1 Sets

After working through this section, you should be able to:

(a) use set notation;
(b) determine whether two given sets are equal and whether one given set is a subset of another;
(c) find the *union*, *intersection* and *difference* of two given sets.

1.1 What is a set?

In mathematics we frequently consider collections of objects of various kinds. We may, for example, consider:

- the solutions of a quadratic equation;
- the points on a circle;
- the prime numbers less than 100;
- the vertices of a triangle;
- the domain of a real function.

A *prime number* is an integer n, greater than 1, whose only positive factors are 1 and n; the first few primes are 2, 3, 5, 7, 11, 13, 17.

The concept of a *set* provides the unifying framework needed to investigate such collections systematically.

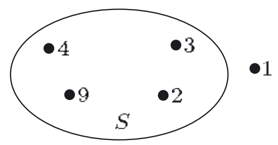
You can think of a **set** as a collection of objects, such as numbers, points, functions, or even other sets. Each object in a set is an **element** or **member** of the set, and the elements **belong to** the set, or are **in** the set.

There is no limitation on the types of object that may appear in a set, provided that the set is specified in a way that enables us to decide, in principle, whether a given object is in the set.

There are many ways of making such a specification. For example, we can define S to be the set of numbers in the list

$4, 9, 3, 2.$

This enables us to decide that the number 2 (say) is in S, but that the number 1 (say) is not in S. We can illustrate this set by a diagram, as shown in the margin; such a diagram is called a **Venn diagram**, after the 19th-century Cambridge mathematician John Venn.



The symbol S is a label for the set, *not* a member of the set. Similar labels will appear in other diagrams.

We can also define a set E by stating

let E be the set of all even integers.

This description enables us to determine whether a given object is in E by deciding whether it is an even integer; for example, 6 is in E, but 5 is not.

Some sets are used so often that special symbols are reserved for them.

$\mathbb{R}$ denotes the set of real numbers.

$\mathbb{R}^*$ denotes the set of non-zero real numbers.

$\mathbb{Q}$ denotes the set of rational numbers.

$\mathbb{Z}$ denotes the set of integers $\ldots, -2, -1, 0, 1, 2, \ldots$.

$\mathbb{N}$ denotes the set of natural numbers $1, 2, 3, \ldots$.

A real number is a number with a decimal expansion (possibly infinite), for example, $\pi = 3.14\ldots$ or 1.1.

A rational number is a real number that can be expressed as a fraction, for example, 14/5 or $-3/4$.

We use the symbol $\in$ to indicate membership of a set; for example, we indicate that 7 is a member of $\mathbb{N}$ by writing

$$7 \in \mathbb{N}.$$

We read this as '7 belongs to $\mathbb{N}$' or '7 is in $\mathbb{N}$'.

We indicate that -9 is *not* a member of $\mathbb{N}$ by writing

$$-9 \notin \mathbb{N}.$$

We read this as '-9 does not belong to $\mathbb{N}$' or '-9 is not in $\mathbb{N}$'.

We also use the symbol $\in$ when we wish to introduce a symbol that stands for an arbitrary element of a set. For example, we write

let $x \in \mathbb{R}$

to indicate that x is an arbitrary (unspecified) member of the set $\mathbb{R}$. We sometimes refer to x as a *real variable*. In general, a variable is a symbol (like x or n) that stands for an arbitrary element of a set.

Exercise 1.1 Which of the following statements are true?

(a) $-2 \in \mathbb{Z}$ (b) $5 \notin \mathbb{N}$ (c) $1.3 \notin \mathbb{Q}$

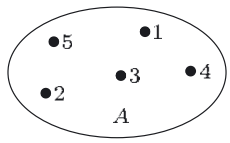
(d) $\frac{1}{2} \in \mathbb{N}$ (e) $-\pi \in \mathbb{R}$ (f) $2 \in \mathbb{Q}$

1.2 Set notation

We now examine some formal ways of specifying a set.

We can specify a set with a small number of elements by listing these elements between a pair of braces (curly brackets). For example, we can specify the set A consisting of the first five natural numbers by

$$A = \{1, 2, 3, 4, 5\}.$$

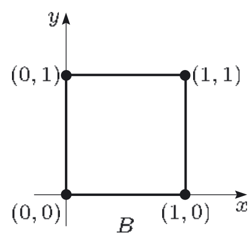


The membership of a set is not affected by the order in which its elements are listed, so we can specify this set A equally well by

$$A = \{5, 2, 1, 4, 3\}.$$

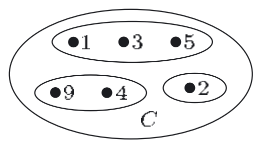
Similarly, we can specify the set B of vertices of the square shown in the margin by

$$B = \{(0,0), (1,0), (1,1), (0,1)\}.$$



We can even specify a set C whose elements are the three sets $\{1, 3, 5\}$, $\{9, 4\}$ and $\{2\}$ by

$$C = \{\{1, 3, 5\}, \{9, 4\}, \{2\}\}.$$



A set with only one element, such as the set $\{2\}$, is called a **singleton**. (Do not confuse the *set* $\{2\}$ with the *number* 2.)

Exercise 1.2 Which of the following statements are true?

(a) $1 \in \{4, 3, 1, 7\}$

(b) $\{-9\} \in \{\{6, 1, 2\}, \{8, 7, 9, 5\}, \{-9\}, \{5, 4\}\}$

(c) $\{9\} \in \{5, 6, 7, 8, 9\}$

(d) $(0, 1) \in \{(1, 0), (1, 4), (2, 4)\}$

(e) $\{0, 1\} \in \{\{0, 1\}, \{1, 4\}, \{2, 4\}\}$

It does not matter if we specify a set element more than once within set brackets; we still describe the set that contains each specified element. For example,

$$\{1, 2, 3, 3\} \quad \text{and} \quad \{1, 2, 3\}$$

describe the same set. However, we usually try to avoid specifying an element more than once.

For a set with a large number of elements, it is not practicable to list all the elements, so we sometimes use three dots (called an *ellipsis*) to indicate that a particular pattern of membership continues. For example, we can specify the set consisting of the first 100 natural numbers by writing

$$\{1, 2, 3, \ldots, 100\}.$$

The use of an ellipsis can be extended to certain infinite sets. For example, we can specify the set of all natural numbers by writing

$$\{1, 2, 3, \ldots\}.$$

One disadvantage of this notation is that the pattern indicated by the ellipsis may be ambiguous. For example, it is not clear whether

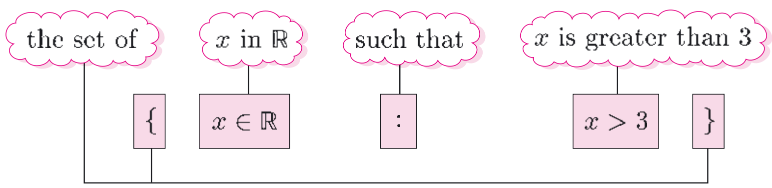
$$\{3, 5, 7, \ldots\}$$

denotes the set of odd prime numbers or the set of odd natural numbers greater than 1. For this reason, this notation can be used only when the pattern of membership is obvious, or where an additional clarifying explanation is given.

An alternative way of specifying a set is to use variables to build up objects of the required type, and then write down the condition(s) that the variables must satisfy. For example, consider the open interval $(3, \infty)$, consisting of all real numbers x such that $x > 3$. Using set notation, we write this as

$$\{x \in \mathbb{R} : x > 3\},$$

which is read as follows:



A set can often be described in several different ways using such set notation. In particular, we can use a letter other than x to denote an arbitrary (general) element of a set; for example, the above interval can also be written as

$$\{r \in \mathbb{R} : r > 3\}.$$

If it is necessary to include more than one condition after the colon, then we write either a comma or the word 'and' between the conditions. So the interval $(0, 1]$ can be written as

$$\{x \in \mathbb{R} : x > 0,\ x \leq 1\} \quad \text{or} \quad \{x \in \mathbb{R} : x > 0 \text{ and } x \leq 1\},$$

although usually we combine the inequalities and write

$$\{x \in \mathbb{R} : 0 < x \leq 1\}.$$

Sometimes it is convenient to specify a set by writing an expression in one or more variables before the colon, and the conditions on the variables after the colon. For example, the set of even integers less than 100 may be specified by

$$\{2k : k \in \mathbb{Z} \text{ and } k < 50\}.$$

Just as when we list the elements of a set, when we use set notation it does not matter if a set element is specified more than once. For example,

$$\{\sin x : x \in \mathbb{R}\} \quad \text{and} \quad \{x \in \mathbb{R} : -1 \leq x \leq 1\}$$

specify the same set.

Set notation is useful when we wish to refer to the set of solutions, called the **solution set**, of one or more equations. For example, the solutions of $x^2 = 1$ form the set

$$\{x \in \mathbb{R} : x^2 = 1\} = \{-1, 1\}.$$

The solution set of an equation depends on the set of values from which the solutions are taken. For example, the solution set of the equation

$$(x - 1)(2x - 1) = 0$$

is

$$\{x \in \mathbb{R} : (x - 1)(2x - 1) = 0\} = \{1, \tfrac{1}{2}\}$$

if we are interested in real solutions, but is

$$\{x \in \mathbb{Z} : (x - 1)(2x - 1) = 0\} = \{1\}$$

if we are interested only in integer solutions. In this unit we assume that solutions are taken from $\mathbb{R}$ unless otherwise stated.

Sometimes an equation has *no* real solutions, so its solution set has no elements. This set is called the **empty set** and is denoted by $\varnothing$. For example,

$$\{x \in \mathbb{R} : x^2 = -1\} = \varnothing.$$

Example 1.1 Use set notation to specify each of the following:

(a) the set of all natural numbers greater than 50;

(b) the set of all real solutions of the equation $x^4 + 8x^2 + 16 = 0$;

(c) the set of all odd integers.

Solution

(a) The elements of this set are the natural numbers n such that $n > 50$. So the set is

$$\{n \in \mathbb{N} : n > 50\}.$$

(b) The elements of this set are the real numbers x that satisfy the given equation. So the set is

$$\{x \in \mathbb{R} : x^4 + 8x^2 + 16 = 0\}.$$

In fact, the given equation has no real solutions, so this set is the empty set $\varnothing$.

(c) An odd integer is one that can be written in the form $2k + 1$, for some integer k. So the set is

$$\{2k + 1 : k \in \mathbb{Z}\}.$$ ■

Exercise 1.3 Use set notation to specify each of the following:

(a) the set of integers greater than -2 and less than 1000;

(b) the closed interval $[2, 7]$;

(c) the set of positive rational numbers with square greater than 2;

(d) the set of even natural numbers;

(e) the set of integer powers of 2.

1.3 Plane sets

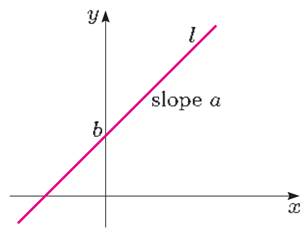
In Unit I1 you met the plane $\mathbb{R}^2$, and saw that each point in the plane can be represented as an ordered pair (x, y) with respect to a given pair of axes. A set of points in $\mathbb{R}^2$ is called a **plane set** or a **plane figure**. Simple examples of plane sets are lines and circles.

Such plane sets occur in many applications of mathematics; for example, in computer graphics.

Lines

Consider a straight line l with slope a and y-intercept b. This line is the set of all points (x, y) in the plane such that $y = ax + b$. Using set notation, we write this as

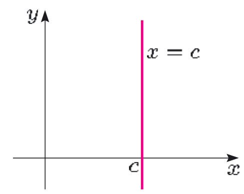
$$l = \{(x, y) \in \mathbb{R}^2 : y = ax + b\}.$$



(We sometimes refer to 'the line $y = ax + b$' as a shorthand way of specifying this set.)

For a line parallel to the y-axis with x-intercept c, we write

$$l = \{(x, y) \in \mathbb{R}^2 : x = c\}.$$



Exercise 1.4

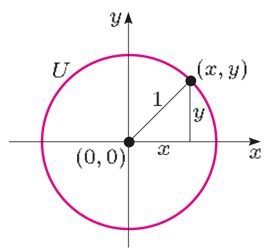
(a) Use set notation to specify the line l with slope 2 that passes through the point $(0, 5)$.

(b) Sketch the line $l = \{(x, y) \in \mathbb{R}^2 : y = 1 - x\}$.

Circles

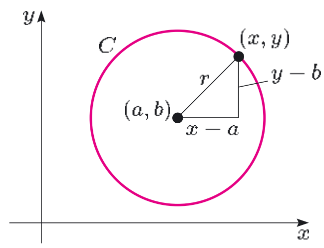
The **unit circle** U is the set of points (x, y) in the plane whose distance from the origin $(0, 0)$ is 1. By Pythagoras' Theorem, these are the points (x, y) for which $x^2 + y^2 = 1$, so, in set notation, the unit circle is written as

$$U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$



In general, the circle C of radius r centred at the point (a, b) is the set of points (x, y) that lie at a distance r from (a, b). By Pythagoras' Theorem, these are the points (x, y) satisfying the equation $(x - a)^2 + (y - b)^2 = r^2$, so, in set notation, this circle is written as

$$C = \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 = r^2\}.$$

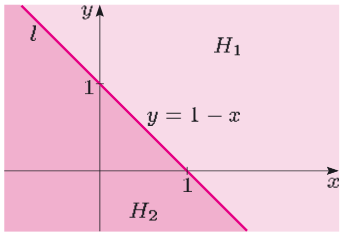


Exercise 1.5

(a) Use set notation to specify the circle C of radius 3 centred at $(1, -4)$.

(b) Sketch the circle $C = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + (y - 3)^2 = 4\}$.

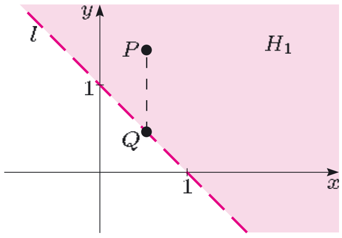
Half-planes, discs and other plane sets



Consider the line

$$l = \{(x, y) \in \mathbb{R}^2 : y = 1 - x\}.$$

This line splits $\mathbb{R}^2$ into three separate parts: the line l itself, the set H_1 of points lying *above* the line, and the set H_2 of points lying *below* the line.



Consider an arbitrary point $P = (x, y)$ in H_1 as shown in the margin. The point $Q = (x, 1 - x)$ lies on the line l, below P, as illustrated, so $y > 1 - x$. Similarly, each point (x, y) in H_2 satisfies $y < 1 - x$. Thus

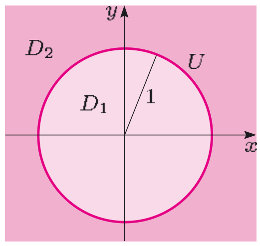
$$H_1 = \{(x, y) \in \mathbb{R}^2 : y > 1 - x\}$$

and

$$H_2 = \{(x, y) \in \mathbb{R}^2 : y < 1 - x\}.$$

(In the diagrams, when a plane set illustrated does not include a boundary line, we draw the boundary line as a broken line.)

The set of points on one side of a line, possibly together with all the points on the line itself, is known as a **half-plane**. A half-plane that does not include the points on the line can be specified using set notation in a similar way to the examples H_1 and H_2 above. The corresponding half-plane that includes the points on the line can be specified by changing the symbol $>$ to $\geq$, or the symbol $<$ to $\leq$.



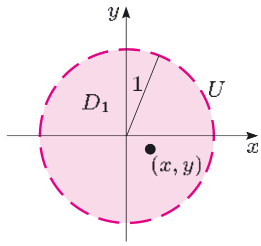
Next consider the unit circle

$$U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

This circle splits $\mathbb{R}^2$ into three separate parts: the circle U itself, the set D_1 of points lying *inside* the circle and the set D_2 of points lying *outside* the circle.

The condition for a point (x, y) to lie inside U is that the distance from the origin is less than 1, so the square of the distance is also less than 1. Thus

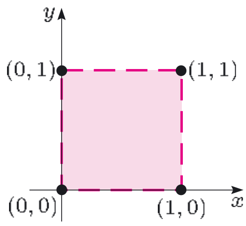
$$D_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$



Similarly,

$$D_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}.$$

The set of points inside a circle, possibly together with all the points on the circle, is known as a **disc**. If we wish to specify the disc consisting of the unit circle and the points inside it, we replace the inequality $<$ by $\leq$ in the set notation specification of D_1 given above.



Now consider the set of points lying inside the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$. This set can be written as

$$\{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < 1\}.$$

If we wish to include the four boundary lines in the set, we replace each symbol $<$ by $\leq$. We would show this set on a diagram by replacing the broken lines in the diagram in the margin by solid lines.

Exercise 1.6 Sketch each of the following plane sets.

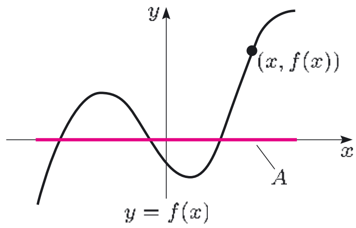
(a) $\{(x, y) \in \mathbb{R}^2 : x < 1\}$

(b) $\{(x, y) \in \mathbb{R}^2 : y < 2 - 2x\}$

(c) $\{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + (y - 2)^2 \leq 4\}$

(d) $\{(x, y) \in \mathbb{R}^2 : x^2 + (y + 3)^2 > 1\}$

We conclude this subsection by considering the graph of a real function. In Unit I1, we sketched the graph of a real function f by plotting points of the form $(x, f(x))$ in $\mathbb{R}^2$, for each element x of the domain A. This suggests the following formal definition of a graph.



Definition The **graph** of a real function $f : A \longrightarrow \mathbb{R}$ is the set

$$\{(x, f(x)) : x \in A\}.$$

Exercise 1.7 Use set notation to specify:

(a) the points in the square with vertices $(0, 1)$, $(2, 1)$, $(2, 3)$, $(0, 3)$, if the boundary is included;

(b) the points on the graph of the function

$$\begin{aligned} f : [0, \infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto 2x^2 + 1. \end{aligned}$$

1.4 Set equality and subsets

Consider the sets $A = \{1, -1\}$ and $B = \{x \in \mathbb{R} : x^2 - 1 = 0\}$. Although these sets are written in different ways, each set contains exactly the same elements, 1 and -1. We say that these sets are *equal*.

Definition Two sets A and B are **equal** if they have exactly the same elements; we write $A = B$.

When two sets each contain a small number of elements, we can usually check whether these elements are the same, and hence decide whether the sets are equal.

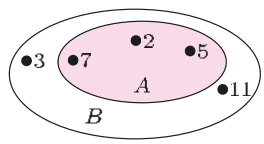
Exercise 1.8 Decide whether each of the following pairs of sets are equal.

(a) $A = \{2, -3\}$ and $B = \{x \in \mathbb{R} : x^2 + x - 6 = 0\}$.

(b) $A = \{k \in \mathbb{Z} : k \text{ is odd and } 2 < k < 10\}$ and $B = \{n \in \mathbb{N} : n \text{ is a prime number and } n < 10\}$.

If two sets each contain more than a small number of elements, it is less easy to check whether they are equal. We shall describe a method for dealing with cases like this after we have introduced an idea that we shall need.

Consider the sets $A = \{7, 2, 5\}$ and $B = \{2, 3, 5, 7, 11\}$. Each element of A is also an element of B. We say that A is a *subset* of B.



Definition A set A is a **subset** of a set B if each element of A is also an element of B. We also say that A is **contained in** B, and we write $A \subseteq B$.

We sometimes indicate that a set A is a subset of a set B by reversing the symbol $\subseteq$ and writing $B \supseteq A$, which we read as 'B **contains** A'. To indicate that A is *not* a subset of B, we write $A \not\subseteq B$. We may also write this as $B \not\supseteq A$, which we read as 'B does not contain A'.

Do not confuse the symbol $\subseteq$ with the symbol $\in$. For example, we write

$$\{1\} \subseteq \{1, 2, 3\},$$

since $\{1\}$ is a *subset* of $\{1,2,3\}$, and

$$1 \in \{1, 2, 3\},$$

since 1 is an *element* of $\{1,2,3\}$.

When we wish to determine whether a given set A is a subset of a given set B, the method that we use depends on the way in which the two sets are defined. If A has a small number of elements, then we check directly by inspection whether each element of A is an element of B. Otherwise, we determine whether an arbitrary element of A fulfils the membership criteria for B, as illustrated by Example 1.2 below.

To show that a given set A is *not* a subset of a given set B, we need to find at least one element of A that does not belong to B. The empty set $\varnothing$ is a subset of *every* set because we cannot find an element in $\varnothing$ which does not belong to the set in question.

Example 1.2 In each of the following cases, determine whether $A \subseteq B$.

(a) $A = \{1, 2, -4\}$ and $B = \{x \in \mathbb{R} : x^5 + 4x^4 - x - 4 = 0\}$.

(b) $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x < 1\}$.

Solution

(a) The elements 1, 2, -4 belong to $\mathbb{R}$, and we can check directly whether they also satisfy the equation $x^5 + 4x^4 - x - 4 = 0$. We have

$$(1)^5 + 4(1)^4 - 1 - 4 = 0, \quad \text{so } 1 \in B,$$

$$(2)^5 + 4(2)^4 - 2 - 4 = 90, \quad \text{so } 2 \notin B.$$

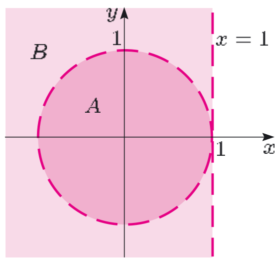
Since $2 \notin B$, we do not need to check whether $-4 \in B$.

So 2 does not belong to B, and hence A is not contained in B.

(b) From the diagram in the margin, it appears that $A \subseteq B$. We cannot check each of the elements of A individually, so we let (x, y) be an arbitrary element of A; then (x, y) is a point of $\mathbb{R}^2$ with $x^2 + y^2 < 1$.

Since $y^2 \geq 0$ for all y, this implies that $x^2 < 1$, and hence that $x < 1$. Thus $(x, y) \in B$.

Since (x, y) is an arbitrary element of A, we conclude that $A \subseteq B$. ■



Exercise 1.9 In each of the following cases, determine whether $A \subseteq B$.

(a) $A = \{(5, 2), (1, 1), (-3, 0)\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x - 4y = -3\}$.

(b) $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : y < 0\}$.

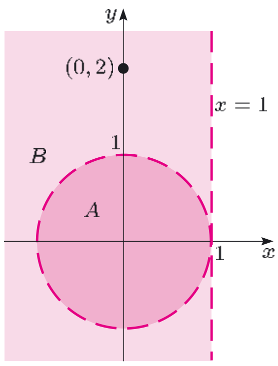
If two sets A and B are equal, then A is a subset of B, and B is a subset of A. If a set A is a subset of a set B that is not equal to B, then we say that A is a **proper subset** of B, and we write $A \subset B$ or $B \supset A$.

In some texts, the symbol $\subset$ is used to mean 'is a subset of' (for which we use the symbol $\subseteq$) rather than 'is a proper subset of'.

To show that a set A is a proper subset of a set B, we must show both that A is a subset of B, and that there is at least one element of B that is not an element of A.

Example 1.3 Show that A is a proper subset of B, where A and B are the sets defined in Example 1.2(b).

Solution We showed in the solution to Example 1.2(b) that $A \subseteq B$. Also, the point $(0, 2)$, for example, lies in B, since its x-coordinate 0 is less than 1, but $(0, 2)$ does not lie in A, since $0^2 + 2^2 = 4 \geq 1$. This shows that A is a proper subset of B. ■



Exercise 1.10 Show that A is a proper subset of B, where A and B are the sets defined in Exercise 1.9(a).

We now return to the question of how we can show that two sets A and B are equal if they have more than a small number of elements. To do this, we show that each set is a subset of the other.

Strategy 1.1 To show that two sets A and B are equal:
show that $A \subseteq B$;
show that $B \subseteq A$.

Example 1.4 Show that the following sets are equal:

$$A = \{(\cos t, \sin t) : t \in [0, 2\pi]\} \quad \text{and} \quad B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

In Unit I1, Section 5, you saw that
$\alpha(t) = (\cos t, \sin t), \quad t \in [0, 2\pi]$,
is a parametrisation of the unit circle, so we expect A and B to be the same set.

Solution First we show that $A \subseteq B$.

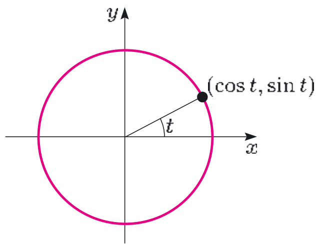
Let (x, y) be an arbitrary element of A; then

$$x = \cos t \quad \text{and} \quad y = \sin t, \quad \text{for some } t \in [0, 2\pi],$$

so

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

This implies that $(x, y) \in B$, so $A \subseteq B$.



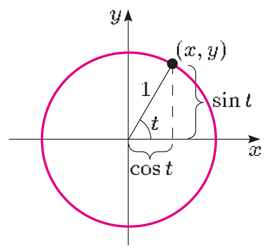
Next we show that $B \subseteq A$.

Let (x, y) be an arbitrary element of B; then

$$x^2 + y^2 = 1.$$

In order to show that (x, y) is an element of A, we need to find a value of $t \in [0, 2\pi]$ such that $(x, y) = (\cos t, \sin t)$. If we take t to be the angle between the x-axis and the line joining the point (x, y) to the origin, then

$$x = \cos t \quad \text{and} \quad y = \sin t.$$



Since $t \in [0, 2\pi]$, it follows that $(x, y) \in A$, so $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, it follows that $A = B$. ■

Exercise 1.11 Show that the following sets are equal:

$$A = \{(t^2, 2t) : t \in \mathbb{R}\} \quad \text{and} \quad B = \{(x, y) \in \mathbb{R}^2 : y^2 = 4x\}.$$

1.5 Counting subsets of finite sets

A *finite* set is a set which has a *finite* number of elements; that is, the number of elements is some natural number, or 0. We saw earlier that in using set notation, we may list the elements of a finite set in any order. For example, the set $\{1, 2, 3\}$ can be written by ordering the elements in six different ways:

The number of elements is 0 in the case of the empty set $\varnothing$.

$$\{1, 2, 3\}, \ \{1, 3, 2\}, \ \{2, 1, 3\}, \ \{2, 3, 1\}, \ \{3, 1, 2\}, \ \{3, 2, 1\}$$

(with each element of the set specified just once).

In general, a set with n elements can be ordered in

$$n \times (n - 1) \times \cdots \times 1 \qquad (1.1)$$

different ways, as there are n choices for the first element, then $n - 1$ choices for the second element, and so on, with just one possibility for the last element.

We denote expression (1.1) by $n!$ (read as 'n factorial').

Definition For any positive integer n,

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1.$$

Also,

$$0! = 1.$$

We define $0!$ to be 1 for convenience, so that results such as $n! = n \times (n-1)!$ are also true for $n = 1$. Also, this definition makes sense because the number of different orderings of the elements of the empty set is 1; we cannot change the order of no elements!

For example, a set with 10 elements can be ordered in

$$10! = 10 \times 9 \times \cdots \times 1 = 3\,628\,800$$

different ways.

A finite set has only finitely many subsets—but how many? Consider, for example, the set $\{1, 2, 3\}$. Below, we list all the subsets of $\{1, 2, 3\}$ in a table, according to the size k of the subsets.

k	subsets of $\{1, 2, 3\}$	number of subsets
0	$\varnothing$	1
1	$\{1\},\ \{2\},\ \{3\}$	3
2	$\{1,2\},\ \{1,3\},\ \{2,3\}$	3
3	$\{1,2,3\}$	1

This table shows that the set $\{1, 2, 3\}$ has $1 + 3 + 3 + 1 = 8$ subsets in all.

Exercise 1.12 List all the subsets of the set $\{1, 2, 3, 4\}$ in a similar table.

We have seen that a set with 3 elements has 8 subsets and a set with 4 elements has 16 subsets. This suggests that a set with n elements has 2^n subsets. To see this, we can argue as follows. Given a set A with n elements, we can associate with each subset of A a string of n symbols, where the kth symbol is a 1 if the kth element of A is in the subset, and a 0 otherwise. For example, if $A = \{1, 2, 3, 4, 5\}$, then the string associated with the subset $\{2, 4, 5\}$ is 01011. There are 2^n such strings (since there are 2 choices for each of the n symbols), so there are 2^n subsets.

We now concentrate on the following related question.

How many subsets with k elements has a set with n elements?

To answer this question, we consider choosing the k elements of the subset in order. There are n choices for the first element of the subset, then $n - 1$ choices for the second element, and so on, with $n - (k - 1) = n - k + 1$ choices for the kth element. Hence the number of ways of choosing k elements *in order* from n elements is

$$n \times (n-1) \times \cdots \times (n-k+1).$$

But some of these $n \times (n-1) \times \cdots \times (n-k+1)$ *ordered* choices give rise to the same subset. In fact, each subset of k elements corresponds to $k!$ ordered choices of k elements. Thus the number of *different* subsets with k elements of a set with n elements is

$$\frac{n \times (n-1) \times \cdots \times (n-k+1)}{k!}.$$

Multiplying the numerator and denominator by $(n-k)!$, we obtain

$$\frac{n!}{k!\,(n-k)!}.$$

We introduce the following notation for this expression.

Definition For any non-negative integers n and k with $k \leq n$,

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

This expression is called a **binomial coefficient**. It is the number of subsets with k elements of a set with n elements.

The expression $\binom{n}{k}$ is read as 'n choose k'. Some texts use the alternative notation ${}^nC_k$, where the 'C' stands for 'combination'. The reason for the name 'binomial coefficient' will become clear in Section 4.

For example, the number of subsets with two elements of a set with three elements is

$$\binom{3}{2} = \frac{3!}{2!\,1!} = 3,$$

as we found in the table on page 14.

A more interesting example is that of a lottery in which participants choose a subset of six numbers from a set of 49 numbers. In this case there are

$$\binom{49}{6} = \frac{49!}{6!\,43!} = \frac{49 \times 48 \times 47 \times 46 \times 45 \times 44}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 13\,983\,816$$

different subsets, or *combinations* as they are commonly called.

We can of course drop the '$\times 1$' in the denominator and write $\dfrac{49 \times 48 \times 47 \times 46 \times 45 \times 44}{6 \times 5 \times 4 \times 3 \times 2}$.

Exercise 1.13 Evaluate $\binom{10}{2}$, $\binom{10}{3}$ and $\binom{11}{3}$, and verify that

$$\binom{10}{2} + \binom{10}{3} = \binom{11}{3}.$$

The result of Exercise 1.13 is a special case of the following general result.

Example 1.5 Prove that that if n and k are positive integers with $1 \leq k \leq n$, then

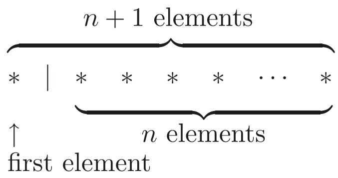
$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

We use this identity in Section 4.

Solution We start with the left-hand side and use successive rearrangements to obtain the right-hand side:

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!\,(n-(k-1))!} + \frac{n!}{k!\,(n-k)!} \\
&= \frac{kn!}{k(k-1)!(n-k+1)!} + \frac{(n-k+1)n!}{k!(n-k)!(n-k+1)} \\
&= \frac{kn!}{k!(n-k+1)!} + \frac{(n-k+1)n!}{k!(n-k+1)!} \\
&= \frac{(k+(n-k+1)) \times n!}{k!\,(n-k+1)!} \\
&= \frac{(n+1) \times n!}{k!\,(n-k+1)!} \\
&= \frac{(n+1)!}{k!\,(n+1-k)!} = \binom{n+1}{k}. \quad \blacksquare
\end{aligned}$$

We can give an alternative proof of the above identity by interpreting the left- and right-hand sides as the results obtained by counting the same thing in two different ways. If we deem one of $n+1$ elements to be the first, then the $\binom{n+1}{k}$ subsets of k elements chosen from these $n+1$ elements consist of $\binom{n}{k-1}$ subsets which include the first element (and $k-1$ other elements), and $\binom{n}{k}$ subsets which do not include the first element.



Such a *combinatorial* or *counting argument* can be spotted only with practice.

Exercise 1.14 Prove the following identity (a) directly, (b) by using a combinatorial argument.

If n and k are positive integers with $0 \leq k \leq n$, then
$$\binom{n}{n-k} = \binom{n}{k}.$$

1.6 Set operations

Consider the two sets $\{2, 3, 5\}$ and $\{1, 2, 5, 8\}$. Using these sets, we can construct several new sets—for example:

- the set $\{1, 2, 3, 5, 8\}$ consisting of all elements belonging to *at least one* of the two sets;
- the set $\{2, 5\}$ consisting of all elements belonging to *both* of the two sets;
- the set $\{3\}$ consisting of all elements belonging to the first set but not the second, and the set $\{1, 8\}$ consisting of all elements belonging to the second set but not the first.

Each of these new sets is a particular instance of a general construction for sets. We now consider them in turn.

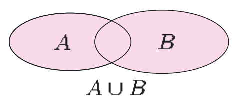
Union

We saw above that if $A = \{2, 3, 5\}$ and $B = \{1, 2, 5, 8\}$, then the set of all elements belonging to at least one of the sets A and B is $\{1, 2, 3, 5, 8\}$. We call this set the *union* of A and B.

More generally, we adopt the following definition.

Definition Let A and B be any two sets; then the **union** of A and B is the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$



Note that the word *or* in this definition is used in the inclusive sense of 'and/or'; that is, the set $A \cup B$ consists of the elements of A and the elements of B, including the elements in both A and B.

In everyday language, an example of 'or' used in the *exclusive* sense is 'Tea or coffee?', since the answer 'Both, please!' is not expected. An example of 'or' used in the *inclusive* sense is 'Milk or sugar?', since in this case you could answer 'Both'.

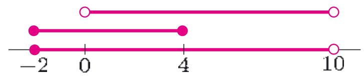
Example 1.6

(a) Simplify $[-2, 4] \cup (0, 10)$.

(b) Express the domain of the function $f(x) = \sqrt{x^2 - 1}$ as a union of intervals.

Solution

(a) The union is the interval $[-2, 10)$.



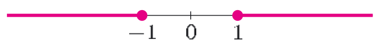
$[-2, 4] \cup (0, 10) = [-2, 10)$

(b) The domain consists of all real numbers x for which $x^2 - 1 \geq 0$: that is, $x^2 \geq 1$, so $x \geq 1$ or $x \leq -1$. Thus the domain of f is the set

$$\{x \in \mathbb{R} : x \leq -1 \text{ or } x \geq 1\}.$$

This is the set of numbers belonging either to the interval $(-\infty, -1]$ or to the interval $[1, \infty)$, and it can therefore be written as

$$(-\infty, -1] \cup [1, \infty). \quad \blacksquare$$



$(-\infty, -1] \cup [1, \infty)$

Exercise 1.15

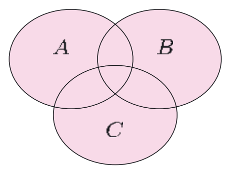
(a) Simplify $(1, 7) \cup [4, 11]$.

(b) Express the domain of the function $f(x) = 1/\sqrt{x^2 - 9}$ as a union of intervals.

(c) Draw a diagram depicting the union of the half-plane $H = \{(x, y) \in \mathbb{R}^2 : y < 0\}$ and the disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.

So far we have defined the union of two sets. We can give a similar definition for the union of any number of sets; for example, the union of three sets A, B and C is the set

$$A \cup B \cup C = \{x : x \in A \text{ or } x \in B \text{ or } x \in C\}.$$



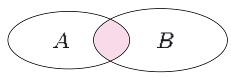
$A \cup B \cup C$

Intersection

We saw above that if $A = \{2, 3, 5\}$ and $B = \{1, 2, 5, 8\}$, then the set of all elements belonging to both of the sets A and B is $\{2, 5\}$. We call this set the *intersection* of A and B.

More generally, we adopt the following definition.

> **Definition** Let A and B be any two sets; then the **intersection** of A and B is the set
>
> $$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$



$A \cap B$

Two sets with no element in common, such as $\{1, 3, 5\}$ and $\{2, 9\}$, are said to be **disjoint**. We write $\{1, 3, 5\} \cap \{2, 9\} = \varnothing$.

Example 1.7

(a) Simplify $[-2, 4] \cap (0, 10)$.

(b) Express the domain of the function $f(x) = 1/\sqrt{4 - x^2} + 1/\sqrt{9 - x^2}$ as an intersection of intervals, and simplify your answer.

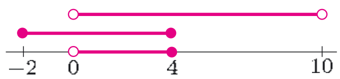
Solution

(a) The intersection is the interval $(0, 4]$.

(b) The domain consists of all real numbers x for which both $4 - x^2 > 0$ and $9 - x^2 > 0$; that is, $x^2 < 4$ and $x^2 < 9$. Thus the domain of f is the set of real numbers x that belong both to the interval $(-2, 2)$ and to the interval $(-3, 3)$. It can therefore be written as

$$(-2, 2) \cap (-3, 3);$$

this is simply the interval $(-2, 2)$. ■



$[-2, 4] \cap (0, 10) = (0, 4]$

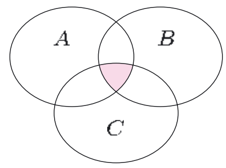
Exercise 1.16

(a) Simplify $(1, 7) \cap [4, 11]$.

(b) Draw a diagram depicting the intersection of the half-plane $H = \{(x, y) \in \mathbb{R}^2 : y < 0\}$ and the disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.

So far we have defined the intersection of two sets. We can give a similar definition for the intersection of any number of sets; for example, the intersection of three sets A, B and C is the set

$$A \cap B \cap C = \{x : x \in A \text{ and } x \in B \text{ and } x \in C\}.$$



$A \cap B \cap C$

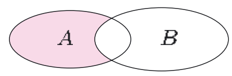
Difference

We saw above that if $A = \{2, 3, 5\}$ and $B = \{1, 2, 5, 8\}$, then the set of all elements belonging to A but not to B is $\{3\}$; we call this set the *difference* $A - B$. Similarly, the set of all elements belonging to B but not to A is $\{1, 8\}$; this set is the *difference* $B - A$.

More generally, we adopt the following definition.

Definition Let A and B be any two sets; then the **difference** between A and B is the set

$$A - B = \{x : x \in A,\ x \notin B\}.$$



$A - B$

Some texts denote the difference between A and B by $A \setminus B$.

Remark Note that $A - B$ is different from $B - A$, when $A \neq B$. Also, for any set A, we have $A - A = \varnothing$.

Example 1.8

(a) Simplify $[-2, 4] - (0, 10)$ and $(0, 10) - [-2, 4]$.

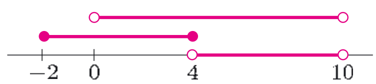
(b) Express the domain of the function $f(x) = 1/(x^2 - 1)$ as a difference between two sets.

Solution

(a) The difference $[-2, 4] - (0, 10)$ is the interval $[-2, 0]$, and the difference $(0, 10) - [-2, 4]$ is $(4, 10)$.

(b) The domain consists of all real numbers x for which $x^2 - 1 \neq 0$; that is, $x \neq 1$ and $x \neq -1$. Thus the domain of f is the difference

$$\mathbb{R} - \{1, -1\}.$$ ■



$(0, 10) - [-2, 4] = (4, 10)$

Exercise 1.17

(a) Simplify $(1,7)-[4,11]$ and $[4,11]-(1,7)$.

(b) Draw diagrams depicting $H-D$ and $D-H$, where H is the half-plane $\{(x,y)\in\mathbb{R}^2 : y<0\}$ and D is the disc $\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 4\}$.

Further exercises

Exercise 1.18 Which of the following statements are true?

(a) $0\in\mathbb{N}$ (b) $0\in\mathbb{Q}$ (c) $-0.6\notin\mathbb{R}$ (d) $37\in\mathbb{Z}$

(e) $20\in\{4,8,12,16\}$ (f) $\{1,2\}\in\{\{2,3\},\{3,1\},\{2,1\}\}$

(g) $\{0\}\in\varnothing$

Exercise 1.19 List the elements of the following sets.

(a) $\{n : n\in\mathbb{N} \text{ and } 2<n<7\}$ (b) $\{x\in\mathbb{R} : x^2+5x+4=0\}$

(c) $\{n\in\mathbb{N} : n^2=25\}$

Exercise 1.20 Use set notation to specify each of the following sets:

(a) the set of integers greater than -20 and less than -3;

(b) the set of non-zero integers which are multiples of 3;

(c) the set of all real numbers greater than 15.

Exercise 1.21 Sketch the following sets in $\mathbb{R}^2$.

(a) $\{(x,y)\in\mathbb{R}^2 : y=4-3x\}$

(b) $\{(x,y)\in\mathbb{R}^2 : (x+1)^2+(y-3)^2=9\}$

(c) $\{(x,y)\in\mathbb{R}^2 : y^2=8x\}$

Exercise 1.22 Sketch the following sets in $\mathbb{R}^2$.

(a) $\{(x,y)\in\mathbb{R}^2 : y<4-3x\}$

(b) $\{(x,y)\in\mathbb{R}^2 : (x+1)^2+(y-3)^2>9\}$

(c) $\{(x,y)\in\mathbb{R}^2 : 0\leq x\leq 2,\ 1\leq y\leq 3\}$

Exercise 1.23 For each of the sets A and B below, determine whether $A\subseteq B$.

(a) $A=\{(0,0),(0,6),(-4,6)\}$ and $B=\{(x,y)\in\mathbb{R}^2 : (x+2)^2+(y-3)^2=13\}$.

(b) $A=\{(x,y)\in\mathbb{R}^2 : x^2+y^2<4\}$ and $B=\{(x,y)\in\mathbb{R}^2 : y<4-8x\}$.

(c) $A=\{(2\cos t, 3\sin t) : t\in[0,2\pi]\}$ and $B=\{(x,y)\in\mathbb{R}^2 : \dfrac{x^2}{4}+\dfrac{y^2}{9}=1\}$.

Exercise 1.24 Show that A is a proper subset of B, where

$$A=\{(x,y)\in\mathbb{R}^2 : x^2+4y^2<1\} \text{ and } B=\{(x,y)\in\mathbb{R}^2 : y<\tfrac{1}{2}\}.$$

Exercise 1.25 For each of the sets A and B below, determine whether $A = B$.

(a) $A = \{1, -1, 2\}$ and $B = \{x \in \mathbb{R} : x^3 - 2x^2 - x + 2 = 0\}$.

(b) $A = \{(2\cos t, 3\sin t) : t \in [0, 2\pi]\}$ and

$$B = \{(x, y) \in \mathbb{R}^2 : \frac{x^2}{4} + \frac{y^2}{9} = 1\}.$$

(c) $A = \{x \in \mathbb{R} : x = \frac{p}{q}, \text{ where } p, q \in \mathbb{N}\}$ and $B = \mathbb{Q}$.

Exercise 1.26 For each of the sets A and B below, find $A \cup B$, $A \cap B$ and $A - B$.

(a) $A = \{0, 2, 4\}$ and $B = \{4, 5, 6\}$.

(b) $A = (-5, 3]$ and $B = [2, 17]$.

(c) $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.