Solutions to the exercises

1.1 **(a)** The denominator of $f(x)$ is $1 - x^2$, which is zero when $x = 1$ and -1, so the domain is the set of all real numbers, excluding 1 and -1.

(b) This function is defined for all real numbers, so the domain is $\mathbb{R}$.

(c) The denominator of $f(x)$ is $x^2 + 5x + 4 = (x + 1)(x + 4)$, which is zero when $x = -1$ and -4, so the domain is the set of all real numbers, excluding -1 and -4.

(d) The denominator of $f(x)$ is $\sqrt{1 - x^2}$, which is zero when $x = 1$ and -1, and is not defined when $x^2 > 1$, that is, when $x > 1$ and $x < -1$, so the domain is the set of all real numbers strictly between -1 and 1.

1.2 **(a)** $(-\infty, -1), \quad (-1, 1), \quad (1, \infty)$

(b) $\mathbb{R} = (-\infty, \infty)$

(c) $(-\infty, -4), \quad (-4, -1), \quad (-1, \infty)$

(d) $(-1, 1)$

2.1 **(a)** In this case $a = 1$, so

$$\begin{aligned} x^2 - 6x + 11 &= (x^2 - 6x) + 11 \\ &= (x - 3)^2 - 9 + 11 \\ &= (x - 3)^2 + 2 \end{aligned}$$

which is always positive.

(b) In this case $a = 3$, so

$$\begin{aligned} 3x^2 + 12x - 1 &= 3(x^2 + 4x) - 1 \\ &= 3(x + 2)^2 - 12 - 1 \\ &= 3(x + 2)^2 - 13 \end{aligned}$$

which is sometimes positive and sometimes negative (for example, positive when $x = 1$ and negative when $x = 0$).

2.2 $f(x) \to 0$ as $x \to -\infty$;
$f(x) \to 0$ as $x \to \infty$;
$f(x) \to \infty$ as $x \to 1^-$;
$f(x) \to -\infty$ as $x \to 1^+$.

2.3 **(a)** $f(x) \to \infty$ as $x \to \infty$;
$f(x) \to \infty$ as $x \to -\infty$.

(b) $f(x) \to \infty$ as $x \to \infty$;
$f(x) \to -\infty$ as $x \to -\infty$.

(c) $f(x) \to -\infty$ as $x \to \infty$;
$f(x) \to -\infty$ as $x \to -\infty$.

(d) $f(x) \to -\infty$ as $x \to \infty$;
$f(x) \to \infty$ as $x \to -\infty$.

2.4 $f(x) = x^4 - 2x^2 + 3$.

1. The domain of f is $\mathbb{R}$.

2. f is even, since, for all x in $\mathbb{R}$,

$$\begin{aligned} f(-x) &= (-x)^4 - 2(-x)^2 + 3 \\ &= x^4 - 2x^2 + 3 = f(x). \end{aligned}$$

3. Using the hint to complete the square shows that $f(x) = (x^2 - 1)^2 + 2 \geq 2$ for all x in $\mathbb{R}$, so f is positive on $\mathbb{R}$. This means that f has no x-intercepts, as $f(x)$ is never zero. The y-intercept is $f(0) = 3$.

4. By step 3, f is positive on $\mathbb{R}$.

5. $$\begin{aligned} f'(x) &= 4x^3 - 4x \\ &= 4x(x^2 - 1) \\ &= 4x(x - 1)(x + 1). \end{aligned}$$

We construct a sign table for $f'(x)$.

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, \infty)$
$4x$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x - 1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x + 1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$-$	0	$+$	0	$-$	0	$+$

From the table, we find that

f is increasing on the intervals $(-1, 0)$ and $(1, \infty)$;

f is decreasing on the intervals $(-\infty, -1)$ and $(0, 1)$;

$f'(x) = 0$ when $x = -1$, 0 and 1.

So

f has stationary points at $x = -1, 0$ and 1.

We deduce that

there is a local minimum at $x = -1$ with $f(-1) = 2$;

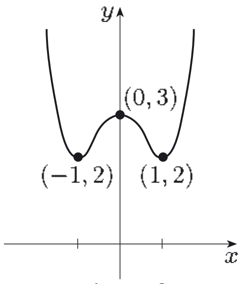
there is a local maximum at $x = 0$ with $f(0) = 3$;

there is a local minimum at $x = 1$ with $f(1) = 2$.

6. The polynomial has degree 4 (even), and the coefficient of x^4 is positive, so

$f(x) \to \infty \quad$ as $x \to \infty$,
$f(x) \to \infty \quad$ as $x \to -\infty$.

This information enables us to sketch the graph.



$y = x^4 - 2x^2 + 3$

2.5 $f(x) = \dfrac{x-3}{2-x}$.

1. The domain of f is $\mathbb{R}$, excluding 2.

2. f is neither even nor odd, since its domain is not symmetric about the origin.

3. $f(x) = 0$ only when $x = 3$, so the only x-intercept is 3.

The y-intercept is $f(0) = -\frac{3}{2}$.

4. We construct a sign table for $f(x)$.

x	$(-\infty, 2)$	2	$(2, 3)$	3	$(3, \infty)$
$x - 3$	$-$	$-$	$-$	0	$+$
$2 - x$	$+$	0	$-$	$-$	$-$
$f(x)$	$-$	$*$	$+$	0	$-$

Thus

f is positive on the interval $(2, 3)$;

f is negative on the intervals $(-\infty, 2)$ and $(3, \infty)$.

5. $f'(x) = \dfrac{(2-x) + (x-3)}{(2-x)^2} = \dfrac{-1}{(2-x)^2}$,

so $f'(x) < 0$ for all x in the domain; that is, f is decreasing on each interval of its domain.

6. The denominator is 0 when $x = 2$, so

the line $x = 2$ is a vertical asymptote.

Also, by step 4,

$f(x) \to -\infty \quad \text{as } x \to 2^-$,

$f(x) \to \infty \quad \text{as } x \to 2^+$.

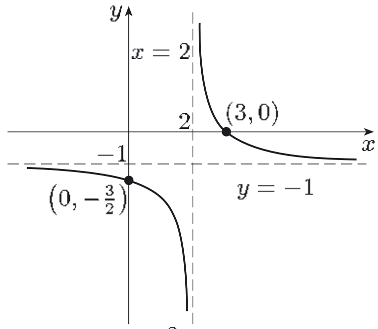
Dividing both numerator and denominator of $f(x)$ by x, we obtain

$$f(x) = \frac{1 - 3/x}{2/x - 1} \to \frac{1-0}{0-1} = -1 \quad \text{as } x \to \pm\infty,$$

so

the line $y = -1$ is a horizontal asymptote.

This information enables us to sketch the graph.



$y = \dfrac{x-3}{2-x}$

2.6 $f(x) = \dfrac{4x+1}{3x-5}$.

1. The domain of f is $\mathbb{R}$, excluding $\frac{5}{3}$.

2. f is neither even nor odd, since its domain is not symmetric about the origin.

3. $f(x) = 0$ when $x = -\frac{1}{4}$, so the x-intercept is $-\frac{1}{4}$. The y-intercept is $f(0) = -\frac{1}{5}$.

4. We construct a sign table for $f(x)$.

x	$(-\infty, -\frac{1}{4})$	$-\frac{1}{4}$	$(-\frac{1}{4}, \frac{5}{3})$	$\frac{5}{3}$	$(\frac{5}{3}, \infty)$
$4x + 1$	$-$	0	$+$	$+$	$+$
$3x - 5$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	0	$-$	$*$	$+$

Thus

f is positive on the intervals $(-\infty, -\frac{1}{4})$ and $(\frac{5}{3}, \infty)$;

f is negative on the interval $(-\frac{1}{4}, \frac{5}{3})$.

5. $f'(x) = \dfrac{(3x-5)4 - (4x+1)3}{(3x-5)^2} = \dfrac{-23}{(3x-5)^2}$,

so $f'(x) < 0$ for all x in the domain; that is, f is decreasing on each interval of its domain.

6. The denominator is 0 when $x = \frac{5}{3}$, so

the line $x = \frac{5}{3}$ is a vertical asymptote.

Also, by step 4,

$f(x) \to -\infty \quad \text{as } x \to \frac{5}{3}^-$,

$f(x) \to \infty \quad \text{as } x \to \frac{5}{3}^+$.

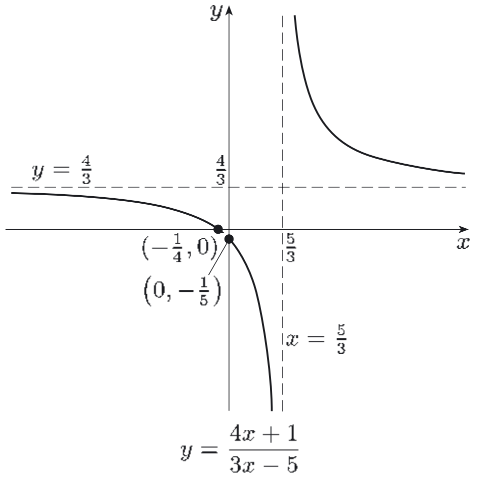
Dividing both numerator and denominator of $f(x)$ by x, we obtain

$$f(x) = \frac{4 + 1/x}{3 - 5/x} \to \tfrac{4}{3} \quad \text{as } x \to \pm\infty,$$

so

the line $y = \frac{4}{3}$ is a horizontal asymptote.

This information enables us to sketch the graph.



2.7 $f(x) = \dfrac{1}{x(x+1)^2}$.

1. The domain of f is $\mathbb{R}$, excluding 0 and -1; it consists of the intervals $(-\infty,-1)$, $(-1,0)$ and $(0,\infty)$.

2. f is neither even nor odd, since the domain is not symmetric about the origin.

3. f has no x-intercepts.

$f(0)$ is not defined, so there is no y-intercept.

4. We construct a sign table for f.

x	$(\infty,-1)$	-1	$(-1,0)$	0	$(0,\infty)$
x	$-$	$-$	$-$	0	$+$
$(x+1)^2$	$+$	0	$+$	$+$	$+$
$f(x)$	$-$	$*$	$-$	$*$	$+$

Thus

f is positive on the interval $(0,\infty)$;

f is negative on the intervals $(-\infty,-1)$ and $(-1,0)$.

5. $$\begin{aligned} f'(x) &= -\frac{(x+1)^2 + 2x(x+1)}{x^2(x+1)^4} \\ &= -\frac{(x+1)(x+1+2x)}{x^2(x+1)^4} \\ &= -\frac{3x+1}{x^2(x+1)^3}, \end{aligned}$$

so

$f'(x) = 0$ when $x = -\frac{1}{3}$.

We construct a sign table for $f'(x)$.

x		-1	$(-1,-\frac{1}{3})$	$-\frac{1}{3}$	$(-\frac{1}{3},0)$	0	
$-(3x+1)$	$+$	$+$	$+$	0	$-$	$-$	$-$
x^2	$+$	$+$	$+$	$+$	$+$	0	$+$
$(x+1)^3$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$-$	$*$	$+$	0	$-$	$*$	$-$

We deduce that

f is increasing on the interval $(-1,-\frac{1}{3})$;

f is decreasing on the intervals $(-\infty,-1)$, $(-\frac{1}{3},0)$ and $(0,\infty)$;

f has a stationary point at $x = -\frac{1}{3}$.

The point $x = -\frac{1}{3}$ is a local maximum with $f(-\frac{1}{3}) = -\frac{27}{4}$.

6. The denominator is 0 when $x = 0$ or $x = -1$, so

the lines $x = 0$ and $x = -1$ are vertical asymptotes.

Also, by step 4,

$f(x) \to -\infty$ as $x \to -1^-$,
$f(x) \to -\infty$ as $x \to -1^+$;
$f(x) \to -\infty$ as $x \to 0^-$,
$f(x) \to \infty$ as $x \to 0^+$.

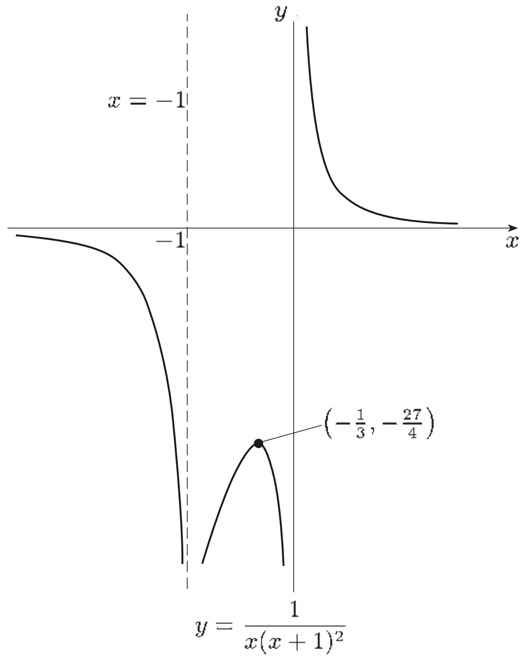
Also,

$f(x) \to 0$ as $x \to \pm\infty$,

so

the line $y = 0$ is a horizontal asymptote.

This information enables us to sketch the graph.



2.8 $f(x) = \frac{1}{5}x^5 - x^3$.

1. The domain of f is $\mathbb{R}$.

2. f is odd, since, for all x in $\mathbb{R}$,
$$\begin{aligned} f(-x) &= \tfrac{1}{5}(-x)^5 - (-x)^3 \\ &= -(\tfrac{1}{5}x^5 - x^3) = -f(x). \end{aligned}$$

3. $f(x) = \frac{1}{5}x^5 - x^3 = x^3(\frac{1}{5}x^2 - 1)$, so

$f(x) = 0$ when $x = 0$ and $x = \pm\sqrt{5}$.

So the x-intercepts are 0 and $\pm\sqrt{5}$, and the y-intercept is 0.

4. We construct a sign table for $f(x)$.

x		$-\sqrt{5}$	$(-\sqrt{5},0)$	0	$(0,\sqrt{5})$	$\sqrt{5}$	
x^3	$-$	$-$	$-$	0	$+$	$+$	$+$
$\frac{1}{5}x^2-1$	$+$	0	$-$	$-$	$-$	0	$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

Thus

f is positive on the intervals $(-\sqrt{5},0)$ and $(\sqrt{5},\infty)$;

f is negative on the intervals $(-\infty,-\sqrt{5})$ and $(0,\sqrt{5})$.

5. $f'(x)=x^4-3x^2=x^2(x^2-3)$, so

$f'(x)=0$ when $x=0$ and $\pm\sqrt{3}$;

$f'(x)>0$ when $x^2>3$;

$f'(x)<0$ when $x^2<3$.

Thus

f is increasing on the intervals $(-\infty,-\sqrt{3})$ and $(\sqrt{3},\infty)$;

f is decreasing on the intervals $(-\sqrt{3},0)$ and $(0,\sqrt{3})$;

f has stationary points at $x=-\sqrt{3},\ 0$ and $\sqrt{3}$.

We deduce that

there is a local maximum at $x=-\sqrt{3}$ with $f(-\sqrt{3})=\frac{6}{5}\sqrt{3}$;

there is a local minimum at $x=\sqrt{3}$; with $f(\sqrt{3})=-\frac{6}{5}\sqrt{3}$;

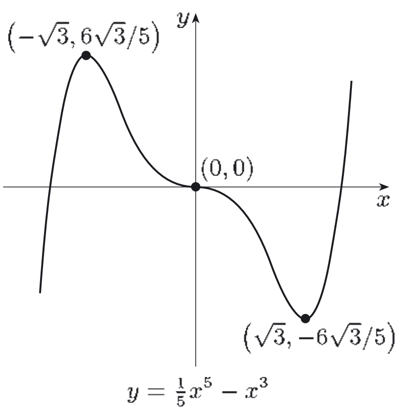
there is a horizontal point of inflection at $x=0$.

6. The polynomial has degree 5 (odd), and the coefficient of x^5 is positive, so

$f(x)\to\infty$ as $x\to\infty$,

$f(x)\to-\infty$ as $x\to-\infty$.

This information enables us to sketch the graph.



2.9 $f(x)=\dfrac{4x+3}{x-7}$.

1. The domain of f is $\mathbb{R}$, excluding 7.

2. f is neither even nor odd, since its domain is not symmetric about the origin.

3. $f(x)=0$ when $x=-\frac{3}{4}$, so the x-intercept is $-\frac{3}{4}$. The y-intercept is $f(0)=-\frac{3}{7}$.

4. We construct a sign table for $f(x)$.

x	$(-\infty,-\frac{3}{4})$	$-\frac{3}{4}$	$(-\frac{3}{4},7)$	7	$(7,\infty)$
$4x+3$	$-$	0	$+$	$+$	$+$
$x-7$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	0	$-$	$*$	$+$

Thus

f is positive on the intervals $(-\infty,-\frac{3}{4})$ and $(7,\infty)$;

f is negative on the interval $(-\frac{3}{4},7)$.

5. $f'(x)=\dfrac{(x-7)4-(4x+3)}{(x-7)^2}=\dfrac{-31}{(x-7)^2}$,

so $f'(x)<0$ for all x in the domain.

Thus f is decreasing on each interval of its domain.

6. The denominator is 0 when $x=7$, so

the line $x=7$ is a vertical asymptote.

Also, from step 4,

$f(x)\to-\infty$ as $x\to 7^-$,

$f(x)\to\infty$ as $x\to 7^+$.

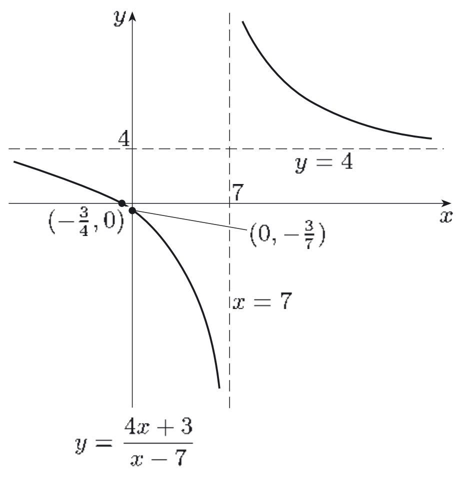
Dividing both numerator and denominator of $f(x)$ by x, we obtain

$$f(x)=\frac{4+3/x}{1-7/x}\to 4 \quad \text{as } x\to\pm\infty,$$

so

the line $y=4$ is a horizontal asymptote.

This information enables us to sketch the graph.



2.10 $f(x)=\dfrac{2x}{x^2+x-2}$.

1. We factorise $f(x)$ as follows:

$$f(x)=\frac{2x}{(x-1)(x+2)}.$$

Thus the domain of f is $\mathbb{R}$, excluding 1 and -2; it consists of the intervals $(-\infty,-2)$, $(-2,1)$ and $(1,\infty)$.

2. f is neither even nor odd, since its domain is not symmetric about the origin.

3. $f(x) = 0$ when $x = 0$, so 0 is both the x-intercept and the y-intercept.

4. We construct a sign table for $f(x)$.

x	$(-\infty,-2)$	-2	$(-2,0)$	0	$(0,1)$	1	$(1,\infty)$
$2x$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x+2$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f(x)$	$-$	$*$	$+$	0	$-$	$*$	$+$

Thus

f is positive on the intervals $(-2, 0)$ and $(1, \infty)$;

f is negative on the intervals $(-\infty, -2)$ and $(0, 1)$.

5. $$f'(x) = \frac{(x^2+x-2)2 - 2x(2x+1)}{(x^2+x-2)^2} = \frac{-2x^2-4}{(x^2+x-2)^2} = \frac{-2(x^2+2)}{(x^2+x-2)^2},$$
so $f'(x) < 0$ for all x in the domain. Thus f is decreasing on each interval of its domain.

6. The denominator is 0 when $x = 1$ and $x = -2$, so

the lines $x = -2$ and $x = 1$ are vertical asymptotes.

Also, from step 4,

$f(x) \to -\infty$ as $x \to -2^-$,

$f(x) \to \infty$ as $x \to -2^+$,

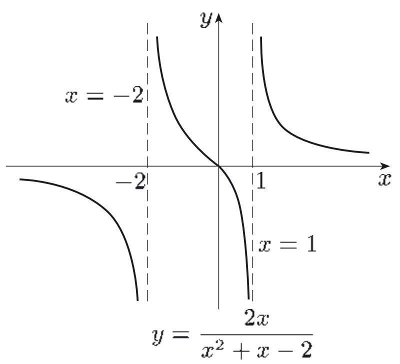
$f(x) \to -\infty$ as $x \to 1^-$,

$f(x) \to \infty$ as $x \to 1^+$.

Dividing both numerator and denominator of $f(x)$ by the dominant term of the denominator, x^2, we obtain
$$f(x) = \frac{2/x}{1 + 1/x - 2/x^2} \to 0 \quad \text{as } x \to \pm\infty,$$
so

the line $y = 0$ is a horizontal asymptote.

This information enables us to sketch the graph.



2.11 $f(x) = \dfrac{1}{\sqrt{1+x^2}}$.

1. The domain of f is $\mathbb{R}$.

2. f is even, since, for all x in $\mathbb{R}$,
$$f(-x) = \frac{1}{\sqrt{1+(-x)^2}} = \frac{1}{\sqrt{1+x^2}} = f(x).$$

3. $f(x) = 0$ has no solution, so there are no x-intercepts.

The y-intercept is $f(0) = 1$.

4. f is positive on $\mathbb{R}$.

5. f is decreasing on the interval $(0, \infty)$;
f is increasing on the interval $(-\infty, 0)$.

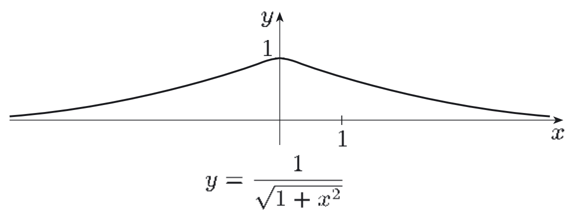
We deduce that there is a local maximum at $x = 0$ with $f(0) = 1$.

6. $f(x) \to 0$ as $x \to \pm\infty$, so

the line $y = 0$ is a horizontal asymptote.

In order to obtain more information about the shape of the curve, we could find another point on the graph, for example $f(1) = 1/\sqrt{2} = \sqrt{2}/2 \simeq 0.707$.

This information enables us to sketch the graph.



3.1 $f(x) = x \cos x$.

1. The function f has domain $\mathbb{R}$.

2. The function f is odd, since
$-x\cos(-x) = -x\cos x$ for all x in $\mathbb{R}$.

It is therefore sufficient initially to consider the features of $f(x)$ only for $x \geq 0$, and then to rotate the graph we obtain about the origin.

3. $f(x) = 0$ when $x = 0$ and when $\cos x = 0$; that is, when $x = 0, \pi/2, 3\pi/2, \ldots$. So the x-intercepts are $0, \pi/2, 3\pi/2, \ldots$ and the y-intercept is 0.

4. For $x > 0$, the intervals on which f is positive or negative alternate between the zeros in the same way as for the cosine function. That is,
f is positive on
$(0, \pi/2), (3\pi/2, 5\pi/2), (5\pi/2, 7\pi/2), \ldots,$
f is negative on
$(\pi/2, 3\pi/2), (5\pi/2, 7\pi/2), \ldots$.

5. $f'(x) = \cos x - x \sin x$, so we omit solving $f'(x) = 0$, as it is not easy.

6. The function has no asymptotes.

7. $-1 \leq \cos x \leq 1$ for all real numbers x, so we have

$-x \leq x\cos x \leq x$ for $x > 0$,

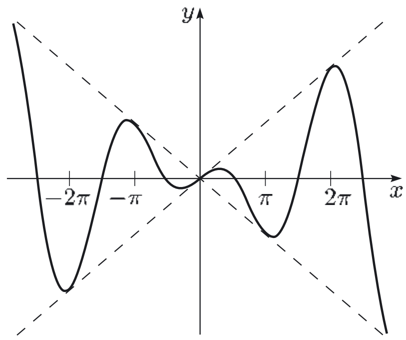
$-x \geq x\cos x \geq x$ for $x < 0$.

These inequalities tell us that

$$-|x| \le f(x) \le |x|, \quad \text{for all real numbers } x,$$

so the graph of f lies between the graphs of the functions $x \longmapsto |x|$ and $x \longmapsto -|x|$. These graphs are the construction lines for this function.

This information enables us to sketch the graph.



$y = x\cos x$

3.2 $f(x) = x + \sin x$

1. The function f has domain $\mathbb{R}$, since both x and $\sin x$ have domain $\mathbb{R}$.

2. $x + \sin x$ is odd, since for all x in $\mathbb{R}$,

$$-x + \sin(-x) = -(x + \sin x).$$

3. $f(0) = 0$, so 0 is both the x-intercept and the y-intercept. There are no other values of x for which $f(x) = 0$.

4. $f(x) \ge 0$ when $x > 0$. Since f is odd, then f is negative for $x < 0$.

5. $f'(x) = 1 + \cos x$, so $f'(x) = 0$ when $\cos x = -1$, that is, when $x = (2k+1)\pi$, for any integer k. At all other points in $\mathbb{R}$, $f'(x) > 0$, so f is increasing on $\mathbb{R}$. It has stationary points where $x = (2k+1)\pi$, but since f is increasing on $\mathbb{R}$ they are neither maxima nor minima, but are horizontal points of inflection.

6. The function has no asymptotes.

7. $-1 \le \sin x \le 1$, for all x in $\mathbb{R}$, so

$$x - 1 \le x + \sin x \le x + 1, \quad \text{for all } x \text{ in } \mathbb{R}.$$

So the graph of f lies between the lines $y = x - 1$ and $y = x + 1$.

Also, for all integers k,

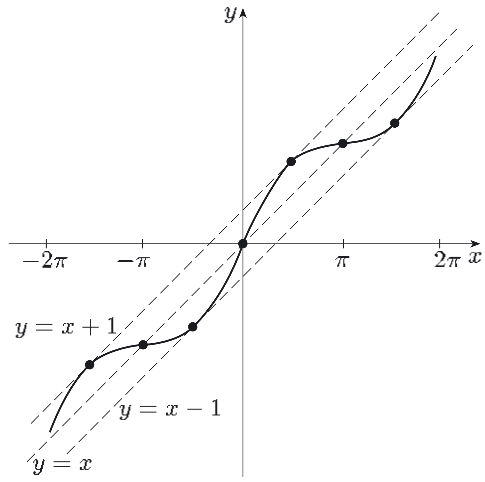
when $x = k\pi$, $\sin x = 0$,
so $f(x) = x$;

when $x = (2k + \frac{1}{2})\pi$, $\sin x = 1$,
so $f(x) = x + 1$;

when $x = (2k + \frac{3}{2})\pi$, $\sin x = -1$,
so $f(x) = x - 1$.

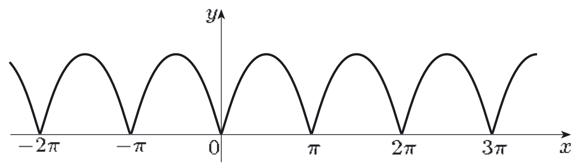
So $y = x - 1$, $y = x + 1$ and $y = x$ can be used as construction lines.

This information enables us to sketch the graph.



$y = x + \sin x$

3.3 Here it is sufficient to notice that the modulus function maps any negative value of $\sin x$ to the corresponding positive value. We therefore obtain the following graph.

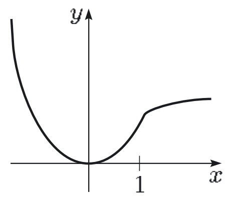


$y = |\sin x|$

There are local maxima with value 1 at $x = \frac{1}{2}(2k+1)\pi$, for any integer k;
there are local minima with value 0 at $x = k\pi$, for any integer k.

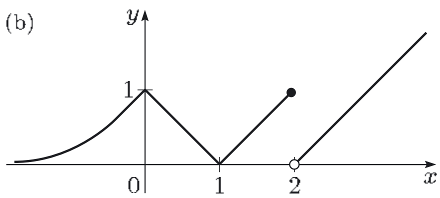
3.4

(a)



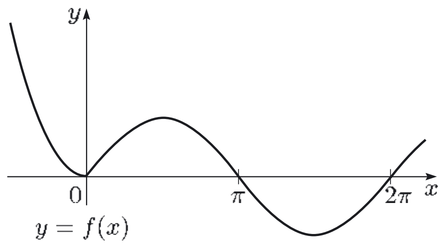
$y = f(x)$

(b)



$y = f(x)$

(c)



$y = f(x)$

3.5 $f(x) = 2\cos x - x$.

1. The domain of f is $\mathbb{R}$.

2. f is not even, odd or periodic.

3. $f(x) = 0$ is not easy to solve, but we know that $f(0) = 2$, which is positive, and $f(\pi/2) = -\pi/2$, which is negative, so there is an x-intercept in the interval $(0, \pi/2)$.

The y-intercept is 2.

4. Since there is only one x-intercept, f is positive when x is less than its value at the intercept, and negative when x is greater than its value at the intercept.

5. $f'(x) = -2\sin x - 1$, so $f'(x) = 0$ when $\sin x = -\frac{1}{2}$; that is, when $x = -\pi/6 + 2k\pi$ or $x = -5\pi/6 + 2k\pi$, for any integer k.

6. The function has no asymptotes.

7. Since

$$-2 \leq 2\cos x \leq 2, \quad \text{for all } x \text{ in } \mathbb{R},$$

then

$$-2 - x \leq 2\cos x - x \leq 2 - x, \quad \text{for all } x \text{ in } \mathbb{R}.$$

Therefore the graph of f lies between the construction lines $y = -2 - x$ and $y = 2 - x$.

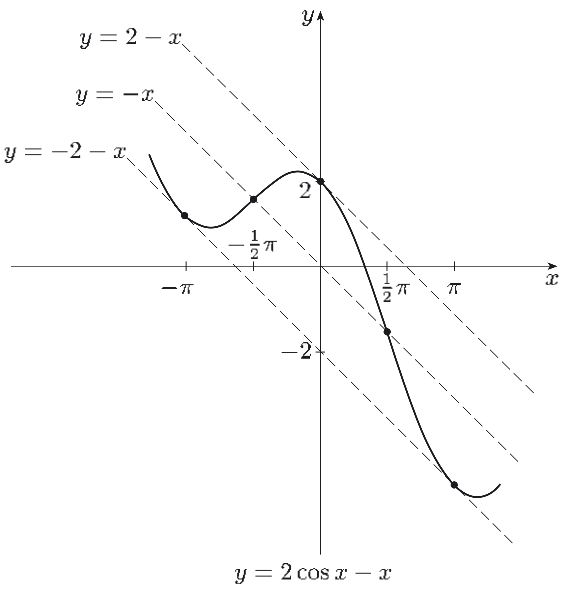
Also, for any integer k,

when $x = 2k\pi$, $\cos 2k\pi = 1$,
so $f(x) = 2 - x$;

when $x = (2k+1)\pi$, $\cos(2k+1)\pi = -1$,
so $f(x) = -2 - x$;

when $x = (k + \frac{1}{2})\pi$, $\cos(k + \frac{1}{2})\pi = 0$,
so $f(x) = -x$.

Thus we obtain the following graph.



3.6 $f(x) = \dfrac{\cos x}{x}$.

1. The domain of f is $\mathbb{R}$, excluding 0.

2. f is odd, since, for all x in the domain,

$$f(-x) = \frac{\cos(-x)}{-x} = \frac{\cos x}{-x} = -\frac{\cos x}{x} = -f(x).$$

3. $f(x) = 0$ whenever $\cos x = 0$; that is, $f(x) = 0$ when $x = (k + \frac{1}{2})\pi$, for any integer k. $f(0)$ is not defined, so there is no y-intercept.

5. The equation $f'(x) = 0$ is not easy to solve, so we omit it.

6. The denominator is 0 when $x = 0$, so

the line $x = 0$ is a vertical asymptote,

and

$$f(x) \to \infty \quad \text{as } x \to 0^+,$$
$$f(x) \to -\infty \quad \text{as } x \to 0^-.$$

7. We know that $-1 \leq \cos x \leq 1$ for all x in $\mathbb{R}$.

For $x > 0$, we have $\dfrac{1}{x} > 0$, so

$$-\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x};$$

and for $x < 0$, we have $\dfrac{1}{x} < 0$, so

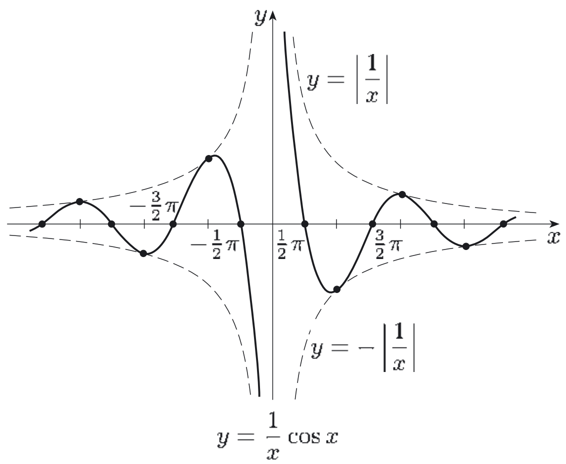
$$-\frac{1}{x} \geq \frac{\cos x}{x} \geq \frac{1}{x}.$$

Hence

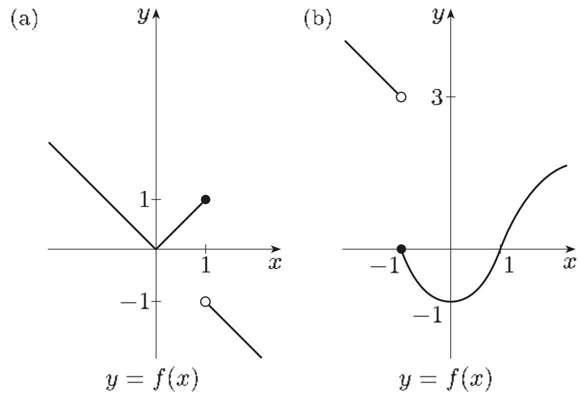
$$-\left|\frac{1}{x}\right| \leq \frac{\cos x}{x} \leq \left|\frac{1}{x}\right|, \quad \text{for all non-zero } x \text{ in } \mathbb{R}.$$

So the graph of f lies between the graphs $y = -|1/x|$ and $y = |1/x|$, the construction lines, and meets these graphs when $\cos x = \pm 1$.

Thus we obtain the following graph.



3.7



4.1 **(a)** $e^x(e^x + e^{-x}) = e^{2x} + e^x e^{-x} = e^{2x} + 1$

(b) $(e^{2x} - e^{-2x})/e^x = e^x - e^{-3x}$

(c) $(e^{5x} + e^{-5x})(e^{5x} - e^{-5x})$
$= e^{10x} - e^{5x}e^{-5x} + e^{-5x}e^{5x} - e^{-10x}$
$= e^{10x} - 1 + 1 - e^{-10x}$
$= e^{10x} - e^{-10x}$

4.2 **(a)** $\cosh^2 x - \sinh^2 x$
$= \frac{1}{4}(e^x + e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2$
$= \frac{1}{4}((e^{2x} + 2e^x e^{-x} + e^{-2x}) - (e^{2x} - 2e^x e^{-x} + e^{-2x}))$
$= \frac{1}{4}(e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x})$
$= \frac{1}{4} \times 4 = 1$

(b) $\cosh x \cosh y + \sinh x \sinh y$
$= \frac{1}{2}(e^x + e^{-x})\frac{1}{2}(e^y + e^{-y})$
$\quad + \frac{1}{2}(e^x - e^{-x})\frac{1}{2}(e^y - e^{-y})$
$= \frac{1}{4}(e^x e^y + e^x e^{-y} + e^{-x}e^y + e^{-x}e^{-y})$
$\quad + \frac{1}{4}(e^x e^y - e^x e^{-y} - e^{-x}e^y + e^{-x}e^{-y})$
$= \frac{1}{4}(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-(x+y)})$
$\quad + \frac{1}{4}(e^{x+y} - e^{x-y} - e^{-x+y} + e^{-(x+y)})$
$= \frac{1}{2}(e^{x+y} + e^{-(x+y)})$
$= \cosh(x + y)$

(c) $\sinh x \cosh y + \cosh x \sinh y$
$= \frac{1}{2}(e^x - e^{-x})\frac{1}{2}(e^y + e^{-y})$
$\quad + \frac{1}{2}(e^x + e^{-x})\frac{1}{2}(e^y - e^{-y})$
$= \frac{1}{4}(e^x e^y + e^x e^{-y} - e^{-x}e^y - e^{-x}e^{-y})$
$\quad + \frac{1}{4}(e^x e^y - e^x e^{-y} + e^{-x}e^y - e^{-x}e^{-y})$
$= \frac{1}{4}(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-(x+y)})$
$\quad + \frac{1}{4}(e^{x+y} - e^{x-y} + e^{-x+y} - e^{-(x+y)})$
$= \frac{1}{2}(e^{x+y} - e^{-(x+y)})$
$= \sinh(x + y)$

4.3 Let $f(x) = \cosh x = \frac{1}{2}(e^x + e^{-x})$; then

$$f'(x) = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x.$$

Let $g(x) = \sinh x = \frac{1}{2}(e^x - e^{-x})$; then

$$g'(x) = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x.$$

Thus

$$\cosh' = \sinh \quad \text{and} \quad \sinh' = \cosh\,.$$

These are similar to the trigonometric derivatives

$$\cos' = -\sin \quad \text{and} \quad \sin' = \cos,$$

but differ by a minus sign in the first one.

4.4 **(a)**
$$\begin{aligned} f(-x) &= \tanh(-x) \\ &= \frac{e^{-x} - e^x}{e^{-x} + e^x} \\ &= -\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) \\ &= -\tanh x = -f(x), \end{aligned}$$

so tanh is an odd function.

(b) $f(x) = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

Dividing both numerator and denominator by e^x (non-zero for all x in $\mathbb{R}$), we obtain

$$f(x) = \frac{1 - e^{-2x}}{1 + e^{-2x}}.$$

(c) $f(x) = \tanh x = \dfrac{\sinh x}{\cosh x}$.

Differentiating the quotient, we obtain

$$f'(x) = \frac{\cosh x(\cosh x) - \sinh x(\sinh x)}{\cosh^2 x},$$

and using the result from Exercise 4.2(a), we obtain

$$f'(x) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$$

Since $\operatorname{sech}^2 x$ is positive for all x in $\mathbb{R}$, it follows that

$$f'(x) > 0, \quad \text{for all } x \text{ in } \mathbb{R}.$$

4.5 **(a)** Since $\cosh 2x = \cosh^2 x + \sinh^2 x$ and $\cosh^2 x - \sinh^2 x = 1$, we have

$$\cosh 2x = \frac{\cosh^2 x + \sinh^2 x}{\cosh^2 x - \sinh^2 x}.$$

Dividing both numerator and denominator by $\cosh^2 x$ (which is never 0), we obtain

$$\cosh 2x = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}.$$

(b) Since $\sinh 2x = 2 \sinh x \cosh x$ and $\cosh^2 x - \sinh^2 x = 1$, we obtain

$$\sinh 2x = \frac{2 \sinh x \cosh x}{\cosh^2 x - \sinh^2 x}.$$

Dividing both numerator and denominator by $\cosh^2 x$, we obtain

$$\sinh 2x = \frac{2 \tanh x}{1 - \tanh^2 x}.$$

4.6 $f(x) = \sinh x$.

1. $\sinh x$ has domain $\mathbb{R}$.

2. $\sinh x$ is odd, since

$$\begin{aligned}\sinh(-x) &= \tfrac{1}{2}(e^{-x} - e^{-(-x)}) \\ &= \tfrac{1}{2}(e^{-x} - e^{x}) \\ &= -\tfrac{1}{2}(e^{x} - e^{-x}) = -\sinh x.\end{aligned}$$

3. $\sinh x = \frac{1}{2}(e^x - e^{-x}) = 0$ when $e^x = e^{-x}$; so the only zero of $\sinh x$ is 0. So 0 is both the x-intercept and the y-intercept.

4. From the graphs of $y = e^x$ and $y = e^{-x}$, we observe that

$$\sinh x > 0, \quad \text{for } x > 0,$$
$$\sinh x < 0, \quad \text{for } x < 0.$$

5. From Exercise 4.3, we know that

$$\sinh' x = \cosh x.$$

Also, we know that

$$\cosh x \geq 1, \quad \text{for all } x \text{ in } \mathbb{R},$$

so $\sinh x$ is strictly increasing on $\mathbb{R}$, and so has no local maxima or local minima.

Since $\sinh x' = \cosh x$ and $\cosh 0 = 1$, the graph of $\sinh x$ has slope 1 at the origin.

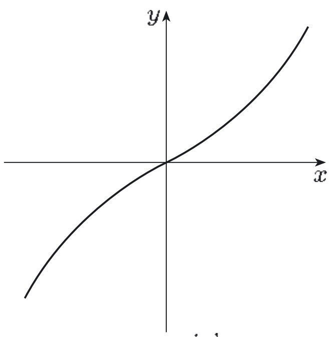
6. $e^x \to \infty$ as $x \to \infty$ and $e^{-x} \to 0$ as $x \to \infty$, so

$$f(x) \to \infty \quad \text{as } x \to \infty.$$

Since $\sinh x$ is an odd function,

$$f(x) \to -\infty \quad \text{as } x \to -\infty.$$

This information enables us to sketch the graph.



$y = \sinh x$

4.7 $f(x) = \tanh x = \dfrac{\sinh x}{\cosh x}$.

1. $\tanh x$ has domain $\mathbb{R}$, since $\cosh x$ never takes the value 0.

2. $\tanh x$ is odd, by Exercise 4.4(a).

3. $\cosh x \geq 1$ for all x in $\mathbb{R}$, so the zeros of $\tanh x$ are the same as those of $\sinh x$, namely 0. So 0 is both the x-intercept and the y-intercept.

4. $\tanh x$ is positive where $\sinh x$ is positive, namely on $(0, \infty)$;

$\tanh x$ is negative where $\sinh x$ is negative, namely on $(-\infty, 0)$.

5. We know, from Exercise 4.4(c), that $f'(x) > 0$ for all x in $\mathbb{R}$, so there are no local maxima or local minima.

Since $f'(0) = \operatorname{sech}^2(0) = 1$, the graph of $\tanh x$ has slope 1 at the origin.

6. Since $\cosh x \geq 1$, the denominator is never zero, so there are no vertical asymptotes.

Since $\tanh x = \dfrac{1 - e^{-2x}}{1 + e^{-2x}}$ (from Exercise 4.4(b)), and since $e^{-2x} \to 0$ as $x \to \infty$, we have $\tanh x \to \dfrac{1 - 0}{1 + 0}$ as $x \to \infty$, Since

$$\tanh x \to 1 \quad \text{as } x \to \infty.$$

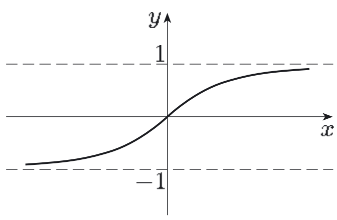
Since tanh is an odd function,

$$\tanh x \to -1 \quad \text{as } x \to -\infty.$$

Thus

the lines $y = 1$ and $y = -1$
are horizontal asymptotes.

This information enables us to sketch the graph.



$y = \tanh x$

4.8 $\operatorname{cosech} x = \dfrac{1}{\sinh x}$.

1. $\sinh x = 0$ when $x = 0$, so $\operatorname{cosech} x$ is not defined at 0. Thus $\operatorname{cosech} x$ has domain $\mathbb{R}$, excluding 0.

2. $\operatorname{cosech} x$ is odd, since $\sinh x$ is odd.

3. and 4.

We know that

$$\sinh x > 0, \quad \text{for } x > 0,$$
$$\sinh x < 0, \quad \text{for } x < 0,$$

so

$$\operatorname{cosech} x > 0, \quad \text{for } x > 0,$$
$$\operatorname{cosech} x < 0, \quad \text{for } x < 0,$$

thus

$\operatorname{cosech} x$ has no zeros.

Also $\operatorname{cosech} x$ is not defined at $x = 0$, so $\operatorname{cosech} x$ has neither x-intercepts nor y-intercepts.

5. $\sinh x$ is increasing on $\mathbb{R}$, so $\operatorname{cosech} x$ is decreasing on the intervals $(-\infty, 0)$ and $(0, \infty)$, and thus has no local maxima or local minima.

6. The graph of $y = \sinh x$ indicates that:

$$\text{when } x = 0, \quad \sinh x = 0;$$
$$\sinh x \to \infty \quad \text{as } x \to \infty;$$
$$\sinh x \to -\infty \quad \text{as } x \to -\infty.$$

But $\operatorname{cosech} x = 1/\sinh x$, and $\sinh x$ is small when x is close to 0, so

the line $x = 0$ is a vertical asymptote.

From steps 3 and 4,

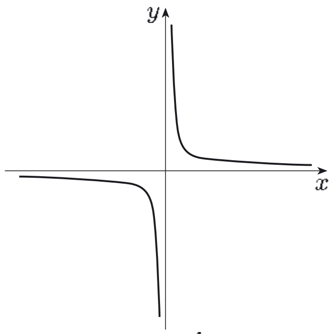
$$\operatorname{cosech} x \to \infty \quad \text{as } x \to 0^+,$$

$$\operatorname{cosech} x \to -\infty \quad \text{as } x \to 0^-.$$

Also,

$$\operatorname{cosech} x \to 0 \quad \text{as } x \to \pm\infty.$$

This information enables us to sketch the graph.



$y = \operatorname{cosech} x$

4.9 $\tanh(x+y) = \dfrac{\sinh(x+y)}{\cosh(x+y)}$

$$= \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y}.$$

Dividing both numerator and denominator by $\cosh x \cosh y$, we obtain

$$\tanh(x+y) = \frac{\dfrac{\sinh x}{\cosh x} + \dfrac{\sinh y}{\cosh y}}{1 + \dfrac{\sinh x \sinh y}{\cosh x \cosh y}} = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$$

4.10 See Exercise 4.7 for the graph of $\tanh x$.
$f(x) = \coth x = 1/\tanh x$.

1. $\tanh x = 0$ when $x = 0$, so the domain of $\coth x$ is the set of all real numbers, excluding 0.

2. $\tanh x$ is odd, so $\coth x$ is odd.

3. $\coth x$ has neither x- nor y-intercepts.

4. We know that

$$\tanh x > 0 \quad \text{when } x > 0,$$

$$\tanh x < 0 \quad \text{when } x < 0,$$

so

$$\coth x > 0 \quad \text{when } x > 0,$$

$$\coth x < 0 \quad \text{when } x < 0.$$

5. $\tanh x$ is increasing on $\mathbb{R}$, so $\coth x$ is decreasing on each interval of its domain, and so has no local maxima or minima.

6. $\tanh x \to 1$ as $x \to \infty$, so

$$\coth x \to 1 \quad \text{as } x \to \infty;$$

$\tanh x \to -1$ as $x \to -\infty$, so

$$\coth x \to -1 \quad \text{as } x \to -\infty.$$

$\tanh x$ is small when x is close to 0, so

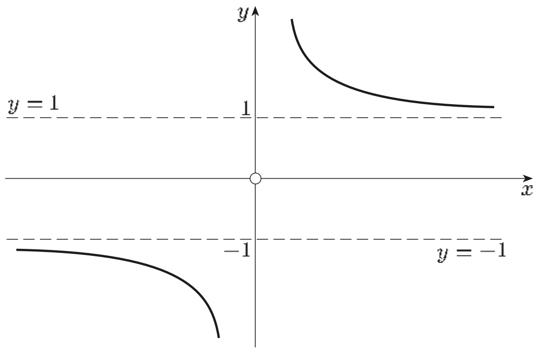
the line $x = 0$ is a vertical asymptote.

Also,

$$\coth x \to \infty \quad \text{as } x \to 0^+,$$

$$\coth x \to -\infty \quad \text{as } x \to 0^-.$$

Thus we obtain the following graph.



$y = \coth x$

5.1 We have

$$x = t - \sin t, \quad y = 1 - \cos t.$$

When $t = \pi/3$,

$$x = \frac{\pi}{3} - \frac{\sqrt{3}}{2} \simeq 1.047 - 0.866 = 0.181,$$

$$y = 1 - \frac{1}{2} = 0.5.$$

When $t = \pi$,

$$x = \pi - 0 \simeq 3.142,$$

$$y = 1 - (-1) = 2.$$

When $t = \pi/2$,

$$x = \pi/2 - 1 \simeq 1.571 - 1 = 0.571,$$

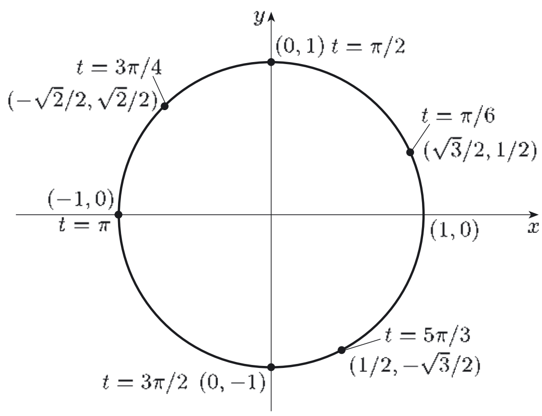
$$y = 1 - 0 = 1.$$

When $t = \pi/6$,

$$x = \frac{\pi}{6} - \frac{1}{2} \simeq 0.524 - 0.5 = 0.024,$$

$$y = 1 - \frac{\sqrt{3}}{2} \simeq 1 - 0.866 = 0.134.$$

5.2



5.3 **(a)** $\alpha(t) = (3\cos t, 3\sin t)$, for t in $[0, 2\pi]$.

(b) $\alpha(t) = (2 + 3\cos t, 1 + 3\sin t)$, for t in $[0, 2\pi]$.

5.4 Let $x = \cos 2\pi t$, $y = \sin 2\pi t$, for t in $[0, 1]$; then

$$x^2 + y^2 = \cos^2 2\pi t + \sin^2 2\pi t = 1,$$

so (x, y) is a point on the unit circle. As t increases from 0 to 1, $2\pi t$ increases from 0 to 2π, so the point (x, y) moves once round the circle.

5.5 (a) $\alpha(t) = (1 + 2t, 2 + 4t)$, for t in $\mathbb{R}$.

(b) $t = \frac{1}{2}$, $t = 3$, $t = -\frac{1}{2}$.

5.6 Eliminating t, we obtain

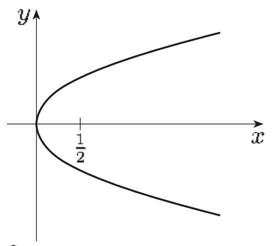
$$t^3 = \frac{x}{p},$$

so

$$y = qt^3 = \frac{q}{p}x,$$

which is the equation of the line through the points $(0, 0)$ and (p, q).

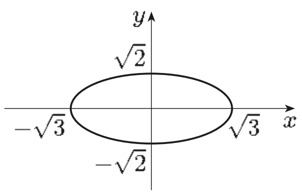
5.7 (a)



$y^2 = 2x$

$\alpha(t) = \left(\frac{1}{2}t^2, t\right)$, for t in $\mathbb{R}$.

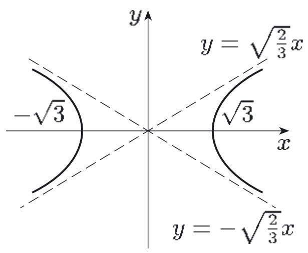
(b)



$$\frac{x^2}{3} + \frac{y^2}{2} = 1$$

$\alpha(t) = (\sqrt{3}\cos t, \sqrt{2}\sin t)$, for t in $[0, 2\pi]$.

(c)



$$\frac{x^2}{3} - \frac{y^2}{2} = 1$$

$\alpha(t) = (\sqrt{3}\sec t, \sqrt{2}\tan t)$, for t in $[-\pi, \pi]$ excluding $-\pi/2$ and $\pi/2$.

5.8 The parametric equations for this curve are

$$x = a\cosh t, \quad y = b\sinh t.$$

We eliminate t by writing

$$x/a = \cosh t, \quad y/b = \sinh t$$

and using the identity

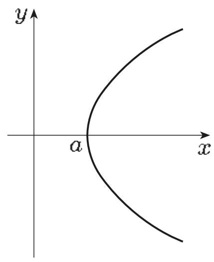
$$\cosh^2 t - \sinh^2 t = 1,$$

to obtain

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the equation for a hyperbola in standard form.

Since $\cosh t$ is always positive, this parametrisation gives only *one* branch of the hyperbola, namely the branch corresponding to positive values of x (because $\cosh t$ takes all values in $[1, \infty)$). Since $\sinh t$ can be positive or negative, we get the *whole* of this branch.



$x = a\cosh t$
$y = b\sinh t$

5.9 (a) $y = \dfrac{1}{x}$. Hence the curve is the graph of the reciprocal function.

(b) $t = x + 1$, so $y = 4 - 3(x + 1) = 1 - 3x$. Hence the curve is a straight line with slope -3 and y-intercept 1.

(c) $t = \dfrac{x}{2}$, so $y = 1 + 3\left(\dfrac{x}{2}\right)^2 = 1 + \dfrac{3x^2}{4}$.

Hence the curve is a parabola, symmetric about the y-axis, with vertex at $(0, 1)$.

5.10 (a)

$$\alpha(0) = (2(1) + 1, 2(0) + 0) = (3, 0)$$

$$\begin{aligned}\alpha(\tfrac{1}{6}\pi) &= (2(\tfrac{1}{2}\sqrt{3}) + \tfrac{1}{2}, 2(\tfrac{1}{2}) + \tfrac{1}{2}\sqrt{3}) \\ &= (\sqrt{3} + \tfrac{1}{2}, 1 + \tfrac{1}{2}\sqrt{3}) \simeq (2.23, 1.87)\end{aligned}$$

$$\begin{aligned}\alpha(\tfrac{1}{3}\pi) &= (2(\tfrac{1}{2}) + (-\tfrac{1}{2}), 2(\tfrac{1}{2}\sqrt{3}) + \tfrac{1}{2}\sqrt{3}) \\ &= (\tfrac{1}{2}, \tfrac{3}{2}\sqrt{3}) \simeq (0.5, 2.60)\end{aligned}$$

$$\alpha(\tfrac{1}{2}\pi) = (2(0) - 1, 2(1) + 0) = (-1, 2)$$

$$\begin{aligned}\alpha(\tfrac{2}{3}\pi) &= (2(-\tfrac{1}{2}) + (-\tfrac{1}{2}), 2(\tfrac{1}{2}\sqrt{3}) + (-\tfrac{1}{2}\sqrt{3})) \\ &= (-\tfrac{3}{2}, \tfrac{1}{2}\sqrt{3}) \simeq (-1.5, 0.87)\end{aligned}$$

$$\begin{aligned}\alpha(\tfrac{5}{6}\pi) &= (2(-\tfrac{1}{2}\sqrt{3}) + (\tfrac{1}{2}), 2(\tfrac{1}{2}) + (-\tfrac{1}{2}\sqrt{3})) \\ &= (-\sqrt{3} + \tfrac{1}{2}, 1 - \tfrac{1}{2}\sqrt{3}) \simeq (-1.23, 0.13)\end{aligned}$$

$$\alpha(\pi) = (2(-1) + 1, 2(0) + 0) = (-1, 0)$$

(b) $$\begin{aligned} f(-t) &= 2\cos(-t) + \cos(-2t) \\ &= 2\cos t + \cos 2t \quad \text{(since cos is even)} \\ &= f(t). \end{aligned}$$

Hence f is even.

$$\begin{aligned} g(-t) &= 2\sin(-t) + \sin(-2t) \\ &= -2\sin t - \sin 2t \quad \text{(since sin is odd)} \\ &= -g(t). \end{aligned}$$

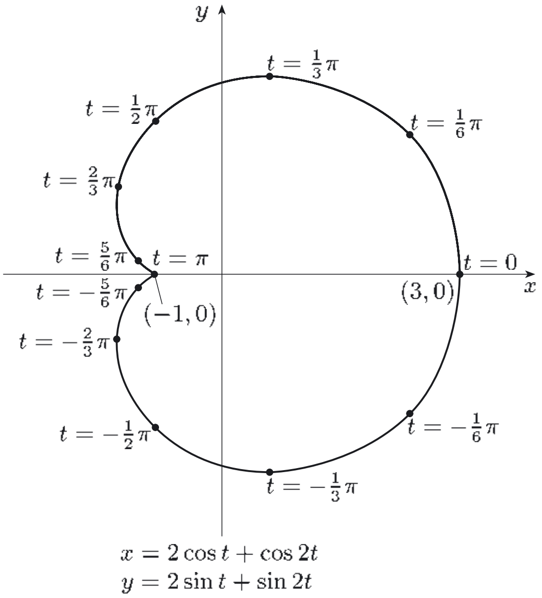
Hence g is odd.

So we have

$$\alpha(t) = (f(t), g(t)) \quad \text{and} \quad \alpha(-t) = (f(t), -g(t)).$$

From this we deduce that the curve is symmetric about the x-axis.

(c) We plot the points obtained in part (a), and use part (b) to complete the curve—a *cardioid*.



Index