5 Symmetry in $\mathbb{R}^3$

After working through this section, you should be able to:

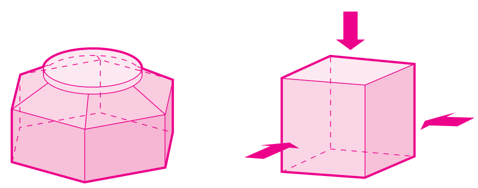
(a) describe the symmetries of some bounded three-dimensional figures;
(b) use two-line symbols to denote symmetries of three-dimensional figures, and to form composites and inverses of such symmetries;
(c) count the number of symmetries of certain polyhedra;
(d) understand why there are exactly five regular polyhedra.

5.1 Pre-programme work: symmetries of a figure in $\mathbb{R}^3$

We use the symbol $\mathbb{R}^3$ to denote three-dimensional space, in which a point is specified by three coordinates x, y, z.

Having considered symmetries of two-dimensional figures, we now extend our ideas to three-dimensional objects.

Some three-dimensional objects have symmetries that are essentially the same as those of a corresponding two-dimensional figure. For example, the six-sided bottle shown below has essentially the same symmetries as a regular hexagon. However, the symmetries of many three-dimensional objects cannot be thought of as essentially the same as the symmetries of any two-dimensional figure. A cube, for instance, has the symmetries of a square when looked at from each of three directions, and (as we shall see) other symmetries as well.



hexagonal bottle cube

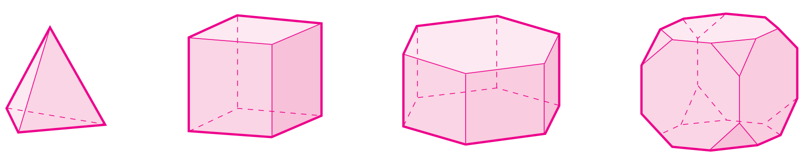
In this section we adapt the definitions of earlier sections to $\mathbb{R}^3$, and consider symmetries of three-dimensional objects.

Definition A **figure** in $\mathbb{R}^3$ is any subset of $\mathbb{R}^3$.

This definition is very general, and includes plane figures as special cases. We shall mainly consider bounded non-planar figures with polygonal faces. Such solids are called *polyhedra*.

A *bounded* figure in $\mathbb{R}^3$ is a figure that can be surrounded by a sphere (of finite radius).

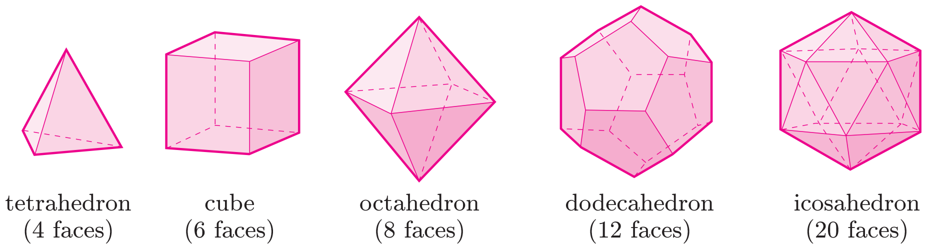
Polyhedra is the Greek for 'many faces'. The singular form is *polyhedron*.



tetrahedron cube hexagonal prism truncated cube

In our examples we restrict attention to *convex polyhedra*: that is, polyhedra without dents or dimples or spikes. Of these, there are five that are of particular interest: the *regular polyhedra* (*Platonic solids*), in which all faces are congruent regular polygons and each vertex is the junction of the same numbers of edges and faces, arranged in the same way, as shown in the following diagram.

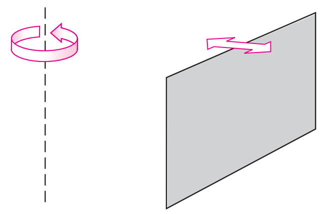
We shall show why there are only five regular polyhedra in Subsection 5.4.



Our initial definitions are almost exactly the same as those for $\mathbb{R}^2$.

> **Definitions** An **isometry** of $\mathbb{R}^3$ is a distance-preserving map $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.
>
> A **symmetry** of a figure F in $\mathbb{R}^3$ is an isometry mapping F onto itself—that is, an isometry $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $f(F) = F$.
>
> Two symmetries of a figure F are **equal** if they have the same effect on F.

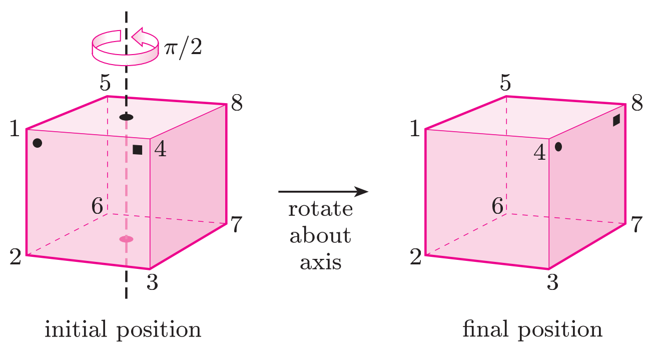
As for $\mathbb{R}^2$, our potential symmetries are rotations (this time about an axis of the figure in $\mathbb{R}^3$), reflections (in a plane), translations and combinations of these isometries. For a bounded figure in $\mathbb{R}^3$ (such as a polyhedron), translations alter the location of the figure and so cannot be symmetries; hence we concentrate on rotations and reflections.



We have to be careful with rotations in $\mathbb{R}^3$, as what is clockwise when looking along an axis of rotation in one direction is anticlockwise when looking along it in the other direction. We often indicate the direction of rotation by an arrow on a diagram.

A **rotation** of F is a symmetry specified by an *axis of symmetry*, a *direction* of rotation and the *angle* through which the figure is rotated.

For example, rotation of the cube through $\pi/2$ about its vertical axis, in the direction indicated, has the following effect.



Rotation arrows in $\mathbb{R}^3$ indicate the direction of rotation; they do not indicate the size of the angle through which the figure is rotated.

We shall adopt and extend our two-line symbols to record symmetries of three-dimensional objects. With the labelling shown above, this rotation is represented by

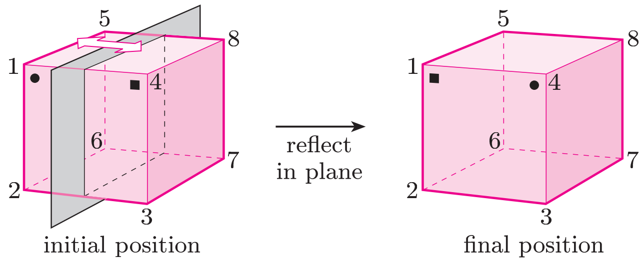
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 7 & 8 & 1 & 2 & 6 & 5 \end{pmatrix}.$$

The identity symmetry

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

can be thought of as a zero rotation.

A **reflection** of F is a symmetry specified by the *plane* in which the reflection takes place.



The two-line symbol for the reflection shown above is

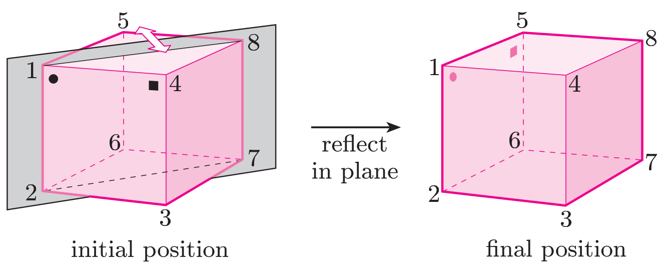
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \end{pmatrix}.$$

Composition of symmetries written as two-line symbols is similar to that for plane figures. For example, the rotation above followed by the reflection above is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 7 & 8 & 1 & 2 & 6 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 6 & 5 & 4 & 3 & 7 & 8 \end{pmatrix}.$$

Remember to work from right to left.

This is a reflection in the diagonal plane passing through the locations labelled 1, 2, 7 and 8, as shown below.

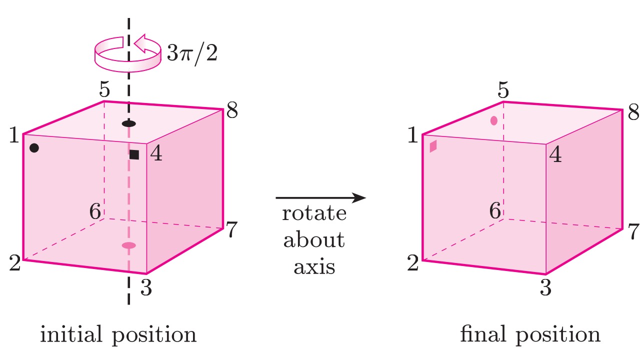


To find the inverse of a symmetry given as a two-line symbol, we turn the symbol upside down. For the rotation above, we have

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 7 & 8 & 1 & 2 & 6 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 3 & 7 & 8 & 1 & 2 & 6 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 2 & 1 & 8 & 7 & 3 & 4 \end{pmatrix}.$$

It is not necessary to reorder the columns into the natural order, but it may help with identifying the symmetry.

This is rotation of the cube through $3\pi/2$ about its vertical axis, in the direction indicated.



Symmetries that we can demonstrate physically with a model (for polyhedra, this means rotations) are called **direct** symmetries, whereas those that we cannot show physically with the model are called **indirect** symmetries. As for plane figures, composition of direct and indirect symmetries follows a standard pattern.

We can make a second model to represent the reflected polyhedron. The images of the polyhedron under an indirect symmetry can then be illustrated by rotating this second model.

direct ∘ direct = direct
direct ∘ indirect = indirect
indirect ∘ direct = indirect
indirect ∘ indirect = direct

∘	direct	indirect
direct	direct	indirect
indirect	indirect	direct

We use $S(F)$ to denote the set of symmetries of a figure F, and $S^+(F)$ to denote the set of direct symmetries of F. The following results apply to figures in $\mathbb{R}^3$, and also to figures in $\mathbb{R}^2$.

Property 5.1

1. The set $S(F)$ of all symmetries of a figure F forms a group under composition.
2. The set $S^+(F)$ of all direct symmetries of a figure F forms a group under composition.
3. If the group $S(F)$ contains both direct and indirect symmetries and there are exactly n direct symmetries, then there are exactly n indirect symmetries. The n indirect symmetries may be obtained by composing each of the n direct symmetries with any one fixed indirect symmetry.

Watch the video programme 'Symmetry counts'.

Video

5.2 Review of the video programme

We begin by looking at various symmetrical objects which we meet in familiar situations.

Turning to mathematical objects, we pose the question: 'How can we count the number of symmetries of each of the regular polyhedra—the tetrahedron, cube, octahedron, dodecahedron and icosahedron?'

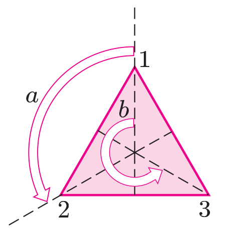
Much of the programme is devoted to answering this question for the tetrahedron.

Symmetries of the equilateral triangle

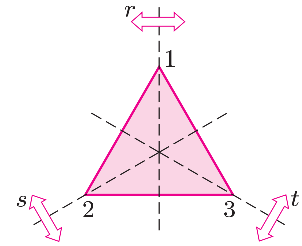
As a first step towards answering the above question, we review our approach to describing and counting symmetries of figures in $\mathbb{R}^2$; in particular, we study the symmetries of the equilateral triangle. This figure has both direct and indirect symmetries.

The direct symmetries are the identity and the two non-trivial rotations, and these form a group. Since there are three direct symmetries, there must be three indirect symmetries. We can find the three indirect symmetries by composing each of the direct symmetries with any one of the indirect symmetries. (See the following diagram.)

See the solution to Exercise 1.9(a).



direct symmetries
$e,\ a,\ b$



indirect symmetries
$r = a \circ s,\ s = e \circ s,\ t = b \circ s$

Thus we obtain the following group table.

$\circ$	e	a	b	r	s	t
e	e	a	b	r	s	t
a	a	b	e	t	r	s
b	b	e	a	s	t	r
r	r	s	t	e	a	b
s	s	t	r	b	e	a
t	t	r	s	a	b	e

$\circ$	direct	indirect
direct	direct	indirect
indirect	indirect	direct

The table for the group of direct symmetries appears as a 'subtable'.

This is another example of 'blocking' in a group table.

We can represent these symmetries as the two-line symbols:

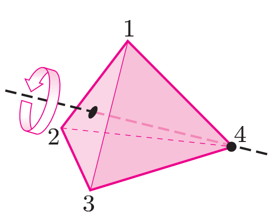
$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$r = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

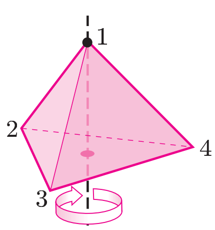
See the solution to Exercise 2.3.

Symmetries of the regular tetrahedron

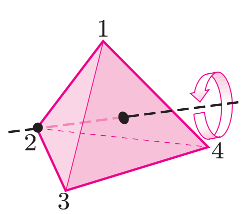
Next we consider the symmetries of the regular tetrahedron. The identity symmetry e leaves the tetrahedron as it is and is a direct symmetry. Also, for each of the four vertices, there are two non-trivial direct symmetries—namely rotations through angles $2\pi/3$ and $4\pi/3$ about an axis through the vertex from the centre of the opposite face. Thus these correspond to anticlockwise rotations of the triangular face opposite the fixed vertex, when viewed from outside the tetrahedron, looking directly at the face.



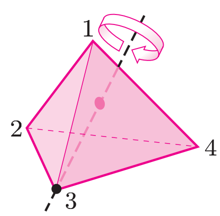
vertex 4 fixed



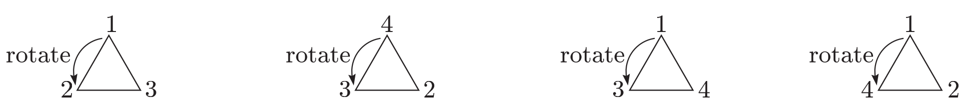
vertex 1 fixed



vertex 2 fixed



vertex 3 fixed



	vertex 4 fixed	vertex 1 fixed	vertex 2 fixed	vertex 3 fixed
$2\pi/3$:	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$
$4\pi/3$:	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$

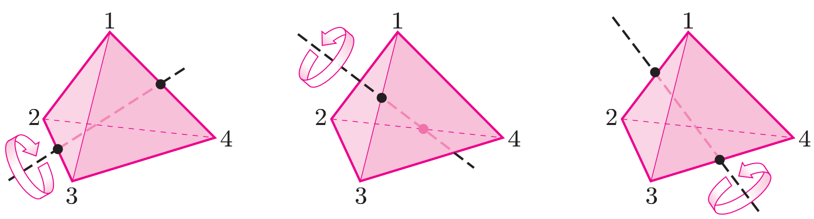
Now we have nine direct symmetries, but are there any more? We know that the set of direct symmetries of the tetrahedron is a group, so it must be closed under composition. So we test for closure by composing some of the symmetries given above.

The symmetry obtained by performing first the rotation through $2\pi/3$ which fixes the vertex at location 1, and then the rotation through $4\pi/3$ which fixes the vertex at location 2, is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$

which is not one of the symmetries already listed. It is a new direct symmetry, which simultaneously interchanges the vertices at locations 1 and 4 and the vertices at locations 2 and 3. Geometrically, it corresponds to a rotation through π about an axis through the midpoints of the opposite edges joining 1 to 4 and 2 to 3.

A similar rotation of the tetrahedron exists for each of the other two pairs of opposite edges. Thus we have found three further direct symmetries.



$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

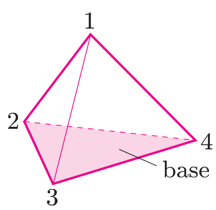
We have now found the following 12 direct symmetries of the tetrahedron.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

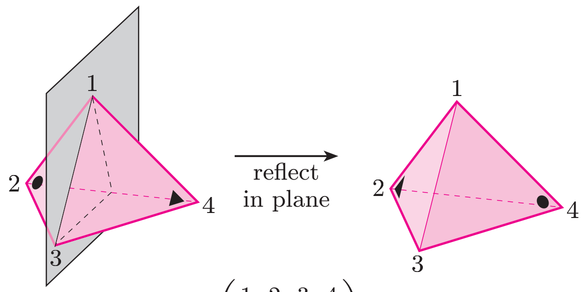
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

We can see that there are only 12 direct symmetries as follows. Imagine picking up the tetrahedron shown below, and placing it down again to occupy the same space it occupied originally, but possibly with the vertices at new locations. We count the number of ways of doing this. We can choose any of the 4 faces as the ‘base’ of the tetrahedron, and then there are 3 ways of placing the tetrahedron on this base, corresponding to the 3 rotational symmetries of the base triangle. Thus altogether there are $4 \times 3 = 12$ ways of placing the tetrahedron—that is, 12 direct symmetries.



There are also indirect symmetries of the tetrahedron—for example, a reflection in the vertical plane through the edge joining the vertices at locations 1 and 3 and the midpoint of the edge joining the vertices at locations 2 and 4.



$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

It follows that the tetrahedron has 12 indirect symmetries, which can be obtained by composing each of the 12 direct symmetries with the above indirect symmetry on the right.

We perform first the indirect symmetry and then the direct symmetry. This is equivalent to applying the twelve direct symmetries to the 'reflected' tetrahedron.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

Thus the tetrahedron has 24 symmetries.

Another way of obtaining the number 24 is as follows. We count the number of ways of replacing the tetrahedron, or the reflected tetrahedron, in the space occupied originally by the tetrahedron, but possibly with the vertices at new locations. We can choose any of the 4 faces to be placed as the base. Each of the 6 symmetries of such a face gives a symmetry of the tetrahedron, so there are 6 ways of replacing the tetrahedron, or the reflected tetrahedron, on this base. Thus there are $4 \times 6 = 24$ symmetries of the tetrahedron.

Symmetries of the regular polyhedra

For any regular polyhedron, each symmetry of a face gives a symmetry of the polyhedron. Therefore an argument similar to that for the tetrahedron shows that the total number of symmetries of a regular polyhedron is the number of faces multiplied by the number of symmetries of each face. Thus we have the following strategy.

Strategy 5.1 To determine the number of symmetries of a regular polyhedron.

1. Count the number of faces.
2. Count the number of symmetries of a face.

Then,

$$\begin{pmatrix} \text{number of} \\ \text{symmetries of} \\ \text{regular polyhedron} \end{pmatrix} = \begin{pmatrix} \text{number of} \\ \text{faces} \end{pmatrix} \times \begin{pmatrix} \text{number of} \\ \text{symmetries of face} \end{pmatrix}.$$

It does not matter whether we do step 1 or step 2 first. In the video, step 2 is done first.

Post-video exercise

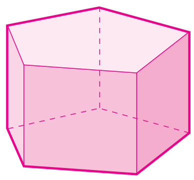
Exercise 5.1 Using Strategy 5.1, show that the cube and the octahedron have 48 symmetries each, and that the dodecahedron and the icosahedron have 120 symmetries each.

5.3 Symmetries of non-regular polyhedra

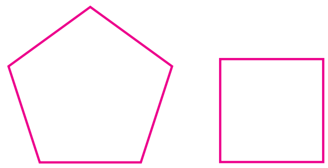
Strategy 5.1 for finding the number of symmetries of a regular polyhedron—multiply the number of faces by the number of symmetries of a face—can be adapted to find the number of symmetries of a non-regular polyhedron. We illustrate the method by two examples.

Pentagonal prism

We consider the pentagonal prism, in which the top and bottom faces are regular pentagons and the vertical faces are squares.



pentagonal prism



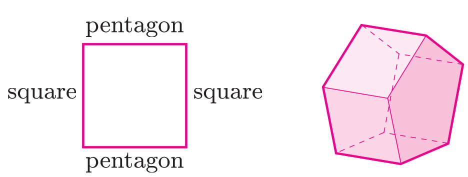
faces

This prism has direct symmetries—for example, we can rotate the prism about a vertical axis. It also has indirect symmetries—for example, we can reflect the prism in the plane that contains a vertical edge and bisects the square face opposite this edge.

To find the number of symmetries of the prism, we can count the number of ways of replacing the prism, or the reflected prism, in the space it occupied originally, but possibly with the vertices at new locations.

In the above figure, the prism is shown with a pentagonal face as its base, so we can choose either of the 2 pentagonal faces to be the base. Each of the 10 symmetries of such a face gives a symmetry of the prism, so there are 10 ways of replacing the prism, or the reflected prism, on this base. Thus there are $2 \times 10 = 20$ symmetries of the prism.

We carried out this calculation by considering one of the pentagonal faces as the base. We can check our answer by considering one of the square faces to be the base, as shown below.



Again we count the number of ways of replacing the prism, or the reflected prism, in the space it originally occupied. We can choose any of the 5 square faces to be the base.

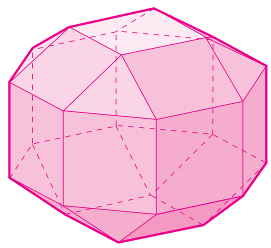
We now have to be careful because only some of the 8 symmetries of the square give symmetries of the prism. For example, one symmetry of the square base is a rotation of $\pi/2$ about its centre, but if we apply the corresponding transformation to the prism as a whole—that is, if we rotate the prism through $\pi/2$ about the vertical axis through the centre of the square base—then the prism does not occupy its original space in $\mathbb{R}^3$, so this is not a symmetry of the prism. Similarly, reflections through the diagonals of the square base do not give symmetries of the prism.

In fact, only 4 of the symmetries of the square base are also symmetries of the prism, namely the identity, rotation through π and reflections in the lines joining midpoints of opposite edges. Thus the number of symmetries of the prism is $5 \times 4 = 20$. This confirms our earlier answer.

Small rhombicuboctahedron

As a second example, we consider the polyhedron shown below. It is called a small rhombicuboctahedron, and it has 18 square faces and 8 faces that are equilateral triangles.

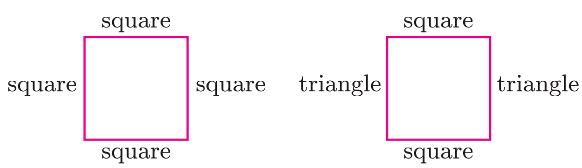
It is not to be confused with the great rhombicuboctahedron, which has 12 square faces, 8 hexagonal faces and 6 octagonal faces.



small rhombicuboctahedron

To find the number of symmetries of the polyhedron, we can count the number of ways of replacing the polyhedron, or the reflected polyhedron, in the space it occupied originally, as shown above with a square face as its base, but possibly with the vertices at new locations.

We immediately come across a new complication—only some of the square faces of the polyhedron can be placed as the base if the polyhedron, or its reflection, is to occupy its original space in $\mathbb{R}^3$. This is because there are two types of square face: in one type, all four edges of the face are joined to other square faces, whereas in the other type, two edges are joined to square faces and two to triangular faces, as shown below.



The small rhombicuboctahedron shown above has a square face of the first type as its base. There are 6 faces of this type in the polyhedron, and we can choose any of these to be placed as the base.

Next we have to determine how many of the eight symmetries of one of these square faces give symmetries of the polyhedron. Consideration of the polyhedron shows that all 8 symmetries do, so the number of symmetries of the polyhedron is $6 \times 8 = 48$.

The strategy for finding the number of symmetries of a non-regular polyhedron is now given.

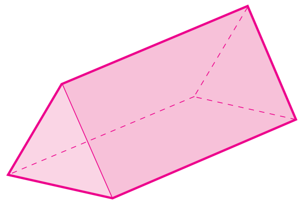
Strategy 5.2 To determine the number of symmetries of a non-regular polyhedron.

1. Select one type of face and count the number of similar faces which are similarly placed in the polyhedron.
2. Count the symmetries of the face within the polyhedron (that is, symmetries of the face that are also symmetries of the polyhedron).

Then,

$$\begin{pmatrix}\text{number of}\\ \text{symmetries of}\\ \text{the polyhedron}\end{pmatrix} = \begin{pmatrix}\text{number of}\\ \text{faces of the}\\ \text{selected type}\end{pmatrix} \times \begin{pmatrix}\text{number of}\\ \text{symmetries of face}\\ \text{that are also}\\ \text{symmetries of}\\ \text{the polyhedron}\end{pmatrix}.$$

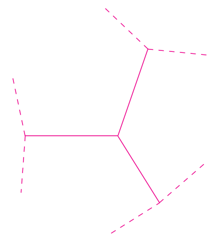
Exercise 5.2 Using Strategy 5.2, calculate the number of symmetries of the following figure. Check your calculations by considering the solid in a different way.



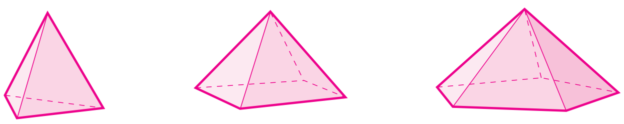
triangular prism

5.4 Platonic solids

The Platonic solids are the convex regular polyhedra in which each face is the same regular polygon and each vertex is the junction of the same number of edges and faces arranged in the same way. In order to produce a solid, we must have at least three edges (and so at least three faces) meeting at each vertex.

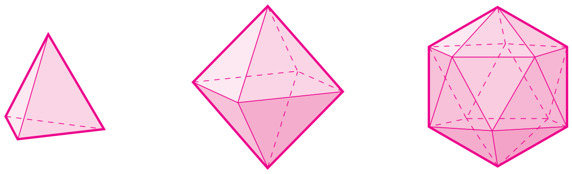


Starting with an equilateral triangle face, we can have three, four or five triangular faces around each vertex, but six equilateral triangles would lie flat, and more than six equilateral triangles would give a non-convex solid.



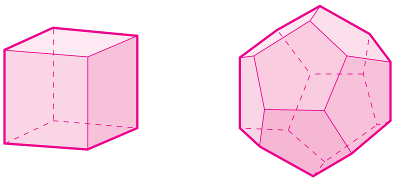
3 faces 4 faces 5 faces

The arrangement of faces at each vertex of the solid must be the same, so we can build up the rest of the solid from the construction at one vertex. The three possibilities above give the tetrahedron, the octahedron and the icosahedron, respectively.



tetrahedron octahedron icosahedron

Next we consider a square as a possible face. Three squares at each vertex give a cube, but four squares would lie flat, and more than four squares would give a non-convex solid.

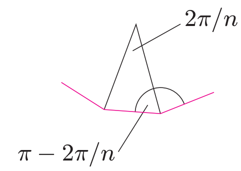


cube dodecahedron

The fifth Platonic solid, the dodecahedron, is formed by using three regular pentagons at each vertex. Four or more regular pentagons around a vertex would give a non-convex solid.

There can be no more such solids because three regular hexagons lie flat, and for any regular polygon with more than six edges, the angle at each vertex is greater than $2\pi/3$, so we cannot fit three together at a vertex without making the solid non-convex.

The angle at a vertex of a regular n-gon is $\pi(n-2)/n$, which is greater than $2\pi/3$ for $n > 6$.



Thus there are precisely five regular polyhedra.

The Platonic solids are so named not because Plato (427–347 BC) discovered them, but because he associated the regular tetrahedron, cube, octahedron and icosahedron with the four elements of fire, earth, air and water, respectively; he associated the dodecahedron with the universe.

Further exercises

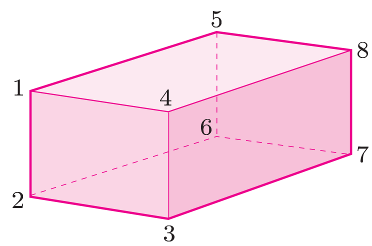
Exercise 5.3 Use Strategy 5.2 to count the number of symmetries of the small rhombicuboctahedron by considering:

(a) a square face, of the second type;

(b) a triangular face.

Exercise 5.4

(a) Use Strategy 5.2 to show that a rectangular block has eight symmetries.



(b) Write down the two-line symbol for each of the eight symmetries, using the labelling shown above.