5 Curves from parameters

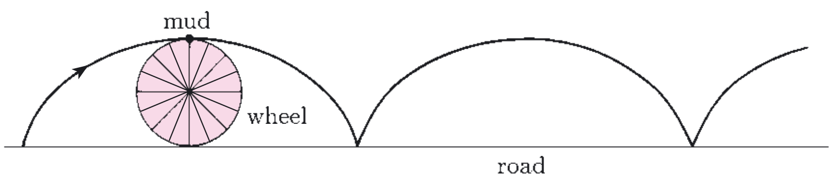
After working through this section, you should be able to:

(a) plot a curve that is specified by a *parametric* representation;
(b) obtain the equation of a curve that is specified by a parametric representation, in simple cases;
(c) sketch a conic whose equation is given in standard form.

5.1 Parametric equations

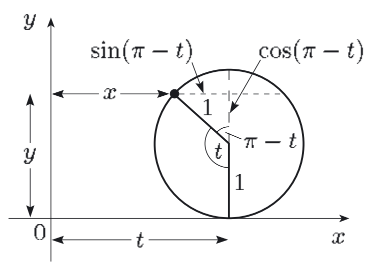
Suppose that you are wheeling a bicycle along a flat road. There is a piece of mud on the outside of one of the tyres. What curve does the piece of mud trace out as the bicycle moves?

Intuitively, we expect to obtain some sort of arch, such as the following, but what is its equation?



Such a curve is called a **cycloid**. Unfortunately, there is no simple formula that directly expresses the vertical coordinate y of a point on the cycloid in terms of the horizontal coordinate x. However, if we track the position of the mud at each moment of time, we can obtain x and y as functions of the time t taken to reach the point (x, y).

We assume for simplicity that the wheel has radius 1 and that the bicycle is being wheeled at a rate of 1 radian per second. Also, when $t = 0$, the mud is at the origin in the diagram. Then the time t (in seconds) is equal to the angle (in radians) between the radius from the mud to the wheel centre and the radius from the road to the wheel centre. From the diagram in the margin, we have



$$x = t - \sin(\pi - t) = t - \sin t,$$
$$y = 1 + \cos(\pi - t) = 1 - \cos t.$$

Thus, the coordinates (x, y) of the mud after t seconds are given by the pair of equations

$$x = t - \sin t, \quad y = 1 - \cos t. \qquad (5.1)$$

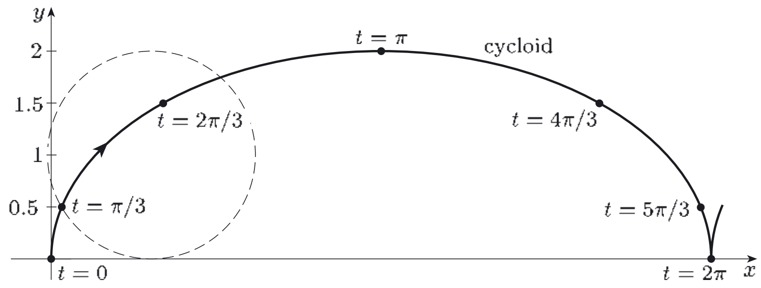
Using these equations, we can calculate the position of the mud at any given time, and draw up a table of approximate function values for the first revolution; for example, it follows from equations (5.1) that when $t = 4\pi/3$,

$$x = 4\pi/3 - \sin(4\pi/3) = 4\pi/3 - (-\sqrt{3}/2) \simeq 5.055,$$
$$y = 1 - \cos(4\pi/3) = 1 - (-1/2) = 1.5.$$

We thus obtain the following table.

Time t	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
Horizontal distance x	0	0.181	1.228	3.142	5.055	6.102	6.283
Vertical distance y	0	0.500	1.500	2.000	1.500	0.500	0

These points are indicated on the cycloid below.



Exercise 5.1 Verify the values of x and y in the table when $t = \pi/3$ and $t = \pi$, and calculate the points on the curve corresponding to $t = \pi/2$ and $t = \pi/6$.

We call the above pair of equations (5.1), giving the values of x and y in terms of t, *parametric equations* for the cycloid, and the variable t is called the *parameter*.

In general, **parametric equations** for a curve have the form

$$x = f(t), \quad y = g(t),$$

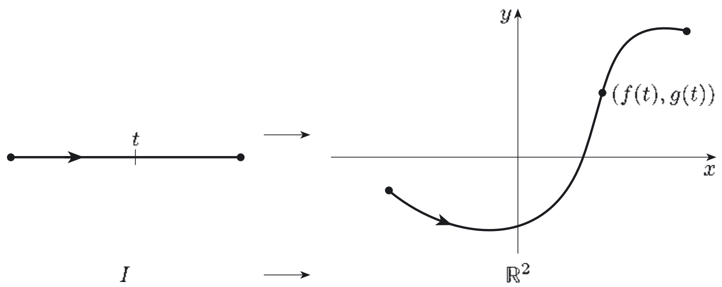
where f and g are real functions of the **parameter** t. Both f and g have the same domain, which is usually an interval I of the real line; for a single arch of the cycloid, an appropriate interval is $[0, 2\pi]$, as we saw above.

Moreover, we can use the functions f and g to define a new function α, whose domain is the interval I and whose rule is $t \longmapsto (f(t), g(t))$, where $t \in I$; the codomain of this function is usually taken to be the Cartesian xy-plane $\mathbb{R}^2$.

As the parameter t moves along the interval I, the image point

$$\alpha(t) = (f(t), g(t)), \quad \text{for } t \text{ in } I,$$

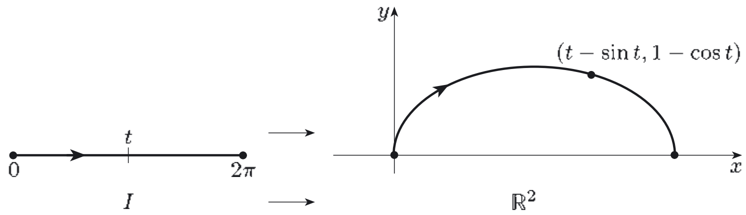
moves along a curve in the codomain $\mathbb{R}^2$.



The function α is called a **parametrisation** of the curve in $\mathbb{R}^2$. For example, the function α defined by

$$\alpha(t) = (t - \sin t, 1 - \cos t), \quad \text{for } t \text{ in } [0, 2\pi],$$

is a parametrisation of an arch of the cycloid (see the following diagram).



5.2 Parametrising conics and other curves

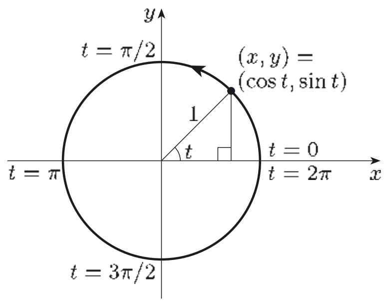
We have seen that parametric equations can be useful for specifying certain curves when there is no simple formula relating the coordinates x and y. They are also useful when the curve in question is not the graph of a function.

To see what we mean, consider the unit circle with equation $x^2 + y^2 = 1$. For a *function*, each value of x must give rise to a *unique* value of y, but here this is not the case; for example, when $x = 0$, then y can be either 1 or -1.

We can avoid this problem by the use of parametric equations.

Parametrising circles

In the diagram in the margin, (x, y) denotes a general point on the unit circle. We can express x and y in terms of the angle t (measured anticlockwise from the positive real axis) by the equations



$$x = \cos t, \quad y = \sin t. \tag{5.2}$$

As the angle t increases from 0 to 2π, the point (x, y) travels once round the circle anticlockwise, starting and ending at the point $(1, 0)$.

In the language of the previous subsection, t is a *parameter* and equations (5.2) are *parametric equations* for the unit circle, giving rise to the *parametrisation*

$$\alpha(t) = (\cos t, \sin t), \quad \text{for } t \text{ in } [0, 2\pi].$$

In this case, we can use the trigonometric identity

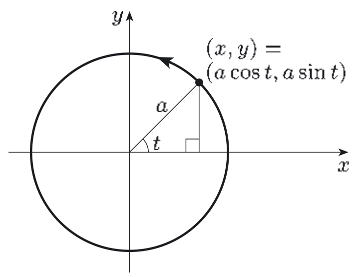
$$\cos^2 t + \sin^2 t = 1$$

to eliminate the parameter t from equations (5.2) and obtain the equation $x^2 + y^2 = 1$, as expected.

Exercise 5.2 Mark on a unit circle the coordinates of the points that correspond to the following values of the parameter t:

$$t = \pi/6, \quad t = \pi/2, \quad t = 3\pi/4, \quad t = \pi, \quad t = 3\pi/2, \quad t = 5\pi/3.$$

Similarly, we can write down a parametrisation for a circle of any radius, centred at the origin. Such a circle has equation $x^2 + y^2 = a^2$, where a is the radius of the circle.



For this circle, we use the parametrisation

$$\alpha(t) = (a \cos t, a \sin t), \quad \text{for } t \text{ in } [0, 2\pi];$$

this corresponds to the parametric equations

$$x = a \cos t, \quad y = a \sin t.$$

In this case, we can eliminate the parameter t by writing $x/a = \cos t$ and $y/a = \sin t$, and using the trigonometric identity $\cos^2 t + \sin^2 t = 1$; this gives the equation $x^2 + y^2 = a^2$, as expected.

Exercise 5.3 Write down a parametrisation for each of the following:

(a) the circle centred at the origin, with radius 3;

(b) the circle with centre $(2, 1)$ and radius 3.

Exercise 5.4 Show that another parametrisation for the unit circle is

$$x = \cos 2\pi t, \quad y = \sin 2\pi t, \quad \text{for } t \text{ in } [0, 1].$$

We now have two different parametrisations for the unit circle traversed once anticlockwise—namely

$$\alpha(t) = (\cos t, \sin t), \quad \text{for } t \text{ in } [0, 2\pi],$$

and

$$\alpha(t) = (\cos 2\pi t, \sin 2\pi t), \quad \text{for } t \text{ in } [0, 1].$$

This illustrates the important fact that a parametrisation of a given curve is not unique. Different parametrisations of a curve may correspond to different modes of traversing the curve.

Parametrising lines

The line through the points $(0, 0)$ and (p, q), where at least one of p, q is not zero, has equation

$$py = qx;$$

or, when $p \neq 0$,

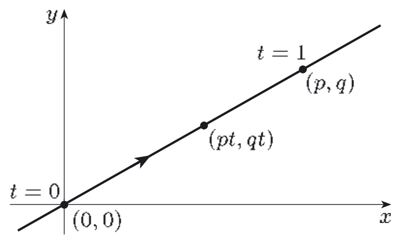
$$y = (q/p)x.$$

For this line, we use the parametrisation

$$\alpha(t) = (pt, qt), \quad \text{for } t \text{ in } \mathbb{R};$$

this corresponds to the parametric equations

$$x = pt, \quad y = qt.$$



The line segment between the two given points corresponds to values of t between 0 and 1.

When $p \neq 0$, we can eliminate the parameter t by writing

$$y = \frac{q}{p}\,pt = \frac{q}{p}x;$$

this gives the equation $y = (q/p)x$, as expected.

When $p = 0$, the parametric equations become

$$x = 0, \quad y = qt,$$

giving the equation $x = 0$.

More generally, the line through the points (p, q) and (r, s), where $r \neq p$, has equation

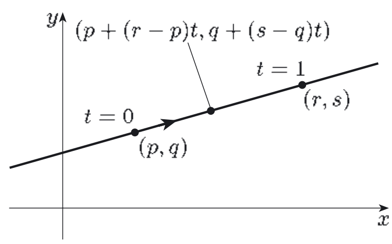
$$y - q = \frac{s - q}{r - p}(x - p).$$

For this line, we use the parametrisation

$$\alpha(t) = (p + (r - p)t, q + (s - q)t), \quad \text{for } t \text{ in } \mathbb{R};$$

this gives the parametric equations

$$x = p + (r - p)t, \quad y = q + (s - q)t.$$



We can eliminate the parameter t to retrieve the equation of the line; we omit the details.

Exercise 5.5

(a) Write down a parametrisation for the line through the points $(1, 2)$ and $(3, 6)$.

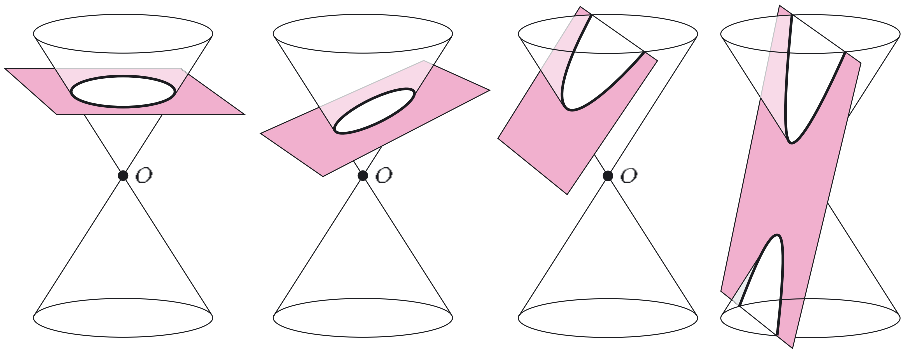
(b) Which values of the parameter t correspond to the points $(2, 4)$, $(7, 14)$ and $(0, 0)$?

Exercise 5.6 Show that we can also parametrise the line through the points $(0, 0)$ and (p, q), where $p \neq 0$, by using the parametric equations

$$x = pt^3, \quad y = qt^3.$$

Parametrising conics

The curves that can be obtained by cutting a plane section through a cone are called *conic sections* or *conics*.



These curves are of three types:

- ellipses,
- parabolas,
- hyperbolas.

Here we present the standard and parametric forms of the equations of these *non-degenerate* conics, together with their main features.

If the plane section passes through the origin, it can also produce a point, a single line and a pair of lines. These are called *degenerate* conics.

A circle is a special case of an ellipse.

Ellipse

The equation of an **ellipse** in standard form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where } a \geq b > 0.$$

When $b = a$, the equation reduces to that of the circle centred at the origin, with radius a, discussed above.

The ellipse has centre $(0, 0)$ and crosses the axes at the points $(a, 0)$, $(0, b)$, $(-a, 0)$ and $(0, -b)$.

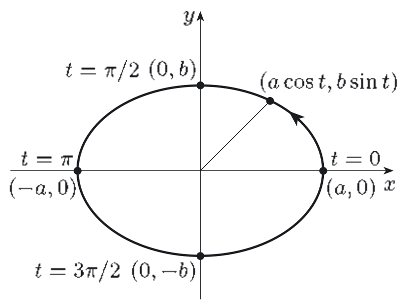
It is unchanged under a rotation through π about the origin, or under a reflection in either of the coordinate axes.

For the ellipse, we use the parametrisation

$$\alpha(t) = (a \cos t, b \sin t), \quad \text{for } t \text{ in } [0, 2\pi];$$

this gives the parametric equations

$$x = a \cos t, \quad y = b \sin t.$$



In this case, we can eliminate the parameter t by writing

$$\frac{x}{a} = \cos t, \quad \frac{y}{b} = \sin t,$$

and using the trigonometric identity $\cos^2 t + \sin^2 t = 1$.

This gives the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

as expected.

Parabola

The equation of a **parabola** in standard form is

$$y^2 = 4ax, \quad \text{where } a > 0.$$

The vertex of the parabola is at $(0, 0)$.

The parabola is unchanged under a reflection in the x-axis.

For the parabola, we use the parametrisation

$$\alpha(t) = (at^2, 2at), \quad \text{for } t \text{ in } \mathbb{R};$$

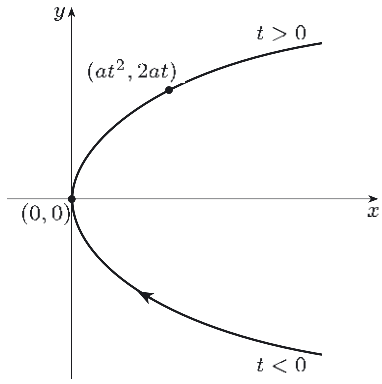
this gives the parametric equations

$$x = at^2, \quad y = 2at.$$

In this case, we can eliminate the parameter t by writing

$$y^2 = (2at)^2 = 4a^2t^2 = 4a \times at^2 = 4ax;$$

this gives $y^2 = 4ax$, as expected.



Hyperbola

The equation of a **hyperbola** in standard form is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{where } a > 0 \text{ and } b > 0.$$

The hyperbola has centre $(0, 0)$ and crosses the x-axis at the points $(a, 0)$ and $(-a, 0)$.

It is unchanged under a rotation through π about the origin, or under a reflection in either of the coordinate axes.

For the hyperbola, we use the parametrisation

$$\alpha(t) = (a \sec t, b \tan t), \quad \text{for } t \text{ in } [-\pi, \pi], \text{ excluding } -\pi/2 \text{ and } \pi/2;$$

this gives the parametric equations

$$x = a \sec t, \quad y = b \tan t.$$

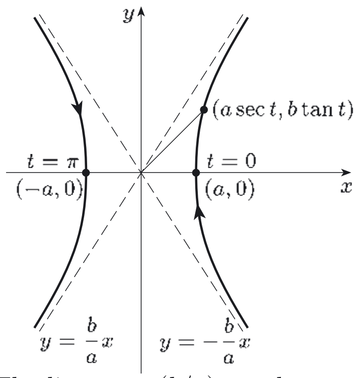
In this case, we can eliminate the parameter t by writing

$$\frac{x}{a} = \sec t, \quad \frac{y}{b} = \tan t,$$

and using the trigonometric identity $\sec^2 t - \tan^2 t = 1$; this gives the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

as expected.



The lines $y = (b/a)x$ and $y = -(b/a)x$ are both asymptotes for the hyperbola. The asymptotes help us to sketch the graph of the hyperbola in the correct position, as described in Sections 1 and 2.

Exercise 5.7 Using the information given above, sketch the following conics.

(a) $y^2 = 2x$ (b) $\dfrac{x^2}{3} + \dfrac{y^2}{2} = 1$ (c) $\dfrac{x^2}{3} - \dfrac{y^2}{2} = 1$

Write down a parametrisation for each conic.

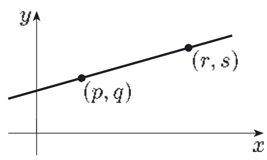
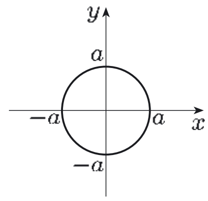
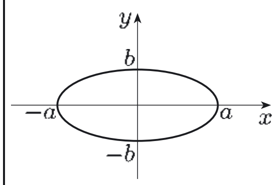
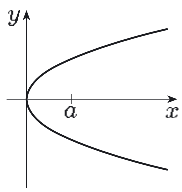
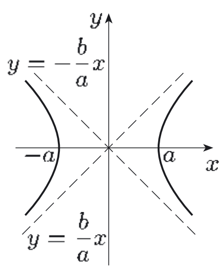
Exercise 5.8 Show that points on the curve with parametrisation

$$\alpha(t) = (a\cosh t, b\sinh t), \quad \text{for } t \text{ in } \mathbb{R},$$

lie on the right-hand half of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

This gives another parametrisation for the hyperbola. However, since $a\cosh t > 0$, we obtain only the right-hand half of the hyperbola (corresponding to $x > 0$).

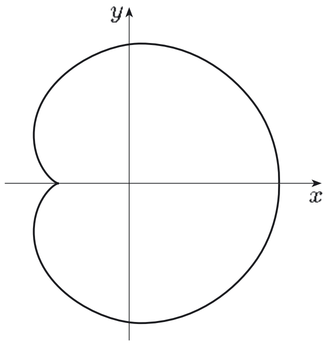
In the following table we summarise the standard parametrisations for lines and conics.

Line through (p, q) and (r, s) $y - q = \dfrac{s-q}{r-p}(x-p)$	$\alpha(t) = (p + (r-p)t, q + (s-q)t)$, for t in $\mathbb{R}$	
Circle centre $(0, 0)$, radius a $x^2 + y^2 = a^2$	$\alpha(t) = (a\cos t, a\sin t)$, for t in $[0, 2\pi]$	
Ellipse in standard form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$\alpha(t) = (a\cos t, b\sin t)$, for t in $[0, 2\pi]$	
Parabola in standard form $y^2 = 4ax$	$\alpha(t) = (at^2, 2at)$, for t in $\mathbb{R}$	
Hyperbola in standard form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$\alpha(t) = (a\sec t, b\tan t)$, for t in $[-\pi, \pi]$, excluding $-\pi/2$ and $\pi/2$ or $\alpha(t) = (a\cosh t, b\sinh t)$, for t in $\mathbb{R}$ (right-hand half only)	

Finally, we give the standard parametrisation for two, more exotic, curves.

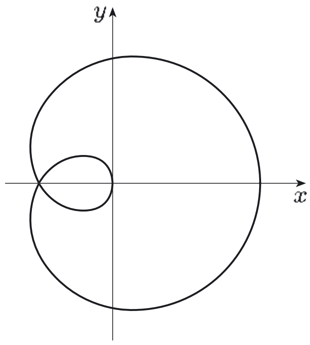
For each of these curves, it is possible, with some effort, to eliminate the parameter t to obtain a single equation involving x and y that is satisfied by all points on the curve; for example, all the points on the cardioid satisfy a polynomial equation in x and y of degree 4. However, the parametric equations for these curves are much more useful, and for many other curves no single equation can be found.

cardioid



$x = 2\cos t + \cos 2t$
$y = 2\sin t + \sin 2t$

trisectrix



$x = \cos t + \cos 2t$
$y = \sin t + \sin 2t$

Further exercises

Exercise 5.9 Identify the curves described by the following parametric equations.

(a) $x = t, \quad y = 1/t$.

(b) $x = t - 1, \quad y = 4 - 3t$.

(c) $x = 2t, \quad y = 1 + 3t^2$.

Exercise 5.10 Consider the parametrisation

$$\alpha(t) = (2\cos t + \cos 2t,\ 2\sin t + \sin 2t).$$

(a) Calculate

$$\alpha(0), \quad \alpha(\tfrac{1}{6}\pi), \quad \alpha(\tfrac{1}{3}\pi), \quad \alpha(\tfrac{1}{2}\pi), \quad \alpha(\tfrac{2}{3}\pi), \quad \alpha(\tfrac{5}{6}\pi), \quad \alpha(\pi).$$

(b) Show that

the function $f(t) = 2\cos t + \cos 2t$ is even,
the function $g(t) = 2\sin t + \sin 2t$ is odd.

What can you deduce about the curve with parametrisation $\alpha(t)$?

(c) Use parts (a) and (b) to sketch the curve with parametrisation $\alpha(t)$.