Introduction to mathematical modelling

The Open University

1 Pollution in the Great Lakes

This section explores a real-world system where mathematical modelling has been used to understand what is happening and to predict what will happen if changes are made. The system concerned is extremely complex but, by keeping things as simple as possible, sufficient information will be extracted to allow a mathematical model of the system to be obtained. Refinements to this simple model will be made in Section 5.

The Great Lakes of North America (Figure 1.1) provide drinking water for tens of millions of people who live in the surrounding area. They also provide a source of food, transport and recreation. In the first half of the twentieth century they were used for dumping sewage and other pollutants. Sources of pollution include industrial waste, agricultural chemicals, acid rain, and oil and chemical spills. The case study concerns the construction of a mathematical model of how the pollution level in a lake varies over time.

The information for this case study comes from two main sources:

Rainey, R.H. (1967) 'Natural displacement of pollution from the Great Lakes', *Science*, **155**, 1242–3;

Thomann, R.V. and Mueller, J.A. (1987) *Principles of Surface Water Quality Modeling and Control*, Harper and Row.

Figure 1.1 The Great Lakes of North America

Now view the first video sequence for this text, 'Pollution in the Great Lakes'.

DVD

Summary of the modelling stages

The multimedia package discussed in detail how the stages of the mathematical modelling process can be applied to the problem of predicting future pollution levels in the Great Lakes. A summary of these details is given below for reference. This summary also includes some information from the video and some additional material not included in the package or on the video.

1 Specify the purpose of the model

Specify purpose

Define the problem

The problem is to predict how long it will take for the level of pollution in a lake to reduce to a target level if all sources of pollution are eliminated. It is intended to use this mathematical model to investigate pollution levels in any one of the Great Lakes, although the model could be used for any polluted lake.

Decide which aspects of the problem to investigate

The model investigates how the pollution level varies with time as clean water flows into the lake and polluted water flows out.

2 Create the model

Create model

State assumptions

The model assumes that:

(a) all sources of pollution have ceased;

(b) the pollutant does not biodegrade in the lake or decay through any other biological, chemical or physical process;

(c) the pollutant is evenly dispersed within the lake at all times;

(d) water flows into and out of the lake at the same constant rate (so that all seasonal effects can be ignored);

(e) all other water gains and losses (e.g. rainfall, evaporation, extraction and seepage) can be ignored;

(f) the volume of water in the lake is constant;

(g) if the mathematical model is to be used for the downstream lakes (Huron, Erie and Ontario), then negligible pollution is flowing into them from the upstream lakes.

The statement that the volume is constant is a consequence of the previous two assumptions, but it is included as an assumption in its own right because of its importance in the modelling process.

Choose variables and parameters

The variables in the model are:

t	the time, in seconds, since all sources of pollution ceased;
$m(t)$	the mass, in kilograms, of pollutant in the lake at time t;
$c(t)$	the concentration, in kilograms per cubic metre, of pollutant in the lake at time t.

The parameters in the model are:

c_{target}	the target concentration level, in kilograms per cubic metre, of pollutant in the lake;
T	the time taken, in seconds, to reduce the concentration of pollutant to the target level c_{target} (that is, $c(T) = c_{\text{target}}$);
V	the volume, in cubic metres, of water in the lake;
r	the water flow rate, in cubic metres per second, into and out of the lake;
$k = r/V$	the proportionate flow rate, in seconds^{-1}.

The essential difference between a variable and a parameter is that a variable is a quantity whose values change during the situation the model describes, while a parameter is a constant of the model (for a given situation).

Formulate mathematical relationships

The relationship between concentration and mass is given by

$$c(t) = \frac{m(t)}{V}.$$

The **input–output principle**,

$$\boxed{\text{accumulation}} = \boxed{\text{input}} - \boxed{\text{output}},$$

is applied to the mass of pollutant during the time interval $[t, t + \delta t]$. This principle is often used in mathematical modelling. You will see another example of its use in Section 4.

The input–output principle is also applied in MST209 *Unit 16*.

By Assumptions (a) and (g), the mass of pollutant entering the lake is zero, so the *input* is zero. By Assumption (b), the pollutant leaves the lake only through the outflow of water. In the time interval $[t, t + \delta t]$, the volume of water leaving the lake is $r\,\delta t$, where r is constant because of Assumptions (d), (e), and (f). Multiplying this by the concentration of pollutant, which is uniform by Assumption (c), gives the mass of pollutant that leaves the lake in that time interval as $rc(t)\delta t$ or $(r/V)m(t)\delta t$. This is the *output*.

The *accumulation* of the mass of pollutant within the lake, over the time interval $[t, t + \delta t]$, is the difference between $m(t + \delta t)$ and $m(t)$, that is, $m(t + \delta t) - m(t)$. Applying the input–output principle therefore gives

$$m(t + \delta t) - m(t) \simeq 0 - \frac{r}{V} m(t)\delta t.$$

It is sometimes useful to replace a group of parameters by a single parameter, particularly when the same group of parameters is often repeated; this is called **reparametrization**. Replacing r/V by k, dividing by δt and then letting $\delta t \to 0$, leads to the differential equation

$$\frac{dm}{dt} = -km(t), \quad \text{where } k = \frac{r}{V}. \tag{1.1}$$

3 Do the mathematics

Do mathematics

Solve equations

The solution to the differential equation for m is

$$m(t) = m(0)e^{-kt}, \quad \text{where } m(0) \text{ is the initial mass of pollutant.}$$

Since $c(t) = m(t)/V$, the pollutant concentration c is given by

$$c(t) = c(0)e^{-kt}, \quad \text{where } c(0) \text{ is the initial concentration of pollutant.}$$

The time, T, taken for the pollutant concentration to reduce to the target level, $c_{\text{target}} = c(T) = c(0)e^{-kT}$, is given by

$$T = -\frac{1}{k} \ln\left(\frac{c_{\text{target}}}{c(0)}\right).$$

Draw graphs

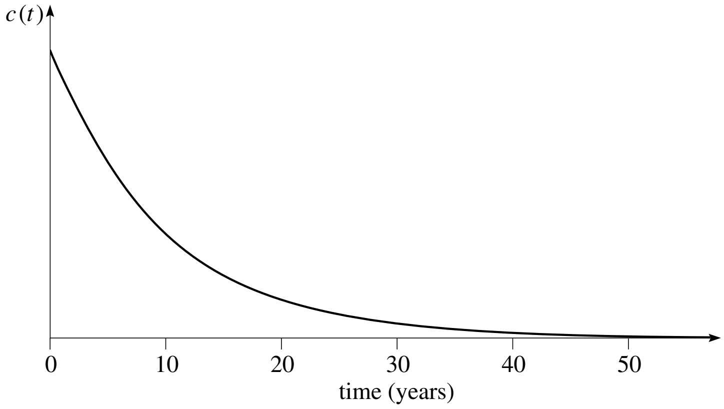

Figure 1.2 Typical graph of concentration against time

Derive results

Suppose that a political decision has been taken to reduce the level of pollution to a tenth of its initial level, so that $c_{\text{target}}/c(0) = \frac{1}{10}$. The corresponding length of time T required is

$$T = -\frac{1}{k}\ln\left(\frac{1}{10}\right) = \frac{\ln 10}{k} \simeq \frac{2.30}{k},$$

so T is inversely proportional to the proportionate flow rate k.

4 Interpret the results

Interpret results

Collect relevant data

The only data required are the values of k. These can be determined for each lake from its volume V and water flow rate r, since $k = r/V$. The data are given in Table 1.1 for all of the Great Lakes.

Table 1.1

Lake	Volume V (10^{12} m^3)	Water flow rate r (10^3 m^3 s^{-1})	$k = r/V$ (10^{-9} s^{-1})
Superior	12.10	2.01	0.166
Michigan	4.92	1.57	0.319
Huron	3.54	5.10	1.441
Erie	0.48	5.90	12.292
Ontario	1.64	6.78	4.134

The units given mean, for example, that the volume of Lake Superior is 12.10×10^{12} m^3, its water flow rate is 2.01×10^3 m^3 s^{-1}, and its proportionate flow rate is 0.166×10^{-9} s^{-1}.

Describe the mathematical solution in words

For Lake Superior, for example, the model predicts that a reduction in the level of pollution to a tenth of its initial level will take

$$T = \frac{\ln 10}{k} = 13.9 \times 10^9 \text{ seconds.}$$

Dividing T by $60 \times 60 \times 24 \times 365.24$ to convert it into years, the model predicts that it will take 439 years for the pollution level in Lake Superior to drop by a factor of ten. A similar calculation for Lake Michigan predicts that it will take 7.22×10^9 seconds, or 229 years, for the pollution level to drop by a factor of ten.

According to Assumption (g) on page 7, only clean water enters the downstream lakes Huron, Erie and Ontario from the upstream lakes, so the model can be used for these lakes as well. Similar calculations then predict that the times required to reduce the pollution level to a tenth

It may be necessary to revisit this assumption later.

of its initial level are 51 years for Lake Huron, 6 years for Lake Erie and 18 years for Lake Ontario.

The reason for the long time periods that are predicted for purging the two upper lakes is their rather low proportionate flow rates. The quantity $V/r = 1/k$ is the *average retention time* for water in the lake, that is, the length of time it takes for an amount of water equal to the volume of water in the lake to flow out of the lake, and hence the time any molecule of water would expect to remain in the lake. For Lake Superior, the average retention time is 191 years, and that for Lake Michigan is 99 years. For the downstream lakes the average retention times are much shorter.

Decide what results to compare with reality

It is perhaps fortunate that the two most polluted lakes in the 1960s were Lake Erie and Lake Ontario, which have relatively small volumes of water and fairly high water flow rates. It was thus possible, as predicted by the model, to clean up these lakes fairly quickly.

You will see a model of the cumulative effects of pollution levels on the whole system in Subsection 6.2.

The model predicts that if significant amounts of pollution are allowed to enter Lake Superior or Lake Michigan, then it will take hundreds of years for the lakes to recover. It could be argued that the mathematical model is quite conservative in its estimation, since it assumes that all sources of pollution have stopped and that the pollutant is completely dispersed in the water, rather than in the flora and fauna or in the sediment at the bottom of the lake. However, since most pollutants biodegrade, it could also be argued that the estimate is rather pessimistic. Even so, a major contamination of either of these lakes, particularly by a pollutant which does not biodegrade, would be a catastrophe for which there would, apparently, be no short-term solution.

It is salutary to note that, in recent years, there have been Fish Advisory Warnings in place for many areas, and in particular for Lake Michigan. People are advised not to eat fish from that lake because of the high level of toxic chemicals in them. Although these chemicals are present in small quantities in the lakes, they tend to concentrate when moving up through the food chain, here leading to fish which accumulate significant quantities of pollutant chemicals. The battle to control pollution in these lakes has not yet been won.

This concludes the summary of the first four modelling stages for this problem. The fifth stage, 'Evaluate the model', will be addressed in Subsection 5.1.

The development of the pollution model for the Great Lakes has highlighted a number of important aspects of mathematical modelling. These include the following.

- By representing quantities using symbols, it is possible to develop models that can be applied in a variety of different situations.
- The simplifying assumptions must be relevant to the model, and should be linked to the mathematical relationships between the variables and parameters.
- Data for the model should be collected after it has been established exactly what data are required.

End-of-section Exercises

Exercise 1.1

In your own words, and without using any equations or symbols, give an outline, or description, of the formulation of the model in this section.

Exercise 1.2

(a) Suppose that a pollutant continues to enter a lake at a constant rate of $q\,\mathrm{kg\,s^{-1}}$ and assume that this pollutant instantaneously diffuses throughout the lake. How would the mathematical model given by Equation (1.1) change?

(b) Show that the mass of pollutant in the lake is now modelled by

$$m(t) = \left(m(0) - \frac{q}{k}\right) e^{-kt} + \frac{q}{k}.$$

What is the long-term effect on the mass of pollutant in the lake?

(c) What happens if, initially, there is no pollution in the lake?

(d) What is the corresponding mathematical model for pollutant concentration?