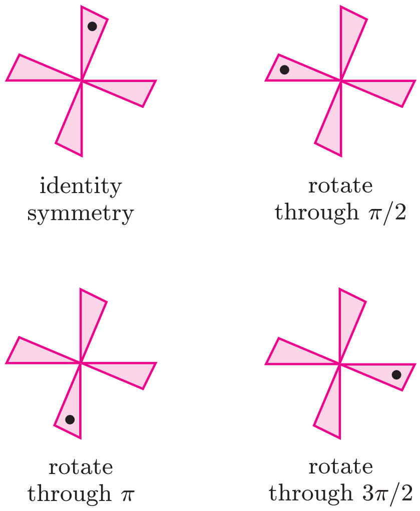
Solutions to the exercises

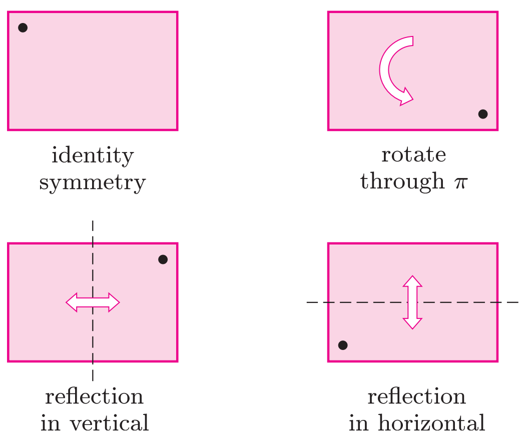
1.1 **(a)** We denote the initial position by a dot at the top.

The symmetries are as follows.



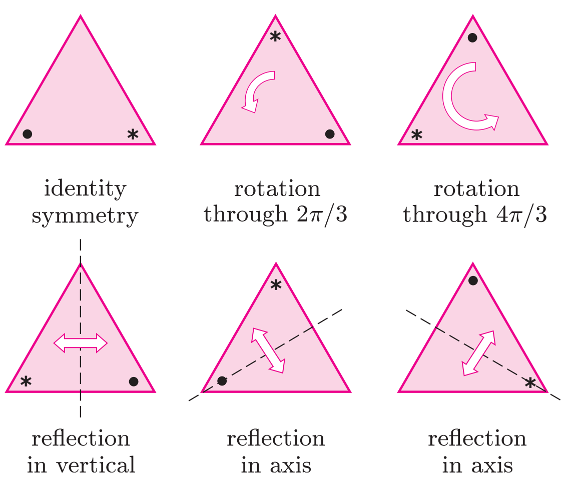
(b) We denote the initial position by a dot in the top left corner.

The symmetries are as follows.

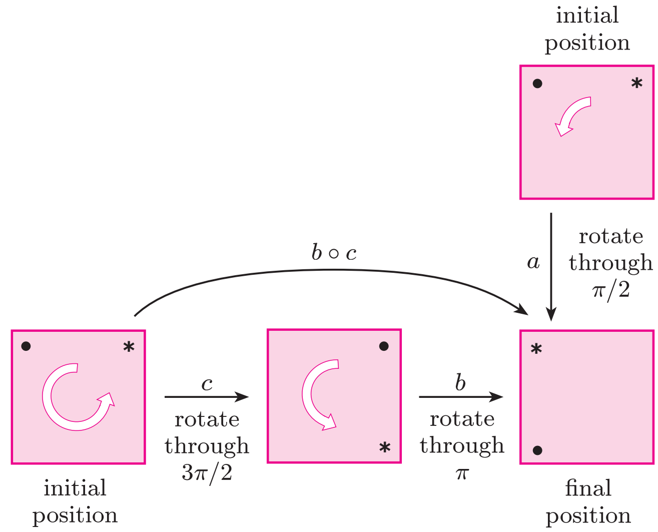


(c) We denote the initial position by symbols near two of the vertices.

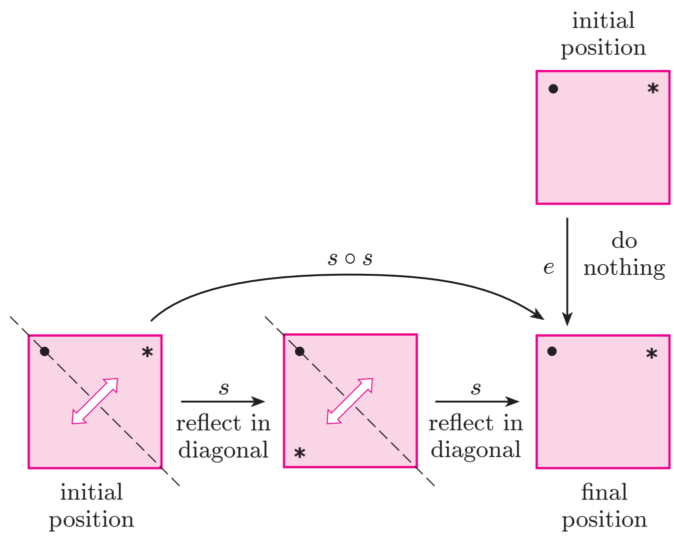
The symmetries are as follows.



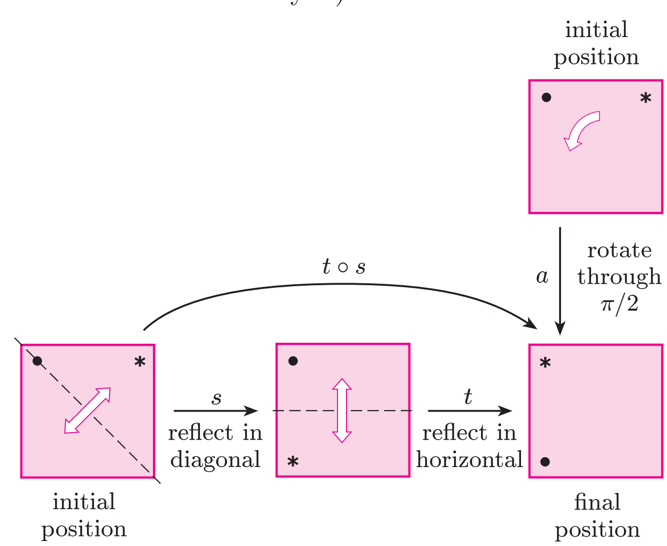
1.2 We find the required composites by drawing diagrams similar to those in the text.



Hence $b \circ c = a$.

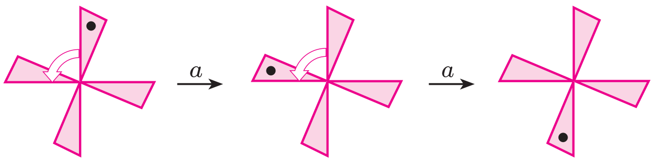


Hence $s \circ s = e$. (A reflection composed with itself is the same as the identity e.)

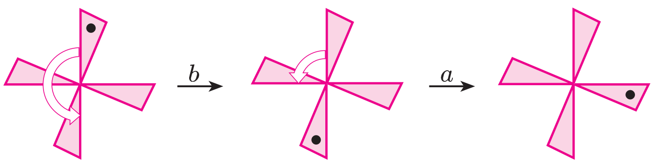


Hence $t \circ s = a$.

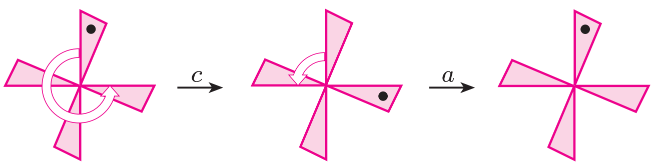
1.3 (a) The required composites are as follows.



Hence $a \circ a = b$.

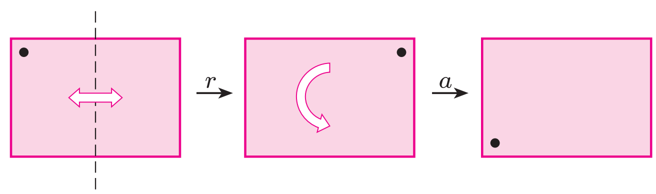


Hence $a \circ b = c$.

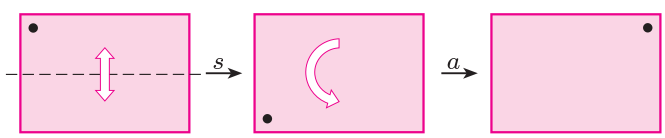


Hence $a \circ c = e$.

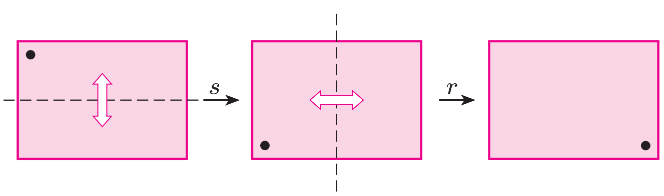
The required composites are as follows.



Hence $a \circ r = s$.

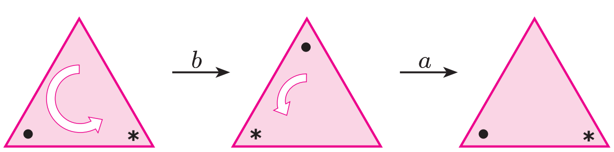


Hence $a \circ s = r$.

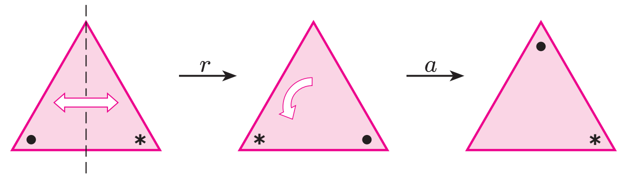


Hence $r \circ s = a$.

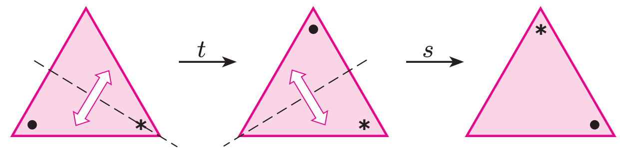
1.4 The required composites are as follows.



Hence $a \circ b = e$.



Hence $a \circ r = t$.



Hence $s \circ t = a$.

1.5 (a) In S(WIND), as in $S(\square)$, the rotations a and c are inverses of each other, and b is self-inverse.

Element	e	a	b	c
Inverse	e	c	b	a

(b) In $S(\boxminus)$, each element is self-inverse.

Element	e	a	r	s
Inverse	e	a	r	s

(c) In $S(\triangle)$, the rotations a and b are inverses of each other, and the other symmetries are self-inverse.

Element	e	a	b	r	s	t
Inverse	e	b	a	r	s	t

1.6 In this exercise we find the composites using the diagrammatic method described in the text. (We do not give the diagrams here.)

First we find $a \circ (t \circ a)$:

$t \circ a = s$ and $a \circ s = t$,

so $a \circ (t \circ a) = t$.

Next we find $(a \circ t) \circ a$:

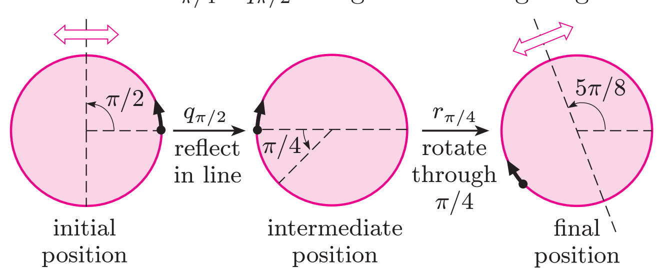
$a \circ t = u$ and $u \circ a = t$,

so $(a \circ t) \circ a = t$.

Hence

$$a \circ t \circ a = a \circ (t \circ a) = (a \circ t) \circ a.$$

1.7 We find $r_{\pi/4} \circ q_{\pi/2}$ using the following diagram.



Hence $r_{\pi/4} \circ q_{\pi/2} = q_{5\pi/8}$.

1.8 Using the formulas, we obtain:

$$r_{\pi/4} \circ r_{\pi/2} = r_{(\pi/4+\pi/2)\ (\mathrm{mod}\ 2\pi)} = r_{3\pi/4},$$

$$q_{\pi/4} \circ q_{\pi/2} = r_{2(\pi/4-\pi/2)\ (\mathrm{mod}\ 2\pi)} = r_{-\pi/2\ (\mathrm{mod}\ 2\pi)} = r_{3\pi/2},$$

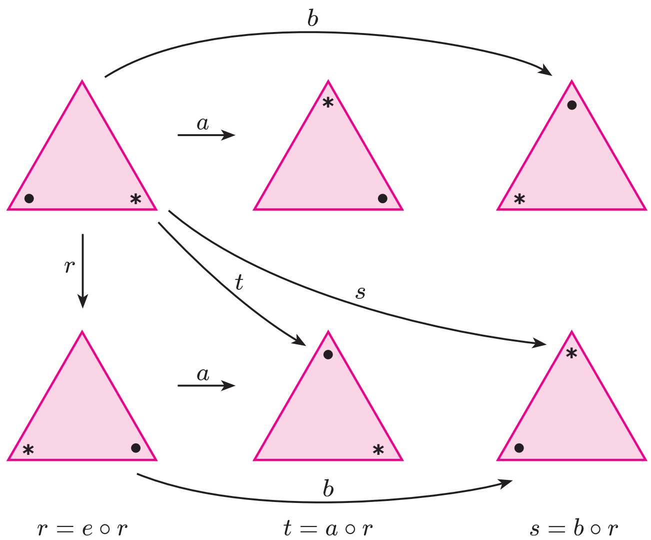
$$q_{\pi/4} \circ r_{\pi/2} = q_{(\pi/4-\frac{1}{2}(\pi/2))\ (\mathrm{mod}\ \pi)} = q_0,$$

$$r_{\pi/4} \circ q_{\pi/2} = q_{(\frac{1}{2}(\pi/4)+\pi/2)\ (\mathrm{mod}\ \pi)} = q_{5\pi/8}.$$

1.9 This solution uses the standard labelling for symmetries introduced in Exercises 1.3 and 1.4.

(a) $S^+(\triangle) = \{e, a, b\}$.

Using the reflection r, we obtain the following diagram.

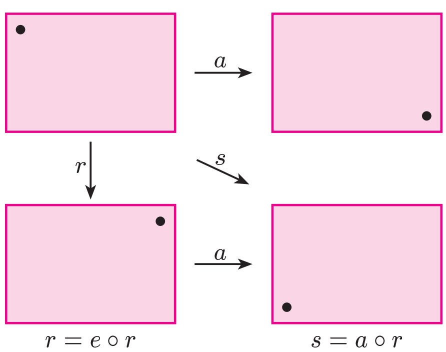


Instead of r, we could have used s or t as the reflection:

$s = e \circ s, \quad r = a \circ s, \quad t = b \circ s,$
$t = e \circ t, \quad s = a \circ t, \quad r = b \circ t.$

(b) $S^+(\square) = \{e, a\}$.

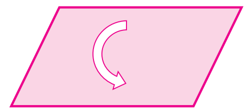
Using the reflection r, we obtain the following diagram.



Alternatively, we could have used the reflection s:

$s = e \circ s, \quad r = a \circ s.$

1.10 (a) Parallelogram

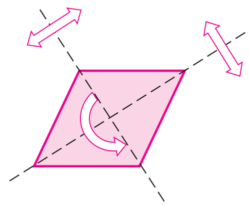


The symmetries are:

the identity;

rotation about the centre through π.

(b) Rhombus



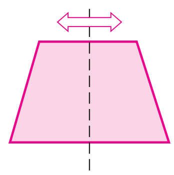
The symmetries are:

the identity;

rotation about the centre through π;

reflection in each diagonal.

(c) Symmetrical trapezium

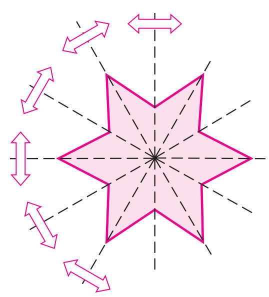


The symmetries are:

the identity;

reflection in the line bisecting the two parallel edges.

(d) Regular star



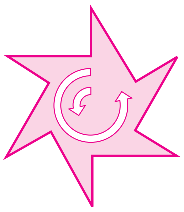
The symmetries are:

six anticlockwise rotations about the centre through 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$ and $5\pi/3$;

six reflections, three in lines joining opposite vertices and three in lines joining opposite reflex angles.

(Only the reflections are shown on the figure.)

(e) Circular sawblade

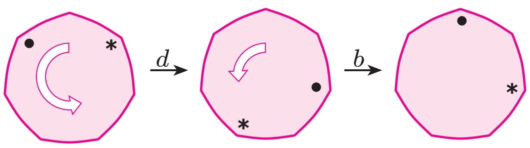


The symmetries are:

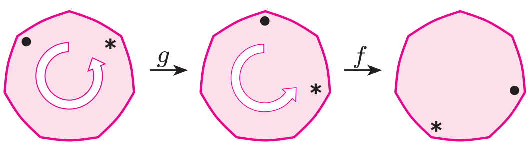
six anticlockwise rotations about the centre through 0, $\pi/3$, $2\pi/3$, π, $4\pi/3$ and $5\pi/3$.

(Two of these rotations, through $2\pi/3$ and $5\pi/3$, are illustrated on the figure. The figure has no reflectional symmetry.)

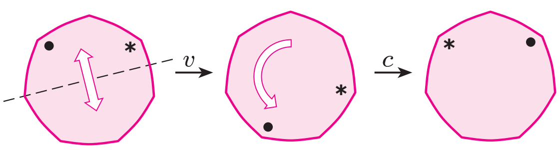
1.11 The required composites are as follows.



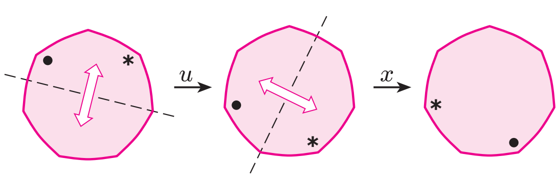
Hence $b \circ d = g$.



Hence $f \circ g = d$.



Hence $c \circ v = r$.



Hence $x \circ u = c$.

1.12

	Rotations	Reflections
Element	e a b c d f g	r s t u v w x
Inverse	e g f d c b a	r s t u v w x

Note that each of the reflections is its own inverse.

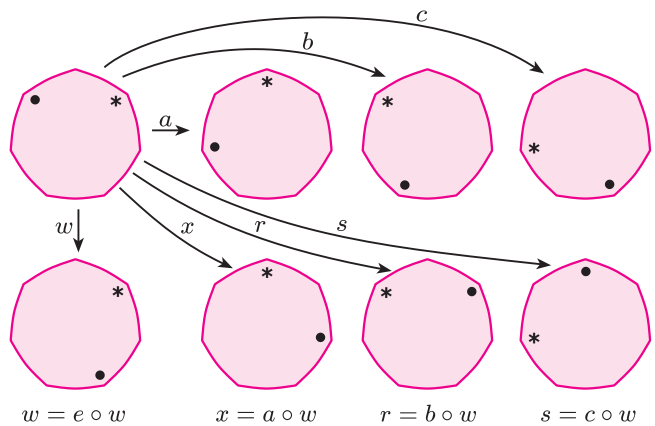
1.13 The direct symmetries are the seven rotations

e, a, b, c, d, f, g.

The indirect symmetries x, r and s are obtained by composing w with the rotations a, b and c, respectively:

$x = a \circ w, \quad r = b \circ w, \quad s = c \circ w.$

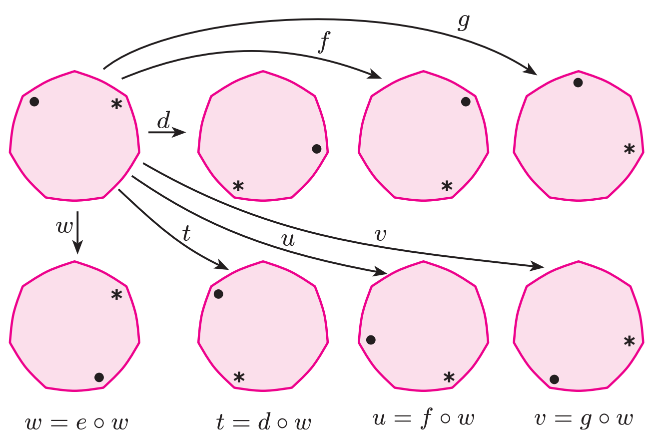
We can picture this as follows.



Similarly, the indirect symmetries t, u and v are obtained by composing w with the rotations d, f and g, respectively:

$t = d \circ w, \quad u = f \circ w, \quad v = g \circ w.$

We can picture this as follows.



2.1 We have

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix},$$

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

2.2 Here

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix},$$

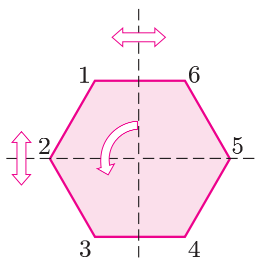
$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

2.3 We have

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$r = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

2.4



(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ represents reflection in the vertical axis of symmetry joining the midpoints of the edges $1, 6$ and $3, 4$.

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$ represents anticlockwise rotation through $2\pi/3$ about the centre.

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$ represents reflection in the horizontal axis of symmetry through locations 2 and 5.

2.5 We have

$$a \circ a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = b,$$

$$b \circ s = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = t,$$

$$s \circ b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = r,$$

$$t \circ s = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = b.$$

2.6 In each case we turn the symbol upside down and reorder the columns.

(a)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & 6 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$$

(b)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 & 6 & 5 & 4 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix}$$

(c)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 5 & 6 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$$

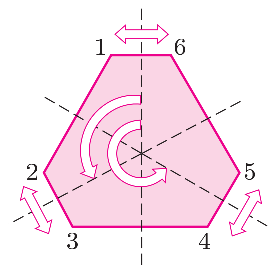
2.7 The Cayley table for $S(\triangle)$ is as follows.

$\circ$	e	a	b	r	s	t
e	e	a	b	r	s	t
a	a	b	e	t	r	s
b	b	e	a	s	t	r
r	r	s	t	e	a	b
s	s	t	r	b	e	a
t	t	r	s	a	b	e

2.8 The Cayley table for $S(\square)$ is as follows.

$\circ$	e	a	r	s
e	e	a	r	s
a	a	e	s	r
r	r	s	e	a
s	s	r	a	e

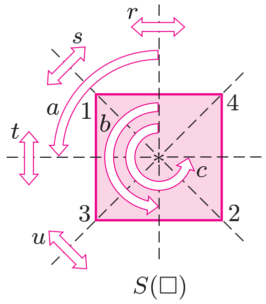
2.9



The symmetries are:

the identity,
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix};$$
rotation about the centre through $2\pi/3$ anticlockwise,
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix};$$
rotation about the centre through $4\pi/3$ anticlockwise,
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix};$$
reflection in the line bisecting the edges joining the locations $1, 6$ and $3, 4$,
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix};$$
reflection in the line bisecting the edges joining the locations $2, 3$ and $5, 6$,
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{pmatrix};$$
reflection in the line bisecting the edges joining the locations $4, 5$ and $1, 2$,
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \end{pmatrix}.$$

2.10 (a)



The two-line symbols are

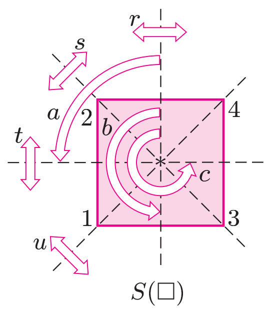
$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix},$$

$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix},$$

$$t = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}.$$

(b)



The two-line symbols are

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix},$$

$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix},$$

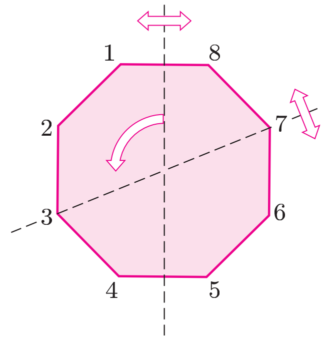
$$t = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}.$$

2.11 The two-line symbols are

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix},$$

$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}.$$

2.12



(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 \end{pmatrix}$ represents anticlockwise rotation through $\pi/2$ about the centre.

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ represents reflection in the axis bisecting the edges joining locations $1, 8$ and $4, 5$.

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 3 & 2 & 1 & 8 & 7 & 6 \end{pmatrix}$ represents reflection in the axis joining locations 3 and 7.

2.13 The two-line symbols are

$$x \circ x = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

$$y \circ y = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = e,$$

$$x \circ y = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

$$y \circ x = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

Using the above results, we obtain

$$(x \circ x) \circ x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = e,$$

$$(x \circ x) \circ y = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = y \circ x,$$

$$y \circ (x \circ x) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = x \circ y.$$

3.1 (a) $(\mathbb{R}, +)$ is a group. The proof is similar to the proof in Frame 3.

We show that the four group axioms hold.

G1 For all $x, y \in \mathbb{R}$,

$$x + y \in \mathbb{R},$$

so $\mathbb{R}$ is closed under $+$.

G2 For all $x \in \mathbb{R}$,

$$x + 0 = x = 0 + x,$$

and $0 \in \mathbb{R}$, so 0 is an identity element.

G3 For each $x \in \mathbb{R}$,

$$x + (-x) = 0 = (-x) + x,$$

and $-x \in \mathbb{R}$, so $-x$ is an inverse of x.

G4 Addition of real numbers is associative.

Hence $(\mathbb{R}, +)$ satisfies the four group axioms, and so is a group.

(b) $(\mathbb{Q}, \times)$ is not a group. This situation is similar to that in Frame 4.

Axioms G1 and G2 hold, and 1 is a multiplicative identity, but axiom G3 fails because 0 has no multiplicative inverse in $\mathbb{Q}$.

(c) $(\mathbb{Q}^*, \times)$ is a group. The proof is similar to the proof in Frame 5.

We show that the four group axioms hold.

G1 For all $x, y \in \mathbb{Q}^*$, we have $x \neq 0$ and $y \neq 0$, so $x \times y \neq 0$ and $x \times y \in \mathbb{Q}^*$, so $\mathbb{Q}^*$ is closed under $\times$.

G2 For all $x \in \mathbb{Q}^*$,

$$x \times 1 = x = 1 \times x,$$

and $1 \in \mathbb{Q}^*$, so 1 is an identity element.

G3 For each $x \in \mathbb{Q}^*$, we have $x \neq 0$, so $1/x$ is defined and is non-zero; hence $1/x \in \mathbb{Q}^*$. Also,

$$x \times \frac{1}{x} = 1 = \frac{1}{x} \times x,$$

so $1/x$ is an inverse of x.

G4 Multiplication of rational numbers is associative.

Hence $(\mathbb{Q}^*, \times)$ satisfies the four group axioms, and so is a group.

3.2 (a) $(\mathbb{Z}, \times)$ is not a group.

$\mathbb{Z}$ is closed under multiplication, and 1 is a multiplicative identity, so axioms G1 and G2 hold.

However, axiom G3 fails because, for example, 2 has no multiplicative inverse in $\mathbb{Z}$, since $\frac{1}{2} \notin \mathbb{Z}$.

(b) $(\mathbb{Z}, -)$ is not a group.

$\mathbb{Z}$ is closed under subtraction, but there is no identity element; for example, there is no integer n such that

$$2 - n = 2 = n - 2,$$

so axiom G2 fails.

Alternatively, we can show that axiom G4 fails using the argument given in Frame 8, part (i). The integers 6, 4 and 1 belong to $\mathbb{Z}$, and

$$6 - (4 - 1) = 6 - 3 = 3,$$

but

$$(6 - 4) - 1 = 2 - 1 = 1.$$

Since $3 \neq 1$, subtraction is not associative on $\mathbb{Z}$.

(c) $(G = \{\text{odd integers}\}, \times)$ is not a group.

The product of two odd integers is odd, so G is closed under $\times$. Also, 1 is odd, so 1 is a multiplicative identity. So axioms G1 and G2 hold.

However, axiom G3 fails because, for example, 3 has no multiplicative inverse in G, since $\frac{1}{3} \notin G$.

(d) $(2\pi\mathbb{Z}, +)$ is a group.

We show that the four group axioms hold.

G1 For all $m, n \in \mathbb{Z}$,

$$2\pi m + 2\pi n = 2\pi(m + n) \in 2\pi\mathbb{Z},$$

so $2\pi\mathbb{Z}$ is closed under $+$.

G2 For all $k \in \mathbb{Z}$,

$$2\pi k + 0 = 2\pi k = 0 + 2\pi k,$$

and $0 = 2\pi \times 0 \in 2\pi\mathbb{Z}$, so 0 is an identity element.

G3 For each $k \in \mathbb{Z}$,

$$2\pi k + 2\pi(-k) = 0 = 2\pi(-k) + 2\pi k,$$

and $2\pi(-k) \in 2\pi\mathbb{Z}$, so $2\pi(-k)$ is an inverse of $2\pi k$.

G4 Addition is associative on $2\pi\mathbb{Z}$.

Hence $(2\pi\mathbb{Z}, +)$ satisfies the four group axioms, and so is a group.

(e) Here the operation is unfamiliar, so we examine the axioms in turn.

The approach is similar to that in Frame 9.

G1 For all $x, y \in \mathbb{R}$,

$$x \circ y = x - y - 1 \in \mathbb{R},$$

so $\mathbb{R}$ is closed under $\circ$.

G2 Is there an identity element $e \in \mathbb{R}$ such that, for each $x \in \mathbb{R}$,

$$x \circ e = x = e \circ x?$$

That is, is there a real number e such that

$$x - e - 1 = x = e - x - 1?$$

Clearly, there is not because

$$x - e - 1 = x \Rightarrow e = -1$$

and

$$e - x - 1 = x \Rightarrow e = 2x + 1.$$

Thus, for each $x \in \mathbb{R}$, we must have

$$e = -1 = 2x + 1.$$

But this is false for $x = 0$, for example, so axiom G2 fails.

Hence $(\mathbb{R}, \circ)$ is not a group.

3.3 (a) Let $x, y, z \in \mathbb{R}$. Then

$$\begin{aligned} & x \circ (y \circ z) \\ &= x \circ (y + z - yz) \\ &= x + (y + z - yz) - x(y + z - yz) \\ &= x + y + z - xy - xz - yz + xyz \qquad \text{(S.1)} \end{aligned}$$

and

$$\begin{aligned} & (x \circ y) \circ z \\ &= (x + y - xy) \circ z \\ &= (x + y - xy) + z - (x + y - xy)z \\ &= x + y + z - xy - xz - yz + xyz. \qquad \text{(S.2)} \end{aligned}$$

Expressions (S.1) and (S.2) are the same, so $\circ$ is associative on $\mathbb{R}$.

(b) Let $x, y, z \in \mathbb{R}$. Then

$$\begin{aligned} & x \circ (y \circ z) \\ &= x \circ (y - z + yz) \\ &= x - (y - z + yz) + x(y - z + yz) \\ &= x - y + z + xy - xz - yz + xyz \qquad \text{(S.3)} \end{aligned}$$

and

$$\begin{aligned} & (x \circ y) \circ z \\ &= (x - y + xy) \circ z \\ &= (x - y + xy) - z + (x - y + xy)z \\ &= x - y - z + xy + xz - yz + xyz. \qquad \text{(S.4)} \end{aligned}$$

Expressions (S.3) and (S.4) are not the same.

For example, if $x = 0$, $y = 1$ and $z = 2$, then

$$\begin{aligned} 0 \circ (1 \circ 2) &= 0 \circ (1 - 2 + 2) \\ &= 0 \circ 1 \\ &= 0 - 1 + 0 = -1 \end{aligned}$$

but

$$\begin{aligned} (0 \circ 1) \circ 2 &= (0 - 1 + 0) \circ 2 \\ &= (-1) \circ 2 \\ &= -1 - 2 - 2 = -5, \end{aligned}$$

so $\circ$ is not associative on $\mathbb{R}$.

(If you can see that $\circ$ is not associative, then it is not necessary to calculate expressions (S.3) and (S.4)); it is sufficient to produce a specific counter-example.)

3.4 We show that $(\mathbb{R} - \{-1\}, \circ)$ satisfies the group axioms.

The proof of axiom G1 needs care.

G1 For all $x, y \in \mathbb{R} - \{-1\}$,

$$x \circ y = x + y + xy \in \mathbb{R}.$$

To show that $x + y + xy \in \mathbb{R} - \{-1\}$, we need to show that, for all $x, y \in \mathbb{R} - \{-1\}$,

$$x + y + xy \neq -1.$$

Now, if x, $y \in \mathbb{R}$, and

$$x + y + xy = -1,$$

then

$$x(1 + y) = -(1 + y).$$

So EITHER $x = -1$, OR $1 + y = 0$, so $y = -1$.

However, the given set does not contain -1, so neither of these conclusions is possible. Thus $x \circ y$ cannot equal -1.

Hence, if $x, y \in \mathbb{R} - \{-1\}$, then $x \circ y \in \mathbb{R} - \{-1\}$, so $\mathbb{R} - \{-1\}$ is closed under $\circ$.

G2 In Frame 9 we showed that 0 is an identity element for $\circ$ on $\mathbb{R}$, and $0 \in \mathbb{R} - \{-1\}$, so 0 is an identity element for $\circ$ on $\mathbb{R} - \{-1\}$.

G3 In Frame 9 we showed that, for each element x in $\mathbb{R} - \{-1\}$, $x \circ y = e$, where $y = -x/(1 + x)$. Here, $x \neq 1 + x$, so $y \neq -1$. Also, $x \circ y = y \circ x$, so each element x in $\mathbb{R} - \{-1\}$ has an inverse $-x/(1 + x)$ in $\mathbb{R} - \{-1\}$.

G4 Let $x, y, z \in \mathbb{R} - \{-1\}$. Our working in Frame 8 shows that $\circ$ is associative on $\mathbb{R} - \{-1\}$.

Hence $(\mathbb{R} - \{-1\}, \circ)$ satisfies the four group axioms, and so is a group.

3.5 (a) This situation is similar to that in Frame 11.

The Cayley table for $(\mathbb{Z}_5, +_5)$ is as follows.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

We show that the four group axioms hold.

G1 No new elements are needed to complete the table, so $\mathbb{Z}_5$ is closed under $+_5$.

G2 The first row of the table shows that

$$0 +_5 x = x, \quad \text{for each } x \in \mathbb{Z}_5.$$

The first column of the table shows that

$$x +_5 0 = x, \quad \text{for each } x \in \mathbb{Z}_5.$$

Hence 0 is an identity element.

G3 From the Cayley table above, we see that

$$0 +_5 0 = 0,$$
$$1 +_5 4 = 0 = 4 +_5 1,$$
$$2 +_5 3 = 0 = 3 +_5 2,$$

so

0 is self-inverse,

1 and 4 are inverses of each other,

2 and 3 are inverses of each other.

G4 The operation $+_5$ is associative.

Hence $(\mathbb{Z}_5, +_5)$ satisfies the four group axioms, and so is a group.

(b) This situation is similar to that in Frame 12.

The Cayley table for $(\mathbb{Z}_5, \times_5)$ is as follows.

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Axioms G1 and G2 hold, and 1 is an identity element.

However, there is no 1 in the column labelled 0, so 0 has no inverse, so axiom G3 fails.

Hence $(\mathbb{Z}_5, \times_5)$ is not a group.

(c) In this case, the troublesome 0 has been omitted, and the Cayley table is as follows.

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

We check the four group axioms in turn.

G1 No new elements are needed to complete the table, so $\mathbb{Z}_5^*$ is closed under $\times_5$.

G2 The first row of the table shows that

$$1 \times_5 x = x, \quad \text{for each } x \in \mathbb{Z}_5^*.$$

The first column of the table shows that

$$x \times_5 1 = x, \quad \text{for each } x \in \mathbb{Z}_5^*.$$

Hence 1 is an identity element.

G3 From the table, we see that

$$1 \times_5 1 = 1,$$
$$4 \times_5 4 = 1,$$
$$2 \times_5 3 = 1 = 3 \times_5 2,$$

so

1 and 4 are self-inverse,

2 and 3 are inverses of each other.

G4 The operation $\times_5$ is associative.

Hence $(\mathbb{Z}_5^*, \times_5)$ satisfies the four group axioms, and so is a group.

3.6 (a) This situation is similar to that in Frame 11, and to Exercise 3.5(a).

The Cayley table for $(\mathbb{Z}_6, +_6)$ is as follows.

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

We check the four group axioms in turn.

G1 No new elements are needed to complete the table, so $\mathbb{Z}_6$ is closed under $+_6$.

G2 The row and column labelled 0 repeat the borders of the table, so 0 is an identity.

G3 From the table, we see that each element has an inverse in $\mathbb{Z}_6$.

Element	0	1	2	3	4	5
Inverse	0	5	4	3	2	1

G4 The operation $+_6$ is associative.

Hence $(\mathbb{Z}_6, +_6)$ satisfies the four group axioms, and so is a group.

(b) This situation is similar to that in Frame 12, and to Exercise 3.5(b).

The Cayley table for $(\mathbb{Z}_6, \times_6)$ is as follows.

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Axioms G1 and G2 hold, and 1 is an identity.

However, there is no 1 in the column labelled 0, so 0 has no inverse, and axiom G3 fails.

Hence $(\mathbb{Z}_6, \times_6)$ is not a group.

(c) The Cayley table for $(\mathbb{Z}_7^*, \times_7)$ is as follows.

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

We check the four group axioms in turn.

G1 No new elements are needed to complete the table, so $\mathbb{Z}_7^*$ is closed under $\times_7$.

G2 The row and column labelled 1 repeat the borders of the table, so 1 is an identity.

G3 From the table, we see that each element has an inverse in $\mathbb{Z}_7^*$.

Element	1	2	3	4	5	6
Inverse	1	4	5	2	3	6

G4 The operation $\times_7$ is associative.

Hence $(\mathbb{Z}_7^*, \times_7)$ satisfies the four group axioms, and so is a group.

(d) The Cayley table for $(\{1,-1\},\times)$ is as follows.

$\times$	1	-1
1	1	-1
-1	-1	1

We check the four group axioms in turn.

G1 No new elements are needed to complete the table, so $\{1,-1\}$ is closed under $\times$.

G2 From the table, we see that 1 is an identity element.

G3 Since $1\times 1=1$ and $(-1)\times(-1)=1$, 1 and -1 are both self-inverse.

G4 Multiplication of numbers is associative.

Hence $(\{1,-1\},\times)$ satisfies the four group axioms, and so is a group.

3.7 We show that the four group axioms hold.

G1 For all $a,b\in\mathbb{Q}^+$, we have $a\circ b=\frac{1}{2}ab\in\mathbb{Q}^+$, so $\mathbb{Q}^+$ is closed under the operation $\circ$.

G2 For all $a\in\mathbb{Q}^+$,

$$a\circ 2=a=2\circ a,$$

and $2\in\mathbb{Q}^+$, so 2 is an identity element.

G3 An inverse of a is not obvious, so we assume that an inverse x exists, and try to find it.

We seek $x\in\mathbb{Q}^+$ such that

$$a\circ x=2=x\circ a;$$

that is,

$$\tfrac{1}{2}ax=2=\tfrac{1}{2}xa,$$

so the only possibility is $x=4/a$.

For each $a\in\mathbb{Q}^+$, we have $4/a\in\mathbb{Q}^+$, so $4/a$ is an inverse of a.

G4 For all $a,b,c\in\mathbb{Q}^+$,

$$\begin{aligned}a\circ(b\circ c)&=a\circ(\tfrac{1}{2}bc)\\&=\tfrac{1}{2}a(\tfrac{1}{2}bc)\\&=\tfrac{1}{4}abc\end{aligned}\qquad\text{(S.5)}$$

and

$$\begin{aligned}(a\circ b)\circ c&=(\tfrac{1}{2}ab)\circ c\\&=\tfrac{1}{2}(\tfrac{1}{2}ab)c\\&=\tfrac{1}{4}abc.\end{aligned}\qquad\text{(S.6)}$$

Expressions (S.5) and (S.6) are the same, so $\circ$ is associative on $\mathbb{Q}^+$.

Hence $(\mathbb{Q}^+,\circ)$ satisfies the four group axioms, and so is a group.

3.8 (a) $(\mathbb{C},+)$ is a group. The proof is similar to the proof in Frame 3.

We show that the four group axioms hold.

G1 For all $x,y\in\mathbb{C}$,

$$x+y\in\mathbb{C},$$

so $\mathbb{C}$ is closed under $+$.

G2 For all $x\in\mathbb{C}$,

$$x+0=x=0+x,$$

and $0\in\mathbb{C}$, so 0 is an identity element.

G3 For each $x\in\mathbb{C}$,

$$x+(-x)=0=(-x)+x,$$

and $-x\in\mathbb{C}$, so $-x$ is an inverse of x.

G4 Addition of complex numbers is associative.

Hence $(\mathbb{C},+)$ satisfies the four group axioms, and so is a group.

3.9 We show that the four group axioms hold.

G1 For all integers $m,n\in\mathbb{Z}$,

$$2m+2n=2(m+n)\in 2\mathbb{Z},$$

so $2\mathbb{Z}$ is closed under $+$.

G2 For all $k\in\mathbb{Z}$,

$$2k+0=2k=0+2k,$$

and $0=2\times 0\in 2\mathbb{Z}$, so 0 is an identity in $2\mathbb{Z}$.

G3 For each $k\in\mathbb{Z}$,

$$2k+(-2k)=0=(-2k)+2k,$$

and $-2k=2(-k)\in 2\mathbb{Z}$, so $-2k$ is an inverse of $2k$ in $2\mathbb{Z}$.

G4 Addition of integers is associative.

Hence $(2\mathbb{Z},+)$ satisfies the four group axioms, and so is a group.

3.10 We show that the four group axioms hold.

G1 For all $m,n\in\mathbb{Z}$,

$$2^m\times 2^n=2^{m+n}\in G,$$

so G is closed under $\times$.

G2 For all $k\in\mathbb{Z}$,

$$2^k\times 2^0=2^k=2^0\times 2^k,$$

and $2^0=1\in G$, so 1 is an identity in G.

G3 For each $k\in\mathbb{Z}$,

$$2^k\times 2^{-k}=2^{k-k}=1=2^{-k+k}=2^{-k}\times 2^k,$$

so 2^{-k} is an inverse of 2^k in G.

G4 For all $k, m, n \in \mathbb{Z}$,

$$\begin{aligned} 2^k \times (2^m \times 2^n) &= 2^k \times 2^{m+n} \\ &= 2^{k+(m+n)} \\ &= 2^{k+m+n} \end{aligned} \qquad \text{(S.7)}$$

and

$$\begin{aligned} (2^k \times 2^m) \times 2^n &= 2^{k+m} \times 2^n \\ &= 2^{(k+m)+n} \\ &= 2^{k+m+n}. \end{aligned} \qquad \text{(S.8)}$$

Expressions (S.7) and (S.8) are the same, so $\times$ is associative on G.

Hence $(G, \times)$ satisfies the four group axioms, and so is a group.

3.11 In each case, we begin by completing a Cayley table, and then examine the group axioms.

(a)

$\times_8$	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

G1 No new elements are needed to complete the table, so $\{1, 3, 5, 7\}$ is closed under $\times_8$.

G2 From the table, we see that 1 is an identity, and $1 \in \{1, 3, 5, 7\}$.

G3 From the table, we see that each element in $\{1, 3, 5, 7\}$ is self-inverse, so this set contains an inverse of each element.

G4 Multiplication of numbers is associative.

Hence $(\{1, 3, 5, 7\}, \times_8)$ satisfies the four group axioms, and so is a group.

(b)

$\times_{15}$	1	2	4	7	8	11	13	14
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1

G1 No new elements are needed to complete the table, so $\{1, 2, 4, 7, 8, 11, 13, 14\}$ is closed under $\times_{15}$.

G2 From the table, we see that 1 is an identity, and $1 \in \{1, 2, 4, 7, 8, 11, 13, 14\}$.

G3 From the table, we see that each element has an inverse in the set.

Element	1	2	4	7	8	11	13	14
Inverse	1	8	4	13	2	11	7	14

G4 Multiplication of numbers is associative.

Hence $(\{1, 2, 4, 7, 8, 11, 13, 14\}, \times_{15})$ satisfies the four group axioms, and so is a group.

(c)

$\times_{15}$	1	4	11	14
1	1	4	11	14
4	4	1	14	11
11	11	14	1	4
14	14	11	4	1

G1 No new elements are needed to complete the table, so $\{1, 4, 11, 14\}$ is closed under $\times_{15}$.

G2 From the table, we see that 1 is an identity, and $1 \in \{1, 4, 11, 14\}$.

G3 From the table, we see that each element in $\{1, 4, 11, 14\}$ is self-inverse, so each element has an inverse in the set.

G4 Multiplication of numbers is associative.

Hence $(\{1, 4, 11, 14\}, \times_{15})$ satisfies the four group axioms, and so is a group.

(d)

$+_{10}$	0	2	4	6	8
0	0	2	4	6	8
2	2	4	6	8	0
4	4	6	8	0	2
6	6	8	0	2	4
8	8	0	2	4	6

G1 No new elements are needed to complete the table, so $\{0, 2, 4, 6, 8\}$ is closed under $+_{10}$.

G2 From the table, we see that 0 is an identity element, and $0 \in \{0, 2, 4, 6, 8\}$.

G3 From the table, we see that each element has an inverse in the set.

Element	0	2	4	6	8
Inverse	0	8	6	4	2

G4 Addition of numbers is associative.

Hence $(\{0, 2, 4, 6, 8\}, +_{10})$ satisfies the four group axioms, and so is a group.

3.12 **(a)** $(\mathbb{Z}_9^*, \times_9)$ is not a group.

The operation $\times_9$ is not closed on the set $\mathbb{Z}_9^* = \{1, 2, 3, 4, 5, 6, 7, 8\}$ because, for example, $3 \in \mathbb{Z}_9^*$ and $6 \in \mathbb{Z}_9^*$, but $3 \times_9 6 = 0 \notin \mathbb{Z}_9^*$, so axiom G1 fails.

(b) $(\mathbb{Z}_9, +_9)$ is a group.

The Cayley table is as follows.

$+_9$	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8	0
2	2	3	4	5	6	7	8	0	1
3	3	4	5	6	7	8	0	1	2
4	4	5	6	7	8	0	1	2	3
5	5	6	7	8	0	1	2	3	4
6	6	7	8	0	1	2	3	4	5
7	7	8	0	1	2	3	4	5	6
8	8	0	1	2	3	4	5	6	7

G1 No new elements are needed to complete the table, so $\mathbb{Z}_9$ is closed under $+_9$.

G2 From the table, we see that 0 is an identity element, and $0 \in \mathbb{Z}_9$.

G3 From the table, we see that each element has an inverse in the set.

Element	0	1	2	3	4	5	6	7	8
Inverse	0	8	7	6	5	4	3	2	1

G4 Addition of numbers is associative.

Hence $(\mathbb{Z}_9, +_9)$ satisfies the four group axioms, and so is a group.

(c) $(\mathbb{Z}_{11}^*, \times_{11})$ is a group.

The Cayley table is as follows.

$\times_{11}$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	1	3	5	7	9
3	3	6	9	1	4	7	10	2	5	8
4	4	8	1	5	9	2	6	10	3	7
5	5	10	4	9	3	8	2	7	1	6
6	6	1	7	2	8	3	9	4	10	5
7	7	3	10	6	2	9	5	1	8	4
8	8	5	2	10	7	4	1	9	6	3
9	9	7	5	3	1	10	8	6	4	2
10	10	9	8	7	6	5	4	3	2	1

G1 No new elements are needed to complete the table, so $\mathbb{Z}_{11}^*$ is closed under $\times_{11}$.

G2 From the table, we see that 1 is an identity, and $1 \in \mathbb{Z}_{11}^*$.

G3 From the table, we see that each element has an inverse in the set.

Element	1	2	3	4	5	6	7	8	9	10
Inverse	1	6	4	3	9	2	8	7	5	10

G4 Multiplication of numbers is associative.

Hence $(\mathbb{Z}_{11}^*, \times_{11})$ satisfies the four group axioms, and so is a group.

4.1 (a) The second row and the second column repeat the borders of the table, so the identity is E.

(Here the letters E and O could denote the sets of even and odd integers under addition.)

(b) The first row and the first column repeat the borders of the table, so the identity is D.

(Here the letters D and I could denote the sets of direct and indirect symmetries of a figure with rotational and reflectional symmetries under composition.)

(c) The third row and the third column repeat the borders of the table, so the identity is w.

4.2 The fifth row and the fifth column repeat the borders of the table, so the identity is e.

The table of inverses is as follows.

Element	a	b	c	d	e	f	g	h
Inverse	b	a	d	c	e	f	h	g

4.3 The elements a, b and c do not have inverses.

For example, from row 2 we see that the only possible candidate for a^{-1} is c, since $a \circ c = e$; but from row 4 we see that $c \circ a = d \neq e$, so a has no inverse.

Alternatively, the element e does not appear symmetrically about the main diagonal.

4.4 (a) Suppose that, in a group $(G, \circ)$,

$$x \circ a = x \circ b.$$

The element x has an inverse x^{-1} in G, so composing both sides on the left with x^{-1}, we obtain

$$x^{-1} \circ (x \circ a) = x^{-1} \circ (x \circ b).$$

Hence

$$(x^{-1} \circ x) \circ a = (x^{-1} \circ x) \circ b \quad \text{(associativity)}.$$

But

$$x^{-1} \circ x = e \quad \text{(inverses)},$$

so we have

$$e \circ a = e \circ b,$$

and hence

$$a = b \quad \text{(identity)}.$$

(b) Similarly, if

$$a \circ x = b \circ x,$$

then, composing both sides on the right with x^{-1}, we obtain

$$(a \circ x) \circ x^{-1} = (b \circ x) \circ x^{-1},$$

so

$$a \circ (x \circ x^{-1}) = b \circ (x \circ x^{-1}) \quad \text{(associativity)}.$$

Hence

$$a \circ e = b \circ e \quad \text{(inverses)},$$

so

$$a = b \quad \text{(identity)}.$$

4.5 **(a)** The table of inverses is as follows.

Element	e	a	b	c	d	f	g	h
Inverse	e	a	b	c	d	f	g	h

The group table is symmetrical about the leading diagonal, so this group is Abelian.

(b) The table of inverses is as follows.

Element	e	a	b	c	d	f	g	h
Inverse	e	c	b	a	g	h	d	f

The group is non-Abelian; for example, $a \circ d = f$, but $d \circ a = h$.

4.6 We use the property that in a group table each element must appear once in each row and once in each column.

(a)

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

(b)

	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

4.7 **(a)** The first row and the first column repeat the borders of the table, so the identity is e.

(b) The table of inverses is as follows.

Element	e	a	b	c	d	f	g	h
Inverse	e	c	b	a	h	g	f	d

(c) The group is non-Abelian because the table is not symmetrical about the leading diagonal; for example, $h \circ g = c$, whereas $g \circ h = a$.

4.8 **(a)** The operation $\circ$ is not closed: a new element d occurs in the body of the table, so axiom G1 fails.

(b) Here the operation $\circ$ is closed, and we see from row 2 and column 2 that a is an identity element.

However, a does not occur symmetrically about the main diagonal, so this is not a group.

Alternatively, in row 1 and column 3, b occurs twice so this is not a group.

(c) Again, the operation $\circ$ is closed, and we see from row 1 and column 1 that e is an identity element.

However, e does not appear symmetrically about the main diagonal, so this is not a group.

Alternatively, the operation is not associative because

$$a \circ (b \circ d) = a \circ a = b$$

but

$$(a \circ b) \circ d = d \circ d = e.$$

Hence

$$a \circ (b \circ d) \neq (a \circ b) \circ d,$$

so axiom G4 fails.

(d) Here the only axiom that fails is G4; for example,

$$a \circ (b \circ d) = a \circ f = d$$

but

$$(a \circ b) \circ d = f \circ d = e.$$

Hence

$$a \circ (b \circ d) \neq (a \circ b) \circ d,$$

so axiom G4 fails.

4.9 We know that $|G|$ is even and that, for each element $g \in G$,

EITHER g is self-inverse

OR g and g^{-1} are distinct elements which are inverses of each other.

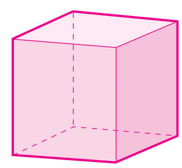
It follows that the number of elements which are self-inverse must be even.

However, e is self-inverse, so there must be at least one element $g \in G$ such that

$$g \circ g = e \quad \text{and} \quad g \neq e.$$

5.1 In each case, we use Strategy 5.1.

Cube

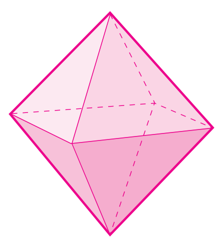


The cube has 6 faces.

Each face of the cube is a square, and so has 8 symmetries (since the order of $S(\square)$ is 8).

It follows from the strategy that the number of symmetries of the cube is $6 \times 8 = 48$.

Octahedron

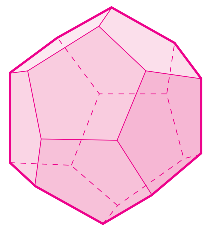


The octahedron has 8 faces.

Each face of the octahedron is an equilateral triangle, and so has 6 symmetries (since the order of $S(\triangle)$ is 6).

It follows from the strategy that the number of symmetries of the octahedron is $8 \times 6 = 48$.

Dodecahedron

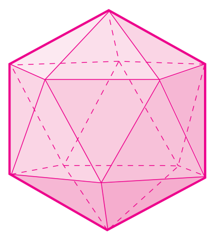


The dodecahedron has 12 faces.

Each face of the dodecahedron is a regular pentagon, and so has 10 symmetries (since the order of $S(⬠)$ is 10).

It follows from the strategy that the number of symmetries of the dodecahedron is $12 \times 10 = 120$.

Icosahedron

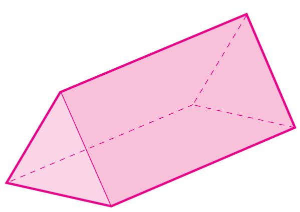


The icosahedron has 20 faces.

Each face of the icosahedron is an equilateral triangle, and so has 6 symmetries.

It follows from the strategy that the number of symmetries of the icosahedron is $20 \times 6 = 120$.

5.2 We use Strategy 5.2.



The triangular prism has two (congruent) equilateral triangle faces and three (congruent) rectangular faces; so there are two ways of applying the strategy.

Consider the equilateral triangle faces.

1. The prism has 2 (congruent) equilateral triangle faces.
2. Each triangular face is an equilateral triangle, and so has 6 symmetries; each of these symmetries is also a symmetry of the whole prism.

It follows from the strategy that the number of symmetries of the triangular prism is $2 \times 6 = 12$.

Alternatively, consider the rectangular faces.

1. The prism has 3 (congruent) rectangular faces.
2. Each rectangular face has 4 symmetries (since $S(▭)$ has order 4); each of these symmetries is also a symmetry of the whole prism.

It follows from the strategy that the number of symmetries of the triangular prism is $3 \times 4 = 12$.

5.3 (a) We consider a square face of the second type as the base—a square face with two edges that are joined to triangular faces and two to square faces. The small rhombicuboctahedron has 12 square faces of this type. Only 4 of the symmetries of this square base give symmetries of the rhombicuboctahedron.

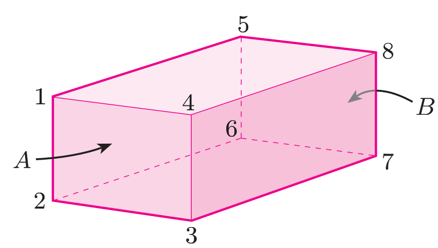
It follows from Strategy 5.2 that the number of symmetries of the rhombicuboctahedron is $12 \times 4 = 48$.

(b) We consider a triangular face as the base. The small rhombicuboctahedron has 8 triangular faces. All 6 symmetries of this equilateral triangle base give symmetries of the polyhedron.

It follows from Strategy 5.2 that the number of symmetries of the polyhedron is $8 \times 6 = 48$.

(As expected, we obtain the same number of symmetries whichever face we consider.)

5.4 (a) Rectangular block



The rectangular block has six faces—three pairs of opposite faces. Opposite faces are congruent rectangles: if faces are not opposite each other, then they are not congruent.

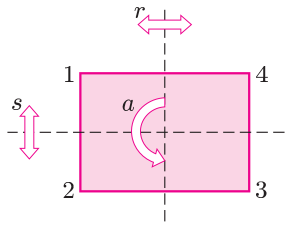
1. Select the 2 congruent faces denoted by A and B in the above diagram.
2. Each of these faces is a rectangle, and so has four symmetries (since the order of $S(▭)$ is 4).

   Each of the symmetries is also a symmetry of the whole block.

It follows from Strategy 5.2 that the number of symmetries of the rectangular block is $2 \times 4 = 8$.

(b) There are two possible approaches here: we give the details of both.

First we list the symmetries which map A to A and B to B. (These are essentially the symmetries of a rectangle.)



$S(\square)$

The two-line symbols for these symmetries are

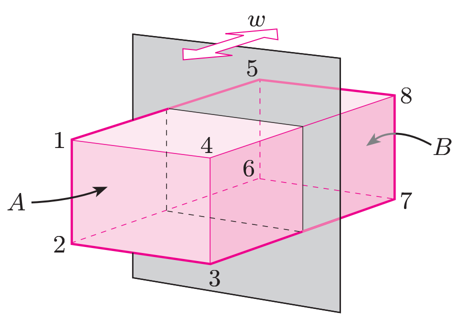
$$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix},$$

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \end{pmatrix},$$

$$r = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \end{pmatrix},$$

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \end{pmatrix}.$$

Now we consider the reflection w in the vertical plane shown below.



The reflection

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \end{pmatrix}$$

maps A to B and B to A, so we obtain the four remaining symmetries by composing each of the above symmetries with w on the right:

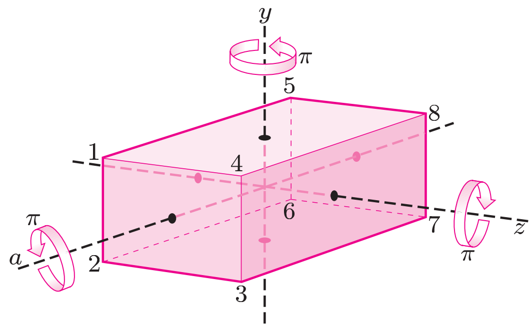
$$w = e \circ w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \end{pmatrix},$$

$$x = a \circ w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \end{pmatrix},$$

$$y = r \circ w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix},$$

$$z = s \circ w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \end{pmatrix}.$$

Alternatively, we find first the direct symmetries and then the indirect symmetries, as in the video programme. The non-trivial direct symmetries a, y and z are anticlockwise rotations through π about the axes shown below.



The two-line symbols for the direct symmetries are

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix},$$

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 \end{pmatrix},$$

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix},$$

$$z = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 8 & 7 & 2 & 1 & 4 & 3 \end{pmatrix}.$$

We obtain the four indirect symmetries by composing each of the direct symmetries with the indirect symmetry w (given above) on the right:

$$w = e \circ w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \end{pmatrix},$$

$$x = a \circ w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \end{pmatrix},$$

$$r = y \circ w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \end{pmatrix},$$

$$s = z \circ w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \end{pmatrix}.$$