}x} - xu = 0,$$

so

$$g(x) = -x \quad \text{and} \quad h(x) = 0.$$

The integrating factor is

$$\begin{aligned} p(x) &= \exp\left(\int(-x)\,\mathrm{d}x\right) \\ &= \exp(-x^2/2) \\ &= e^{-x^2/2}. \end{aligned}$$

Hence the general solution is

$$u = e^{x^2/2}\left(\int 0\ \mathrm{d}x\right)$$
$$= ce^{x^2/2},$$

where c is an arbitrary constant.

Substituting the initial condition $u(0) = 2$ (that is, $u = 2$ when $x = 0$) gives

$$2 = ce^0,$$

so $c = 2$. Hence the solution of the initial value problem is

$$u = 2e^{x^2/2}.$$

(b) After division by t, the given equation can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{2}{t}y = t,$$

so it is of the form in equation (23) with

$$g(t) = \frac{2}{t} \quad \text{and} \quad h(t) = t.$$

The integrating factor is

$$p(t) = \exp\left(\int \frac{2}{t}\,\mathrm{d}t\right)$$
$$= \exp(2\ln t) \quad (\text{since } t > 0)$$
$$= \exp\left(\ln\left(t^2\right)\right)$$
$$= t^2.$$

Thus the general solution is

$$y = t^{-2}\left(\int t^3\,\mathrm{d}t\right)$$
$$= t^{-2}\left(\tfrac{1}{4}t^4 + c\right)$$
$$= \tfrac{1}{4}t^2 + ct^{-2},$$

where c is an arbitrary constant.

Substituting the initial condition $y(1) = 1$ (that is, $y = 1$ when $t = 1$) gives

$$1 = \tfrac{1}{4} + c,$$

so $c = \frac{3}{4}$. Hence the solution of the initial value problem is

$$y = \tfrac{1}{4}t^2 + \tfrac{3}{4}t^{-2}.$$

Solution to Activity 28

(a) The differential equation is

$$\frac{\mathrm{d}T}{\mathrm{d}t} + \lambda T = \lambda T_{\mathrm{a}0}\, e^{-kt}.$$

Comparing this with equation (23), we see that the differential equation is linear with

$$g(t) = \lambda \quad \text{and} \quad h(t) = \lambda T_{\mathrm{a}0}\, e^{-kt}.$$

Hence the integrating factor is

$$p(t) = \exp\left(\int \lambda\,\mathrm{d}t\right) = e^{\lambda t},$$

so the general solution is

$$T = e^{-\lambda t}\left(\int \lambda T_{\mathrm{a}0}\, e^{\lambda t} e^{-kt}\,\mathrm{d}t\right),$$
$$= e^{-\lambda t}\left(\lambda T_{\mathrm{a}0}\int e^{(\lambda-k)t}\,\mathrm{d}t\right),$$
$$= e^{-\lambda t}\left(\lambda T_{\mathrm{a}0}\,\frac{e^{(\lambda-k)t}}{\lambda-k} + A\right),$$
$$= \frac{\lambda T_{\mathrm{a}0}\, e^{-kt}}{\lambda-k} + A\, e^{-\lambda t},$$

where A is an arbitrary constant.

(b) When $k = 0$, the ambient temperature at time t is

$$T_{\mathrm{a}}(t) = T_{\mathrm{a}0}\, e^{-0\times t} = T_{\mathrm{a}0},$$

and so it is a constant. Substituting $k = 0$ into the general solution we obtain

$$T = \frac{\lambda T_{\mathrm{a}0}\, e^{-0\times t}}{\lambda - 0} + A\, e^{-\lambda t} = T_{\mathrm{a}0} + A\, e^{-\lambda t}.$$

But since the constant ambient temperature in Activity 22 was represented by T_{a}, we have

$$T_{\mathrm{a}0} = T_{\mathrm{a}},$$

so the solution in this case is

$$T = T_{\mathrm{a}} + A\, e^{-\lambda t},$$

as in Activity 22.

Solution to Activity 29

The differential equation was

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-1}{x}$$

and its general solution was calculated to be

$$y = 1 + cx.$$

Substituting the general solution into the right-hand side of the differential equation gives

$$\frac{y-1}{x} = \frac{1+cx-1}{x} = c.$$

Differentiating the general solution gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = c,$$

so the two sides of the differential equation are identical, as required.

Solution to Activity 30

(a) This differential equation requires the integrating factor method.

(b) This differential equation can be solved by direct integration, by separation of variables or by the integrating factor method.

(c) This differential equation can be solved by separation of variables or by the integrating factor method.

(d) This differential equation requires the integrating factor method.

(e) This differential equation can be rearranged in the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} - \left(\frac{2}{t} + 1\right) y = t,$$

so it can be solved by the integrating factor method.

(f) This differential equation can be rearranged in the form

$$\frac{\mathrm{d}u}{\mathrm{d}x} = (u^2 - 2)(x + 2),$$

so it can be solved by separation of variables.

Solution to Activity 31

(a) The differential equation $\mathrm{d}r/\mathrm{d}t = -k$ is directly integrable, so the general solution is

$$r = \int (-k)\,\mathrm{d}t$$
$$= -kt + c,$$

where c is an arbitrary constant.

(b) Since the initial radius of the snowball is 100 cm, the initial condition is $r = 100$ when $t = 0$. Substituting this into the general solution, we obtain $c = 100$. Hence the required particular solution is

$$r = 100 - kt.$$

(c) The initial volume of the snowball (in cm^3) is $\frac{4}{3}\pi \times 100^3$. If we denote the radius of the snowball after 2 days by R, then

$$\tfrac{4}{3}\pi R^3 = \tfrac{1}{2} \times \tfrac{4}{3}\pi \times 100^3,$$

from which we find that

$$R = (\tfrac{1}{2})^{1/3} \times 100 = 100 \times 2^{-1/3}.$$

Substituting the fact that $r = R = 100 \times 2^{-1/3}$ when $t = 2$ into the particular solution found in part (b), we obtain

$$100 \times 2^{-1/3} = 100 - 2k.$$

It follows that

$$k = \tfrac{1}{2}\left(100 - 100 \times 2^{-1/3}\right) = 50\left(1 - 2^{-1/3}\right).$$

(d) The snowball disappears when $r = 0$; that is, when $100 - kt = 0$. This occurs when

$$t = \frac{100}{k} = \frac{2}{1 - 2^{-1/3}} = 9.694\ldots.$$

It therefore takes 10 days, to the nearest day, for the snowball to disappear.

Solution to Activity 32

(a) The differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t\cos t$$

is directly integrable, so the solution is given by

$$x = \int t\cos t\,\mathrm{d}t.$$

Integration by parts gives

$$\int t\cos t\,\mathrm{d}t = t\sin t - \int \sin t\,\mathrm{d}t$$
$$= t\sin t + \cos t + c,$$

where c is an arbitrary constant. Hence the general solution is

$$x = t\sin t + \cos t + c.$$

(b) The initial condition is $x = 2$ when $t = 0$. Substituting these values into the general solution gives

$$2 = 0\sin 0 + \cos 0 + c = 1 + c,$$

so $c = 1$. Hence the particular solution is

$$x = t\sin t + \cos t + 1.$$

Solution to Activity 33

The differential equation is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = tx \quad (x > 0).$$

Separating the variables gives

$$\int \frac{1}{x}\,\mathrm{d}x = \int t\,\mathrm{d}t;$$

that is, since $x > 0$,

$$\ln x = \tfrac{1}{2}t^2 + c,$$

where c is an arbitrary constant. To make x the subject of this equation, recall that $\exp(\ln x) = x$. Hence, taking exponentials of both sides gives

$$\begin{aligned} x &= e^{(t^2/2)+c} \\ &= e^c e^{t^2/2} = Ae^{t^2/2}, \end{aligned}$$

where $A = e^c$. The constant A must be positive, since $e^c > 0$ for any value of c, but is otherwise arbitrary. The general solution for $x > 0$, in explicit form, is therefore

$$x = Ae^{t^2/2},$$

where A is an arbitrary positive constant.

The initial condition is $x = 2$ when $t = 0$. Putting $t = 0$ and $x = 2$ into the general solution gives

$$2 = Ae^0,$$

so $A = 2$. Hence the required particular solution is

$$x = 2e^{t^2/2}.$$

Solution to Activity 34

(a) The differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t}{\cos x + x}$$

is separable. Separating the variables gives

$$\int (\cos x + x)\,\mathrm{d}x = \int t\,\mathrm{d}t.$$

After integration, we obtain

$$\sin x + \tfrac{1}{2}x^2 = \tfrac{1}{2}t^2 + c,$$

where c is an arbitrary constant. This is the general solution in implicit form. It cannot be rearranged to give an explicit solution.

The initial condition is $x = 0$ when $t = 1$. Substituting $t = 1$ and $x = 0$ into the general solution gives

$$0 = \tfrac{1}{2} + c,$$

so $c = -\frac{1}{2}$.

Hence the required particular solution, in implicit form, is

$$\sin x + \tfrac{1}{2}x^2 = \tfrac{1}{2}t^2 - \tfrac{1}{2}.$$

(b) To verify the particular solution, we differentiate both sides with respect to t:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sin x + \tfrac{1}{2}x^2) = \frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}t^2 - \tfrac{1}{2}).$$

Using the chain rule, the left-hand side is

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x + \tfrac{1}{2}x^2)\,\frac{\mathrm{d}x}{\mathrm{d}t} = (\cos x + x)\frac{\mathrm{d}x}{\mathrm{d}t},$$

and the right-hand side is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tfrac{1}{2}t^2 - \tfrac{1}{2}) = t.$$

Thus

$$(\cos x + x)\frac{\mathrm{d}x}{\mathrm{d}t} = t.$$

This equation can be rearranged to give

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t}{\cos x + x} \quad (x \geq 0),$$

which is the original differential equation, as required.

Solution to Activity 35

After dividing both sides by e^{3t} and rearranging, the given differential equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}t} + y = e^{-3t},$$

which is a linear first-order differential equation with

$$g(t) = 1 \quad \text{and} \quad h(t) = e^{-3t}.$$

The integrating factor is

$$p(t) = \exp\left(\int 1\,\mathrm{d}t\right) = \exp(t) = e^t.$$

Hence the general solution is

$$y = e^{-t}\left(\int e^t e^{-3t}\,\mathrm{d}t\right) = e^{-t}\left(\int e^{-2t}\,\mathrm{d}t\right).$$

Integrating gives

$$\begin{aligned} y &= e^{-t}\left(-\tfrac{1}{2}e^{-2t} + c\right) \\ &= ce^{-t} - \tfrac{1}{2}e^{-3t}, \end{aligned}$$

where c is an arbitrary constant.

Substituting the initial condition $y(0) = 3$ (that is, $y = 3$ when $t = 0$) into the general solution gives

$$3 = ce^0 - \tfrac{1}{2}e^0 = c - \tfrac{1}{2},$$

so $c = \frac{7}{2}$. Hence the solution of the initial value problem is

$$y = \tfrac{1}{2}(7e^{-t} - e^{-3t}).$$

Solution to Activity 36

(a) First rearrange the differential equation to get it in the form of equation (23),

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{2x}{x^2+1}\right) y = x.$$

This is a linear first-order differential equation with

$$g(x) = \frac{2x}{x^2+1} \quad \text{and} \quad h(x) = x.$$

The integrating factor is therefore

$$p(x) = \exp\left(\int \frac{2x}{x^2+1}\,\mathrm{d}x\right).$$

Notice that $2x/(x^2+1)$ is of the form $f'(x)/f(x)$, so we can use the standard integral

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln|f(x)| + c$$

to give

$$p(x) = \exp(\ln|x^2+1|) = \exp(\ln(x^2+1)) = x^2+1.$$

Here we used the fact that $x^2+1 > 0$ for all x.

So the general solution of the differential equation is

$$y = \frac{1}{x^2+1}\left(\int x(x^2+1)\,\mathrm{d}x\right).$$

Integrating gives

$$y = \frac{1}{x^2+1}\left(\int (x^3+x)\,\mathrm{d}x\right)$$
$$= \frac{1}{x^2+1}\left(\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2 + c\right),$$

where c is an arbitrary constant. This can be simplified to give the general solution

$$y = \frac{x^4+2x^2+4c}{4(x^2+1)},$$

where c is an arbitrary constant.

(b) To apply the initial condition $y(0) = \frac{1}{4}$, we substitute $y = \frac{1}{4}$ and $x = 0$ into the general solution. This gives

$$\frac{1}{4} = \frac{4c}{4},$$

so $c = \frac{1}{4}$. Hence the particular solution is

$$y = \frac{x^4+2x^2+1}{4(x^2+1)},$$

which can be further simplified as follows:

$$y = \frac{(x^2+1)^2}{4(x^2+1)} = \tfrac{1}{4}(x^2+1).$$