4 Hyperbolic functions

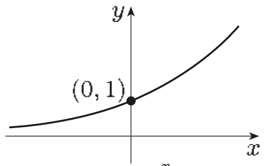
After working through this section, you should be able to:

(a) define the *hyperbolic functions* $\cosh x$, $\sinh x$ and $\tanh x$, and be familiar with their properties;

(b) sketch the graphs of $\cosh x$, $\sinh x$ and $\tanh x$, and their reciprocals.

4.1 Properties of hyperbolic functions

In Subsection 1.2 you met the graph $y = e^x$ of the exponential function, often referred to as exp, which is shown in the margin. The function exp has the following properties.



$y = e^x$

1. The domain of exp is $\mathbb{R}$.

2. exp is not even, odd or periodic.

3. $e^x > 0$ for all x in $\mathbb{R}$, so exp is positive on $\mathbb{R}$.

4. exp is its own derivative—that is, if $f(x) = e^x$, then $f'(x) = e^x$.
   Since $e^x > 0$ for all x in $\mathbb{R}$, exp is increasing on $\mathbb{R}$.

5. $e^0 = 1$, $e^x > 1$ for all $x > 0$ and $e^x < 1$ for all $x < 0$;
   $e^{x+y} = e^x \times e^y$ for all x, y in $\mathbb{R}$.

6. For each positive integer n, $e^x/x^n \to \infty$ as $x \to \infty$.

7. $e^x \to \infty$ as $x \to \infty$ and
   $e^x \to 0$ as $x \to -\infty$.

These properties are explained and discussed in greater detail in Analysis Block A.

We sometimes express property 6 by saying that e^x *grows faster than any polynomial when x is large.*

The main aim of this section is to define and explore some new functions that involve the exponential function. The following exercise gives you some practice in manipulating exponential terms.

Exercise 4.1 Simplify each of the following expressions so that it involves no products or quotients.

(a) $e^x(e^x + e^{-x})$

(b) $(e^{2x} - e^{-2x})/e^x$

(c) $(e^{5x} + e^{-5x})(e^{5x} - e^{-5x})$

Certain combinations of e^x and e^{-x} appear so frequently in mathematics that it is useful to introduce functions that involve these combinations. In this section we use the exponential function to define the hyperbolic functions sinh, cosh and tanh, all with domain $\mathbb{R}$.

The name 'hyperbolic' originates from their use as parametric forms for a hyperbola. (See Section 5.)

- cosh is the *hyperbolic cosine function*, with rule $\cosh x = \dfrac{e^x + e^{-x}}{2}$;
- sinh is the *hyperbolic sine function*, with rule $\sinh x = \dfrac{e^x - e^{-x}}{2}$;
- tanh is the *hyperbolic tangent function*, with rule $\tanh x = \dfrac{\sinh x}{\cosh x}$.

Pronounce 'cosh' as it is spelled.

Pronounce 'sinh' as 'sinsh' or 'shine'.

Pronounce 'tanh' as 'tansh' or 'than' (as in 'thank').

Although the hyperbolic functions may seem to have no connection with the trigonometric functions, the similar names arise from various properties of these functions that are similar to those of trigonometric functions.

The next two exercises demonstrate some similarities between the hyperbolic functions cosh and sinh and the trigonometric functions cos and sin.

Exercise 4.2 Using the above definitions, prove the following.

(a) $\cosh^2 x - \sinh^2 x = 1$

(b) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

(c) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

(In parts (b) and (c), start from the right-hand side.)

Here $\cosh^2 x$ and $\sinh^2 x$ are abbreviations for $(\cosh x)^2$ and $(\sinh x)^2$, respectively.

Exercise 4.3 Find the derivatives of the functions $\cosh x$ and $\sinh x$, and compare your answers with the derivatives of $\cos x$ and $\sin x$.

As you might expect, we can also define three other hyperbolic functions:

$$\operatorname{sech} x = \frac{1}{\cosh x} \quad \text{just as} \quad \sec x = \frac{1}{\cos x},$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} \quad \text{just as} \quad \operatorname{cosec} x = \frac{1}{\sin x},$$

$$\coth x = \frac{1}{\tanh x} \quad \text{just as} \quad \cot x = \frac{1}{\tan x}.$$

Pronounce 'sech' as 'sesh' or 'sheck'.

Pronounce 'cosech' as 'co-sesh' or 'co-sheck'.

Pronounce 'coth' to rhyme with 'moth'.

You are now asked to investigate the properties of the function tanh.

Exercise 4.4 Let $f(x) = \tanh x$.

(a) Show that f is an odd function.

(b) Show that

$$f(x) = \frac{1 - e^{-2x}}{1 + e^{-2x}}.$$

(c) Show that $f'(x) = \operatorname{sech}^2 x$, and deduce that $f'(x) > 0$ for all x in $\mathbb{R}$.

For comparison, the derivative of tan is $\sec^2$.

The following table compares some identities satisfied by the hyperbolic functions with the corresponding identities for trigonometric functions. The hyperbolic identities are identical to the trigonometric ones, apart from a change of sign whenever a product of two sine terms occurs.

The functions tan, cosec and cot all contain a sine term.

Hyperbolic functions	**Trigonometric functions**
cosh is even: $\cosh(-x) = \cosh x$ sinh is odd: $\sinh(-x) = -\sinh x$ tanh is odd: $\tanh(-x) = -\tanh x$	cos is even: $\cos(-x) = \cos x$ sin is odd: $\sin(-x) = -\sin x$ tan is odd: $\tan(-x) = -\tan x$
$\cosh^2 x - \sinh^2 x = 1$ $1 - \tanh^2 x = \operatorname{sech}^2 x$ $\coth^2 x - 1 = \operatorname{cosech}^2 x$	$\cos^2 x + \sin^2 x = 1$ $1 + \tan^2 x = \sec^2 x$ $\cot^2 x + 1 = \operatorname{cosec}^2 x$
$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$ $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$ $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$	$\sin(x+y) = \sin x \cos y + \cos x \sin y$ $\cos(x+y) = \cos x \cos y - \sin x \sin y$ $\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$
$\sinh 2x = 2 \sinh x \cosh x$ $\cosh 2x = \cosh^2 x + \sinh^2 x$ $= 2\cosh^2 x - 1$ $= 1 + 2\sinh^2 x$ $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$	$\sin 2x = 2 \sin x \cos x$ $\cos 2x = \cos^2 x - \sin^2 x$ $= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ $\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

Finally, we ask you to use this table to obtain two further identities.

Exercise 4.5 Show that

(a) $\cosh 2x = \dfrac{1 + \tanh^2 x}{1 - \tanh^2 x}$;

(b) $\sinh 2x = \dfrac{2\tanh x}{1 - \tanh^2 x}$.

4.2 Graphs of hyperbolic functions

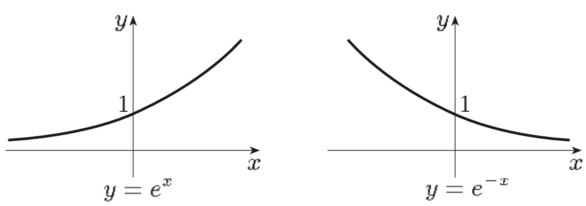
We now turn our attention to sketching the graphs of the hyperbolic functions. We shall see that they bear little or no resemblance to the graphs of the corresponding trigonometric functions.

Example 4.1 Sketch the graph of the function

$f(x) = \cosh x$.

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

Solution The graphs of $y = e^x$ and $y = e^{-x}$ are as follows.



Since $\cosh x = \frac{1}{2}(e^x + e^{-x})$, we have to 'take the average' of these graphs: for each value of x, the required value is halfway between the values for these graphs.

We use Strategy 2.1.

1. $f(x) = \cosh x$ has domain $\mathbb{R}$.
2. f is even, since

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^{x}}{2} = \cosh x, \quad \text{for all } x \text{ in } \mathbb{R}.$$

3. To find any x-intercepts of f we have to solve the equation

$$(e^{-x} + e^{x})/2 = 0.$$

However e^x and e^{-x} are positive for all x in $\mathbb{R}$, so

$\cosh x$ is positive for all x in $\mathbb{R}$.

This means that the entire graph lies above the x-axis, so it has no x intercepts.

$f(0) = (e^0 + e^{-0})/2 = (1+1)/2 = 1$, so the y-intercept is 1.

4. As shown in step 3, f is positive for all x in $\mathbb{R}$.
5. $f'(x) = \sinh x = \frac{1}{2}(e^x - e^{-x})$, which is positive when $x > 0$, negative when $x < 0$ and zero when $x = 0$, so

f is decreasing on the interval $(-\infty, 0)$;
f is increasing on the interval $(0, \infty)$;
f has a local minimum at 0, with value $\cosh(0) = 1$.

The graphs on page 41 illustrate the following properties:
$e^x > e^{-x}$, for $x > 0$,
$e^x < e^{-x}$, for $x < 0$.
We deduce that
$\sinh x > 0$, for $x > 0$,
$\sinh x < 0$, for $x < 0$.

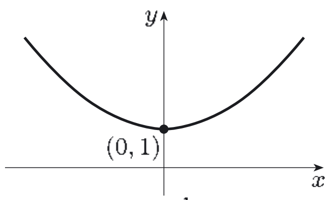
6. Since $e^x \to \infty$ as $x \to \infty$ and $e^{-x} \to 0$ as $x \to \infty$,

$\cosh x \to \infty \quad$ as $x \to \infty$,

and, since cosh is even,

$\cosh x \to \infty \quad$ as $x \to -\infty$.

This information enables us to produce the following sketch.



$y = \cosh x$

So the graph of the cosh function bears little resemblance to that of the cosine function; for example,

$\cosh x \geq 1, \quad$ for all x in $\mathbb{R}$,

whereas

$-1 \leq \cos x \leq 1, \quad$ for all x in $\mathbb{R}$.

Nor is there much similarity between the graphs of the sinh and tanh functions and those of the sine and tangent functions, as you will discover by working through the next two exercises.

Exercise 4.6 Sketch the graph of the function

$f(x) = \sinh x.$

Exercise 4.7 Use the results of Exercise 4.4 to sketch the graph of the function

$$f(x) = \tanh x.$$

Using the properties of the functions cosh and sinh, we can now sketch the graphs of their reciprocals, sech and cosech.

Example 4.2 Sketch the graph of the function

$$f(x) = \operatorname{sech} x = \frac{1}{\cosh x}.$$

Solution We use Strategy 2.1.

1. f has domain $\mathbb{R}$, since $\cosh x$ is never 0.
2. f is an even function, since $\cosh x$ is an even function.
3. $\cosh x \geq 1$ for all x in $\mathbb{R}$, so

   $$0 < \operatorname{sech} x \leq 1, \quad \text{for all } x \text{ in } \mathbb{R}.$$

   So f has no x-intercepts.

   $f(0) = \operatorname{sech} 0 = \frac{1}{\cosh 0} = \frac{1}{1} = 1$. So the y-intercept is 1.
4. From step 3, f is positive for all $x \in \mathbb{R}$.
5. Since cosh x is decreasing on $(-\infty, 0)$, and increasing on $(0, \infty)$,

   $$\operatorname{sech} x \text{ is increasing on } (-\infty, 0), \text{ and decreasing on } (0, \infty);$$

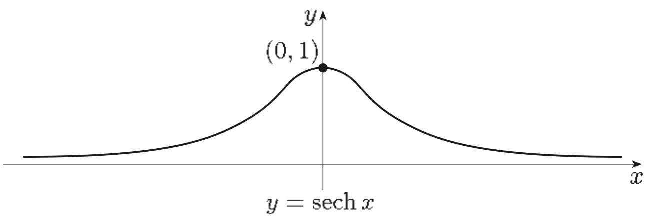
   thus

   $$\operatorname{sech} x \text{ has a local maximum at 0 with value } \operatorname{sech}(0) = 1.$$
6. Since $\cosh x \to \infty$ as $x \to \pm\infty$,

   $$\operatorname{sech} x \to 0 \text{ as } x \to \pm\infty.$$

   So $y = 0$ is a horizontal asymptote.

This information enables us to produce the following sketch.



Exercise 4.8 Sketch the graph of the function

$$f(x) = \operatorname{cosech} x.$$

Further exercises

Exercise 4.9 Prove that

$$\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$$

Exercise 4.10 Sketch the graph of the function $f(x) = \coth x$.