2 Representing symmetries

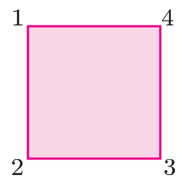
After working through this section, you should be able to:

(a) use a *two-line symbol* to represent a symmetry;
(b) describe geometrically the symmetry of a given figure which corresponds to a given two-line symbol;
(c) find the composite of two symmetries given as two-line symbols;
(d) find the inverse of a symmetry given as a two-line symbol;
(e) write down a *Cayley table* for the set of symmetries of a plane figure;
(f) appreciate how certain properties of the set of symmetries of a figure feature in a Cayley table.

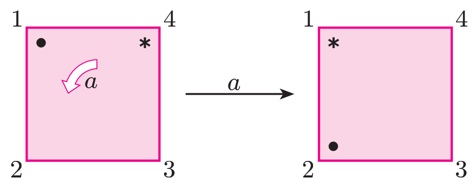
2.1 Two-line symbol

So far we have represented symmetries of plane figures by letters, and used diagrams or models to work out composites. This method is illuminating but time consuming. In this section we introduce an algebraic notation which permits us to manipulate and compose symmetries easily, but at the expense of geometric intuition.

We illustrate the notation in terms of the symmetries of a square. In the diagram we have labelled the locations of the vertices of the square with the numbers 1, 2, 3 and 4. In our discussion we consider these numbers as being fixed to the background plane. So the number 1 is always at the top left-hand corner of the square in this example. It does not label the vertex of the square, and so does not move when we apply a symmetry to the square. We use these numbers to record the effect of a symmetry as follows.



We first consider the effect of the symmetry a (rotation through $\pi/2$ about the centre).



The symmetry a maps the vertices as follows.

	shorthand
vertex at location 1 to location 2	$1 \longmapsto 2$
vertex at location 2 to location 3	$2 \longmapsto 3$
vertex at location 3 to location 4	$3 \longmapsto 4$
vertex at location 4 to location 1	$4 \longmapsto 1$

We can think of a as a function mapping the set $\{1, 2, 3, 4\}$ to itself.

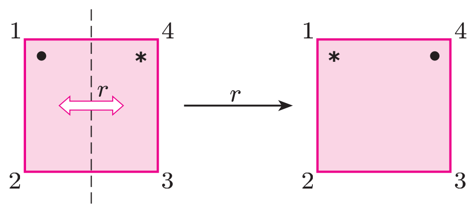
$$a : \begin{cases} 1 \longmapsto 2 \\ 2 \longmapsto 3 \\ 3 \longmapsto 4 \\ 4 \longmapsto 1 \end{cases} \quad \text{or} \quad a : \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 4 & 1 \end{array}$$

Strictly, a symmetry does not act on the numbers 1, 2, 3, 4. We use 1, 2, 3, 4 here as shorthand for 'the vertex of the square at location 1', etc.

We model our new notation for a on the second version, omitting the arrows and enclosing the numbers in parentheses: thus we write

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

The symmetry r has the following effect.



Thus the symmetry r

interchanges the vertices at locations 1 and 4,

interchanges the vertices at locations 2 and 3.

So, in our new notation, we write

$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

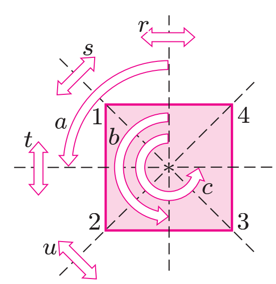
The identity symmetry e leaves all the vertices at their original locations, so we write

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

We refer to this notation for recording a symmetry of a plane figure as the *two-line symbol* for the symmetry. To specify a symmetry in this form, we must first give a picture of the figure with labelled locations.

Example 2.1 For the square with vertices at locations 1, 2, 3 and 4, labelled as shown, describe geometrically the symmetry represented by the two-line symbol

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$



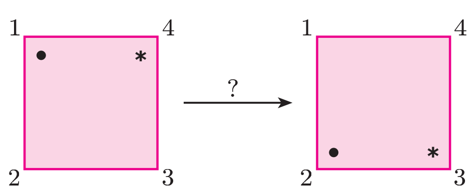
$S(\square)$

Solution This two-line symbol represents a symmetry that

interchanges the vertices at locations 1 and 2,

interchanges the vertices at locations 3 and 4.

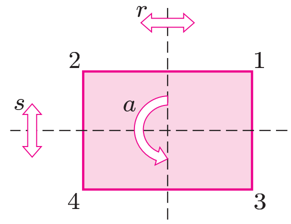
Thus the symmetry has the following effect.



So the two-line symbol represents reflection in the horizontal axis—the symmetry t. ■

Exercise 2.1 Find the two-line symbols representing the remaining symmetries of the square, namely b, c, s and u, using the labelling of locations given in Example 2.1.

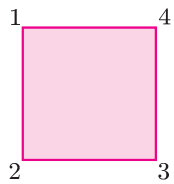
Exercise 2.2 Find the two-line symbols representing each of the four symmetries of the labelled rectangle in the margin. (Do not forget e.)



$S(\square)$

There is no universal way of labelling the locations of a plane figure, so the two-line symbols depend on the choice of labels. For example, for the square and rectangle above, reflection in the vertical axis is represented by a different two-line symbol in each case because we have used different systems for labelling the locations of the vertices (anticlockwise around the square, but across the top and bottom of the rectangle).

Usually, we try to maintain an anticlockwise labelling of the locations of the vertices, starting at the top left.



Now we define formally the two-line symbol representing a symmetry of a polygonal figure.

Definition Let f be a symmetry of a polygonal figure F which moves the vertices of the figure F originally at the locations labelled $1, 2, 3, \ldots, n$ to the locations labelled $f(1), f(2), f(3), \ldots, f(n)$, respectively.

The **two-line symbol** representing f is

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ f(1) & f(2) & f(3) & \ldots & f(n) \end{pmatrix}.$$

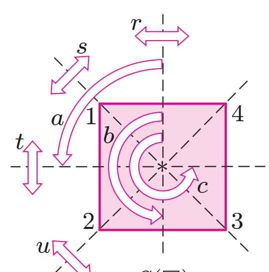
We cannot choose $f(1)$, $f(2)$, … arbitrarily here, as f must be a symmetry of F.

Remarks

1. The order of the columns in the symbol is not important, although we often use the natural order to aid recognition. For example, using the location labels as shown, we would normally write the two-line symbols for the eight symmetries of the square as follows.

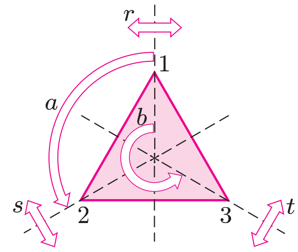
In natural order we would write $\begin{pmatrix} 2 & 4 & 3 & 1 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ as $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$.

rotations	reflections
$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$	$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$
$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$	$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$
$b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$	$t = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$
$c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$	$u = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$



$S(\square)$

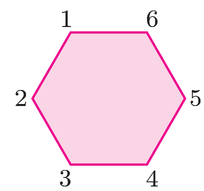
2. Not all two-line symbols represent symmetries of a particular figure. For example, with our choice of labelling, $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$ is not a symmetry of the square because there is no symmetry which interchanges the vertices at locations 2 and 3, and leaves fixed the vertices at locations 1 and 4.



$S(\triangle)$

Exercise 2.3 Using the given labelling for the locations of the vertices, write down the two-line symbol for each of the symmetries of the equilateral triangle.

Exercise 2.4 The following two-line symbols represent symmetries of the hexagon shown. Describe each symmetry geometrically.



(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$

2.2 Composing symmetries

One advantage of the two-line notation for symmetries is that it is easy to form composites. For example, let us form the composite $r \circ a$ of symmetries of the square. We have

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$

which are shorthand for the functions

$$a : \begin{cases} 1 \longmapsto 2 \\ 2 \longmapsto 3 \\ 3 \longmapsto 4 \\ 4 \longmapsto 1 \end{cases} \quad \text{and} \quad r : \begin{cases} 1 \longmapsto 4 \\ 2 \longmapsto 3 \\ 3 \longmapsto 2 \\ 4 \longmapsto 1 \end{cases}.$$

So the composite $r \circ a$ is given by

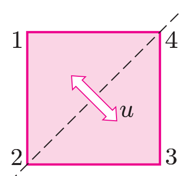
Remember that $r \circ a$ means a first, then r.

$$\begin{array}{ccccc} 1 & \longmapsto & 2 & \longmapsto & 3 \\ 2 & \longmapsto & 3 & \longmapsto & 2 \\ 3 & \longmapsto & 4 & \longmapsto & 1 \\ 4 & \longmapsto & 1 & \longmapsto & 4 \\ & a & & r & \end{array}; \text{ that is, } \begin{array}{ccc} 1 & \longmapsto & 3 \\ 2 & \longmapsto & 2 \\ 3 & \longmapsto & 1 \\ 4 & \longmapsto & 4 \\ & r \circ a & \end{array}.$$

Thus the two-line symbol for $r \circ a$ is

$$r \circ a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix};$$

this is reflection in the diagonal shown—the symmetry u.



Composing the functions r and a as we did above suggests a method of combining two-line symbols for symmetries without drawing diagrams.

To determine $r \circ a$, the composite of two symmetries r and a written as two-line symbols, we reorder the columns of the symbol for r to make its top line match the order of the bottom line of the symbol for a. We then read off the two-line symbol for the composite $r \circ a$ as the top line of the symbol for a and the bottom line of the symbol for r.

For example, in $S(\square)$,

$$\begin{aligned} r \circ a &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 1 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} = u. \end{aligned}$$

After some practice, you may find that you can compose two-line symbols without reordering the columns.

For example, to find $r \circ a$ in $S(\square)$, we first write down

$$r \circ a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}.$$

Then we find the entries of the bottom row in turn.

Now a sends 1 to 2 and r sends 2 to 3, so the composite sends 1 to 3:

$$r \circ a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & & & \end{pmatrix}.$$

Also a sends 2 to 3 and r sends 3 to 2, so the composite sends 2 to 2:

$$r \circ a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & & \end{pmatrix}.$$

The final two entries are found in the same way: a sends 3 to 4 and r sends 4 to 1, so the composite sends 3 to 1; a sends 4 to 1 and r sends 1 to 4, so the composite sends 4 to 4:

$$r \circ a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

As we have seen already, the order of composition is important. For example,

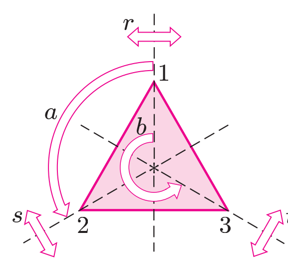
$$a \circ r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = s,$$

so

$$r \circ a \neq a \circ r.$$

Exercise 2.5 Using the two-line symbols for the symmetries of the equilateral triangle found in Exercise 2.3, find the following composites:

$$a \circ a, \quad b \circ s, \quad s \circ b, \quad t \circ s.$$



$S(\triangle)$

2.3 Finding the inverse of a symmetry

We saw in Section 1 that every symmetry has an inverse which 'undoes' the effect of the symmetry. There is an easy way to write down the inverse of a symmetry given by a two-line symbol. Let us consider again the symmetry a of the square:

Property 1.3

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

This is shorthand for the function

$$a : \begin{cases} 1 \longmapsto 2 \\ 2 \longmapsto 3 \\ 3 \longmapsto 4 \\ 4 \longmapsto 1 \end{cases} \quad \text{with inverse} \quad a^{-1} : \begin{cases} 2 \longmapsto 1 \\ 3 \longmapsto 2 \\ 4 \longmapsto 3 \\ 1 \longmapsto 4 \end{cases},$$

The inverse of a function is obtained by reversing the arrows of the function.

so

$$a^{-1} = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = c.$$

Reversing the arrows in the mapping a is equivalent to reading the two-line symbol for a from the bottom to the top. In other words, to find the inverse of a, we turn the two-line symbol upside down. Reordering the columns in the symbol into the natural order is optional but it may make the inverse easier to recognise.

For example,

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad \text{so} \quad a^{-1} = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \text{(turn the symbol upside down).}$$

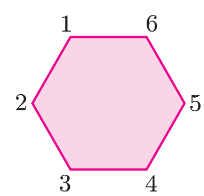
$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \quad \text{(reorder the columns).}$$

Exercise 2.6 Find the inverse of each of the following symmetries of a regular hexagon, given as a two-line symbol.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$

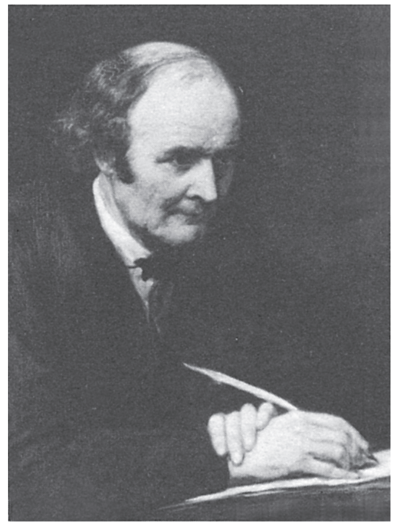


2.4 Cayley tables

Finally in this section, we consider a way of recording composites of symmetries. This is by means of a **Cayley table**. To form the Cayley table for the elements of a set $S(F)$ of symmetries, we list the elements of $S(F)$ across the top and down the left-hand side of a square array.

	e	f	g	$\cdots$	x	y	z
e							
f							
g							
$\vdots$							
x							
y							
z							

The order in which we choose to list the elements is not important, but it is important to use the same ordering across the top and down the side. Normally we put the identity symmetry e first, as shown above. This square array enables us to list every possible composite of pairs of elements in $S(F)$. However, this is practicable only if $S(F)$ is a small set, and is not feasible for $S(\bigcirc)$, which is infinite!



Arthur Cayley (1821–1895) was the leading British algebraist of the nineteenth century. He helped to lay the groundwork for the abstract theory of groups and he developed the algebra of matrices and determinants.

For any two elements x and y of $S(F)$, the composite $x \circ y$ is recorded in the cell in the row labelled x and the column labelled y.

	$\cdots$	y	$\cdots$
$\vdots$		$\vdots$	
x	$\cdots$	$x \circ y$	$\cdots$
$\vdots$		$\vdots$	

Note that x is on the left both in the composite and in the border of the table. Of course, the composite $x \circ y$ is the result of performing first the symmetry y and then the symmetry x.

We have found many of the composites of elements of $S(\square)$ already; for example, $a \circ t = u$, $t \circ a = s$ and $r \circ a = u$. The complete Cayley table for $S(\square)$ is as follows.

$\circ$	e	a	b	c	r	s	t	u
e	e	a	b	c	r	s	t	u
a	a	b	c	e	s	t	u	r
b	b	c	e	a	t	u	r	s
c	c	e	a	b	u	r	s	t
r	r	u	t	s	e	c	b	a
s	s	r	u	t	a	e	c	b
t	t	s	r	u	b	a	e	c
u	u	t	s	r	c	b	a	e

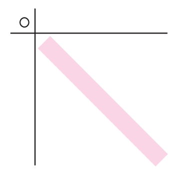
The Cayley table illustrates a number of properties of $S(\square)$ to which we shall refer later.

Closure No new elements are needed to complete the table because every composite is one of the eight symmetries.

Identity The row and column labelled by the identity e repeat the borders of the table.

Inverses The identity e occurs when an element is composed with its inverse, so e appears once in each row and once in each column. Also, e appears symmetrically in the table.

Self-inverse elements When a self-inverse element is composed with itself, the identity e appears on the leading diagonal of the table. Conversely, when e appears on the leading diagonal, the corresponding element is self-inverse.



The *leading diagonal* or *main diagonal* is the diagonal from top left to bottom right.

Direct and indirect symmetries We have chosen to list the direct symmetries, e, a, b and c, first, followed by the indirect symmetries, r, s, t and u. This leads to a 'blocking' of the Cayley table, as illustrated below.

$\circ$	e	a	b	c	r	s	t	u
e	e	a	b	c	r	s	t	u
a	a	b	c	e	s	t	u	r
b	b	c	e	a	t	u	r	s
c	c	e	a	b	u	r	s	t
r	r	u	t	s	e	c	b	a
s	s	r	u	t	a	e	c	b
t	t	s	r	u	b	a	e	c
u	u	t	s	r	c	b	a	e

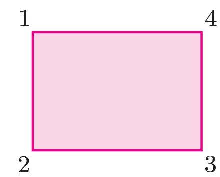
$\circ$	direct	indirect
direct	direct	indirect
indirect	indirect	direct

We shall see an example of 'blocking' in another Cayley table in Section 5.

Exercise 2.7 Using the two-line symbols from Exercise 2.3 to work out the composites, construct the Cayley table for the symmetries of an equilateral triangle.

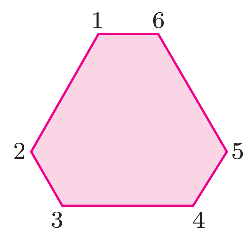
You found some of these composites in Exercise 2.5, and also in Exercise 1.4.

Exercise 2.8 Using the labelling of the rectangle shown, construct the Cayley table for $S(\square)$.

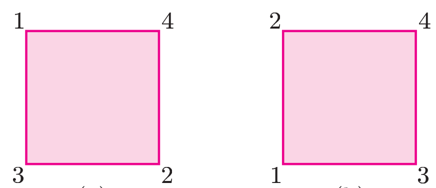


Further exercises

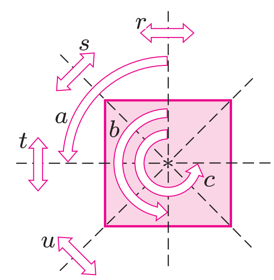
Exercise 2.9 Describe geometrically the symmetries of the (non-regular) hexagon shown in the margin. (The sides joining 1 to 6, 2 to 3 and 4 to 5 all have the same length, as do the sides joining 1 to 2, 3 to 4 and 5 to 6.) Write down the two-line symbol for each symmetry.



Exercise 2.10 Write down the two-line symbol for each of the eight symmetries of a square for each of the following labellings.

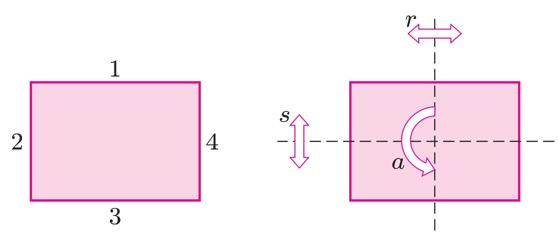


(a) (b)



$S(\square)$

Exercise 2.11 In the following figure, the labels 1, 2, 3 and 4 refer to the locations of the four edges of the figure (instead of the vertices).



The position of the rectangle may be specified by the locations of the four edges instead of the four vertices. So our definition of two-line symbol still makes sense if we replace vertices by edges.

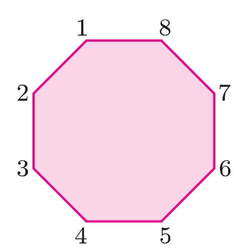
For this labelling of the rectangle, write down the two-line symbol for each symmetry of the rectangle.

Exercise 2.12 For the labelling of the regular octagon shown in the margin, interpret geometrically each of the following two-line symbols.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 2 & 1 & 8 & 7 & 6 \end{pmatrix}$



Exercise 2.13 Let

$$x = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

Find the two-line symbol for each of the following composites:

$$x \circ x, \quad y \circ y, \quad x \circ y, \quad y \circ x.$$

Hence show that

$$(x \circ x) \circ x = e, \quad (x \circ x) \circ y = y \circ x \quad \text{and} \quad y \circ (x \circ x) = x \circ y.$$