M343
Applications of probability

Modelling events in time and space

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1.1

Contents

Random processes

1 What is a random process?

In this section, the basic ideas of random processes and the notation used to represent them are introduced. Several examples are described in Subsection 1.1, and the Bernoulli process is discussed in Subsection 1.2.

1.1 Basic ideas

Examples 1.1 to 1.4 illustrate some fundamental ideas concerning random processes.

Example 1.1 *The gambler's ruin*

Two players, Adam and Ben, with initial capital £k and £$(a-k)$, respectively (where a and k are positive integers and $a > k$), engage in a series of games that involve some element of chance. After each game, the loser pays the winner £1. The series of games continues until one player has lost all his money, in which case he is said to be ruined.

This situation can be described in terms of a sequence of random variables. Let X_n represent Adam's capital (in £) after n games. The series of games ends when either $X_n = 0$ (in which case Adam is ruined) or $X_n = a$ (Ben is ruined). Suppose that $k = 4$ and $a = 7$, and that Adam's capital at the start and after each subsequent game is given by the sequence

$$4, 5, 6, 5, 4, 3, 4, 5, 6, 5, 6, 7.$$

This is a **realisation** of the sequence of random variables $\{X_n\}$ for $n = 0, 1, \ldots, 11$. In this case the series of games ends when Ben is ruined after the eleventh game.

If Adam and Ben had played again, the results of their games would probably have produced a different sequence of values from the one given above; that is, a different realisation of the sequence of random variables $\{X_n\}$ would have been obtained. Note that the distribution of each X_n (other than X_0) depends on chance. It also depends on the value of X_{n-1}: for example, if $X_{n-1} = 4$, then the value of X_n can be only either 3 or 5. ♦

The sequence of random variables $\{X_n; n = 0, 1, \ldots\}$ described in Example 1.1 is an example of a **random process** or **stochastic process**. Essentially, a random process is a sequence of random variables that is developing, usually over time, though as you will see later, it can be developing in space. The term 'random process' is also used for the whole underlying situation that is being observed.

The word 'stochastic' is derived from a Greek word meaning 'to aim at'. The evolution of its meaning is uncertain, but it now means 'pertaining to chance' or 'random'. It is pronounced sto-kas-tic.

In Example 1.1, Adam's capital was observed only at specific instants of time, immediately after each game had been completed. A process that is observed continuously during an interval of time is described in Example 1.2.

Example 1.2 *Customers in a village shop*

For two hours, a record was kept of the number of customers in a village shop. Figure 1.1 shows the number of customers in the shop at time t ($0 \leq t \leq 2$), where t is measured in hours.

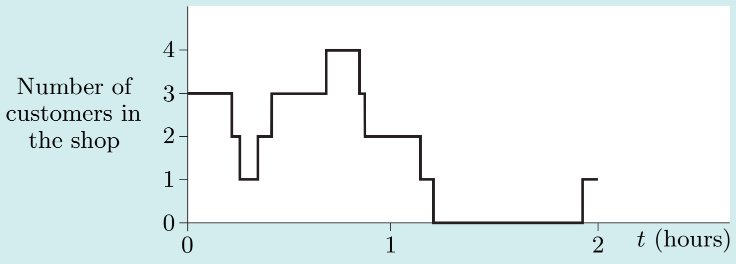

Figure 1.1 The number of customers in a shop

Each rise in the graph represents a customer arriving, and each fall, a customer leaving. At the start of the period of observation, there were three customers in the shop, and during the second hour there was a period of about 40 minutes when the shop was empty.

This example is a particular case of a *queueing process*; such processes are studied in *Book 4*.

If a record had been kept at the same time of day on another day, it would almost certainly have been different from the one represented in Figure 1.1: it would have been a different realisation of the stream of random variables $\{X(t); 0 \leq t \leq 2\}$, where $X(t)$ is the number of customers in the shop at time t. On each occasion that such records are kept, a realisation is obtained of a developing situation – the number of customers in the shop at time t.

For any fixed t, the distribution of $X(t)$ will depend on several factors, such as the arrival rate of customers and the time a customer spends in the shop. It also depends on the starting value. For example, in the realisation shown in Figure 1.1, $X(0) = 3$, so it would be very unlikely that $X(0.01) = 0$, because it is unlikely that all three customers would leave in under a minute. On the other hand, if the shop had been empty at time 0, then it would be quite likely that $X(0.01) = 0$. ♦

In the gambler's ruin example, the process is observed only at specific points in time. This process is said to be a **discrete-time random process**. On the other hand, the number of customers in the village shop is a **random process in continuous time**. In both of the examples, the random variables have taken only discrete non-negative integer values. Two processes for which the random variables are continuous are described in Examples 1.3 and 1.4.

Example 1.3 *Replacing light bulbs*

Suppose that there is a supply of light bulbs whose lifetimes are independent and identically distributed. One bulb is used at a time, and as soon as it fails it is replaced by a new one. Let W_n represent the time at which the nth bulb fails and is replaced. Then the random process $\{W_n; n = 1, 2, \ldots\}$ gives the sequence of replacement times. Note that the 'time' variable in this example is n, the number of the bulb, so this is a discrete-time random process. However, the lifetime of a bulb is a continuous random variable, so W_n, the replacement time of the nth light bulb, is continuous. ♦

This is an example of a *renewal process*; such processes are discussed in *Book 5*.

Example 1.4 The water level in a reservoir

The level of water in a reservoir depends on supply, in the form of rain or melting snow, on demand, which is the water used by the community served by the reservoir, and on other minor factors, such as evaporation from the surface. Both supply and demand may be assumed to be continuous random variables. Let $L(t)$ be a random variable representing the level of water in the reservoir at time t, where t is measured in years. The level of the reservoir can be observed at any time $t \geq 0$, so $\{L(t); t \geq 0\}$ is a continuous-time random process. A typical realisation of the process $\{L(t); t \geq 0\}$ is shown in Figure 1.2.

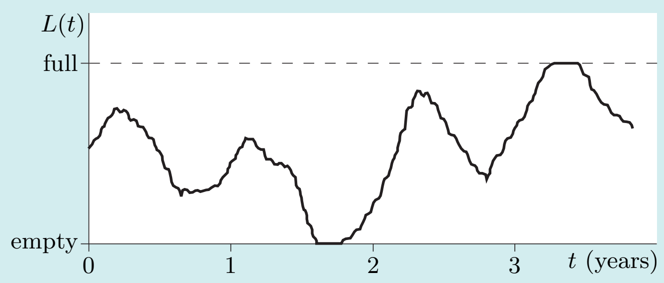

Figure 1.2 A realisation of $\{L(t); t \geq 0\}$, where $L(t)$ is the water level in a reservoir

The water level $L(t)$ tends to reach a maximum around March after the winter rain, and to fall to a minimum in the autumn, because there is more demand (watering gardens, etc.) in the summer, but less supply. However, there is variation from year to year. The water level may fall to zero in a drought or remain at a maximum for a period of time when the reservoir is full and an overflow is in operation. In this example, the random variable $L(t)$ is continuous and the random process develops in continuous time. ♦

Mathematically, a random process is a collection $\{X(t)\}$ or $\{X_n\}$ of random variables defined in either continuous time or discrete time. For a random process in continuous time, the time variable is usually defined for an interval of the real line, most commonly $-\infty < t < \infty$ or $t \geq 0$, in which case the collection is written as $\{X(t); t \in \mathbb{R}\}$ or $\{X(t); t \geq 0\}$. The range of t is called the **time domain**. For a random process in discrete time, the time variable is usually defined at integer points, and the process is written as $\{X_n; n = 0, 1, 2, \ldots\}$. In this case, the time domain is $\{0, 1, 2, \ldots\}$. Thus typically a random process with a discrete time domain is written as $\{X_n; n = 0, 1, 2, \ldots\}$, and a random process with a continuous time domain is written as $\{X(t); t \geq 0\}$.

The set of values that may be taken by the random variables in a random process is called the **state space** of the process. A state space can be either discrete or continuous. In Example 1.2, $X(t)$ is the number of customers in a village shop at time t, so the state space of $\{X(t); 0 \leq t \leq 2\}$ is $\{0, 1, 2, \ldots\}$. The state space is discrete in this example. In Example 1.4, $L(t)$ is the level of water in a reservoir, which is a continuous non-negative variate, so the state space of $\{L(t); t \geq 0\}$ is $\{l : l \geq 0\}$. The state space is continuous in this example.

Activity 1.1 Time domains and state spaces

For each of the random processes described in Examples 1.1 and 1.3, identify the time domain and say whether it is discrete or continuous. Also write down the state space and say whether it is discrete or continuous.

Activity 1.2 More time domains and state spaces

A shop is open between 9 am and 6 pm on weekdays. Two random processes associated with the daily business of the shop are as follows.

(a) $\{A_n; n = 1, 2, \ldots\}$, where A_n is the amount (in £) spent by the nth customer who enters the shop during the day.

(b) $\{B(t); 0 \leq t \leq 9\}$, where $B(t)$ is the number of items sold by t hours after the shop opens.

For each process, identify the time domain and say whether it is discrete or continuous. Also write down the state space and say whether it is discrete or continuous.

1.2 The Bernoulli process

The term **Bernoulli trial** is used to describe a single statistical experiment in which there are two possible outcomes; these outcomes are referred to as 'success' and 'failure'. A sequence of independent Bernoulli trials for which the probability of success remains constant from trial to trial is called a **Bernoulli process**. The formal definition of a Bernoulli process is given in the following box.

Bernoulli process

A **Bernoulli process** is a sequence of Bernoulli trials in which:

- ◇ trials are independent;
- ◇ the probability of success remains the same from trial to trial.

Examples of Bernoulli processes include: a sequence of rolls of a die where success is a 'six'; testing items off a production line where success is a 'good' item and failure is a defective item; and treating successive patients arriving at a hospital where success is a 'cure'.

For a Bernoulli process, the idea of trials occurring in order, one after the other, is crucial.

There are several different sequences of random variables associated with a Bernoulli process. The simplest one is $\{X_n; n = 1, 2, \ldots\}$, where $X_n = 1$ if the nth trial results in success, and $X_n = 0$ if it is a failure. Each X_n has a Bernoulli distribution with parameter p, the probability of success at any trial. The random variables $X_1, X_2, \ldots$ are independent and identically distributed, and the distribution of X_n does not depend on n or on the results of previous trials.

The 'time' variable, n, is discrete; it denotes the number of the trial. The random variables are also discrete-valued; the state space is $\{0, 1\}$. Therefore the random process $\{X_n; n = 1, 2, \ldots\}$ has a discrete time domain and a discrete state space.

The real time variable, the time between trials, is not considered in a Bernoulli process; it is irrelevant, for example, exactly when patients arrive at the hospital.

A typical realisation of the process $\{X_n; n = 1, 2, \ldots\}$ is

$$0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 0.$$

In this realisation, successes occur at trials 4, 6, 7, 9, 10 and 11, and failures at the other trials. In this realisation there were 13 trials.

Another sequence of random variables associated with a Bernoulli process is $\{Y_n; n = 1, 2, \ldots\}$, where

$$Y_n = X_1 + X_2 + \cdots + X_n.$$

This sequence specifies the number of successes that have occurred after n trials have been completed. The realisation of the sequence $\{Y_n; n = 1, 2, \ldots, 13\}$ corresponding to the realisation of $\{X_n; n = 1, 2, \ldots, 13\}$ is

$$0, 0, 0, 1, 1, 2, 3, 3, 4, 5, 6, 6, 6.$$

This realisation is represented in Figure 1.3.

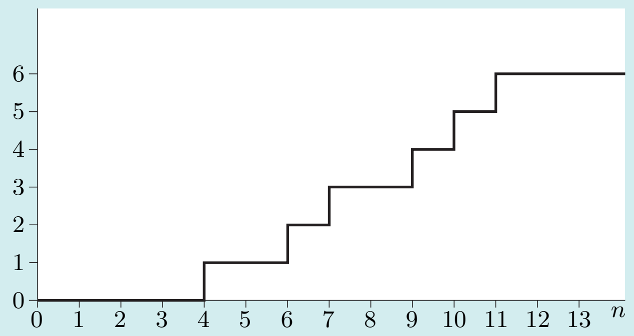

Figure 1.3 A realisation of $\{Y_n; n = 1, 2, \ldots, 13\}$ from a Bernoulli process

Note that, since the random variable Y_n is discrete, the graph of any realisation of the sequence $\{Y_n; n = 1, 2, \ldots\}$ appears as an increasing step function.

Activity 1.3 The distribution of Y_n

(a) What is the unconditional distribution of Y_n?

(b) Specify the conditional distribution of Y_n given $Y_{n-1} = y$.

Activity 1.4 Another sequence

Suggest a third sequence of random variables associated with a Bernoulli process – that is, another random process. You should use the notation for random processes in your answer. Write down the distribution of the random variables in the sequence.

Say whether the time domain of your random process is discrete or continuous, write down the state space, and say whether the state space is discrete or continuous.

Example 1.5 The British national lottery

A gambler who buys a single ticket each Saturday for the British national lottery has probability $p = 1/13\,983\,816$ of winning the jackpot (or a share of it) each week. This probability is unaltered from Saturday to Saturday, and what happens on any Saturday is independent of what has occurred on previous Saturdays. The Bernoulli random variable X_n is defined to take the value 1 if the gambler wins the jackpot at the nth attempt, and 0 otherwise. Then over a period of (say) 20 Saturdays, the random process $\{X_n; n = 1, 2, \ldots, 20\}$ is a discrete-valued discrete-time random process that may be modelled by a Bernoulli process with parameter p. ♦

Example 1.6 Births of boys and girls in a family

At first sight, a Bernoulli process might appear to be a useful model for the sequence of boys and girls in a family – scoring 1 for a girl, say, and 0 for a boy. Then a family of five with four sisters and a baby brother might be represented by the sequence $1, 1, 1, 1, 0$. The probability p of a female child may be estimated from data.

For some purposes, the model may be adequate. Indeed, departures from the two defining assumptions of the Bernoulli process are typically rather hard to detect. But extended analyses of available data suggest that the assumption of independence from birth to birth is not valid. Nature has a kind of memory, and the gender of a previous child affects to some degree the probability distribution for the gender of a subsequent child. ♦

Activity 1.5 Modelling the weather

Observers classified each day in a three-week interval as either 'Wet' or 'Dry' according to some rule for summarising the weather on any particular day. The 21-day sequence was as follows.

Wet Wet Wet Dry Dry Wet Wet Dry Wet Wet Wet
Wet Dry Dry Dry Wet Wet Wet Dry Dry Dry

Scoring 1 for a wet day and 0 for a dry day, the sequence may be written more conveniently as

1 1 1 0 0 1 1 0 1 1 1 1 0 0 0 1 1 1 0 0 0.

This is a realisation of the random process $\{X_n; n = 1, 2, \ldots, 21\}$, where the random variable X_n is discrete and takes the value 0 or 1, so that X_n is a Bernoulli random variable.

The time variable n denotes the number of the day in the sequence. Thus the random process $\{X_n; n = 1, 2, \ldots\}$ has a discrete time domain and a discrete state space.

Explain whether or not a Bernoulli process is likely to be a good model for the daily weather.

Activity 1.5 raises an important point: probability models are not usually 'right' or 'wrong': they may be better described as adequate or inadequate. However, M343 is concerned primarily with the development and application of models, rather than with testing their adequacy.

Summary of Section 1

In this section, a random process has been defined to be a sequence of random variables, and the notation for random processes has been introduced. You have seen that the time domain and the state space of a random process can each be either discrete or continuous. The Bernoulli process has been discussed. It has been noted that the term 'random process' is used to describe a whole physical process as well as a sequence of random variables associated with a physical process.

2 Further examples

This section consists of further examples of random processes, many of which will be discussed in detail later in the module. At this stage, you are expected to note the type of situation that may be modelled by a random process, to become accustomed to identifying sequences of random variables, to recognise whether the time domain and the state space of a random process are discrete or continuous, and to practise using the notation for random processes.

Example 2.1 A ticket queue

Many probability models have been developed to describe various queueing situations. The simplest situation is that of a ticket queue at a box office where customers arrive and join the end of the queue; they eventually reach the front and are served, then they leave. Sometimes an arriving customer will find that there is no queue and is served immediately.

To define the queueing process completely, the arrival and service mechanisms must be specified. The commonest and simplest assumption is that customers arrive at random. Frequently, the service time is assumed to have an exponential distribution. ♦

Activity 2.1 Random processes for a queue

There are several random processes associated with the queueing model described in Example 2.1. The basic random process is $\{Q(t); t \geq 0\}$, where $Q(t)$ is the number of people in the queue at time t. Other processes include $\{L_n; n = 1, 2, \ldots\}$, where L_n is the number of people in the queue when the nth customer arrives, and $\{W_n; n = 1, 2, \ldots\}$, where W_n is the time that the nth customer has to wait before being served.

For each of these processes, say whether the time domain and the state space are discrete or continuous, and write down the state space.

The simple queueing model in Example 2.1 can be extended in many ways. Possibilities include having more than one server, arrival according to an appointments system, arrival of customers in batches, and 'baulking'. ('Baulking' is a term used to describe the phenomenon of a long queue discouraging customers from joining.)

Several *queueing processes* are developed and analysed in *Book 4*.

Example 2.2 Machine breakdowns

A machine can be in one of two states: working or under repair. As soon as it breaks down, work begins on repairing it; and as soon as the machine is repaired, it starts working again.

For $t \geq 0$, let $X(t)$ be a random variable such that $X(t) = 1$ if the machine is working and $X(t) = 0$ if it is under repair. Then the sequence $\{X(t); t \geq 0\}$ is a random process with continuous time domain and having the discrete state space $\{0, 1\}$. ♦

Activity 2.2 Another random process

Suggest another random process associated with the model for machine breakdowns described in Example 2.2. Say whether the time domain is discrete or continuous. Write down the state space and say whether it is discrete or continuous.

The model for machine breakdowns in Example 2.2 could be extended to include several machines, and questions could then be asked about how many machines are working at any time or how many mechanics are required to prevent a build up of broken-down machines. The model could include a third state: broken and awaiting repair.

Example 2.3 The card-collecting problem

With every petrol purchase, an oil company gives away a card portraying an important event in the history of the petroleum industry. There are 20 such cards, and on each occasion the probability of receiving any particular card is 1/20. For an individual customer, one sequence of random variables associated with this situation is $\{X_n; n = 1, 2, \ldots\}$, where $X_n = 1$ if the card received at his nth purchase is a new one for his collection, and $X_n = 0$ if his nth card is a replica of one he has already. Both the time domain and the state space are discrete. The state space is $\{0, 1\}$. ♦

Activity 2.3 Collecting cards

(a) Suppose that the customer in Example 2.3 has i different cards after $n - 1$ purchases. Write down the distribution of X_n in this case.

(b) Explain whether or not the card-collecting process is a Bernoulli process.

(c) Identify two other random processes (sequences of random variables) associated with the card-collecting process. For each sequence, say whether the time domain and the state space are discrete or continuous. In each case, give the state space.

Example 2.4 The weather

Suppose that, at a particular location, the weather is classified each day as either wet or dry according to some specific criterion – perhaps wet if at least 1 mm of rain is recorded, otherwise dry. Weather tends to go in spells of wet or dry, and a possible model is that the weather on any one day depends only on the weather the previous day. For example, if it rains today, then the probability that it will rain tomorrow is $\frac{3}{5}$, and the probability that it will be dry is $\frac{2}{5}$; on the other hand, if it is dry today, then the probability that it will be dry (wet) tomorrow is $\frac{7}{10}$ $\left(\frac{3}{10}\right)$.

This is an example of a *Markov chain.* Such models are discussed in *Book 3.*

The random variable X_n could be defined to take the value 0 if it is wet on day n, and 1 if it is dry on day n. The sequence $\{X_n; n = 0, 1, 2, \ldots\}$ is a random process with a discrete time domain and a discrete state space. The time domain is $\{0, 1, 2, \ldots\}$ and the state space is $\{0, 1\}$.

The sorts of question that arise include the following. If it is wet on Monday, what is the probability that it will be wet the following Thursday? What proportion of days will be wet in the long run? If it is wet today, how long is it likely to be before the next wet day? ♦

Activity 2.4 *The weather model*

The model described in Example 2.4 can be thought of as a sequence of trials. Explain why it is *not* a Bernoulli process.

Example 2.5 *Family surnames*

In a community, a surname is passed down from generation to generation through male offspring only. Suppose that each man has a number of sons. This number is a random variable taking the values $0, 1, 2, \ldots$. Each man reproduces independently of all others.

One ancestor (patriarch) has a number of sons who form the first generation. Each of these has sons who form the second generation, and so on. Let the random variable X_n represent the number of men in the nth generation, with $X_0 = 1$ denoting the original ancestor. Then $\{X_n; n = 0, 1, 2, \ldots\}$ is an example of a *branching process*. The time domain, which represents the generation number, is discrete. The state space is $\{0, 1, 2, \ldots\}$, which is also discrete.

Branching processes are discussed in *Book 3*.

The questions that are of interest include the following. What is the distribution of the size of the nth generation? And will the family surname survive, or will it eventually become extinct? ♦

Example 2.6 *The spread of a disease*

Suppose that an infectious disease is introduced into a community and spreads through it. At any time, each member of the community may be classified as belonging to just one of four categories: healthy but susceptible to the disease; having the disease and infectious; recovered and immune from a further attack; dead. These categories can be called S_1, S_2, S_3, S_4, respectively.

A person may pass from S_1 to S_2 after contact with someone in S_2. Anyone in S_2 will eventually go to either S_3 or S_4. Four sequences of random variables, $\{S_1(t); t \geq 0\}$, $\{S_2(t); t \geq 0\}$, $\{S_3(t); t \geq 0\}$, $\{S_4(t); t \geq 0\}$, can be defined, where $S_i(t)$ is the number of people in category S_i at time t. Each of these processes is defined for continuous time, and its state space is discrete. If the disease starts in a community of size N with a single infectious person, then $S_1(0) = N - 1$, $S_2(0) = 1$, $S_3(0) = 0$, $S_4(0) = 0$. If, at some time t, $S_2(t) = 0$, then the disease will spread no further. There is a relationship between the four variates: if no one enters or leaves the community, then $S_1(t) + S_2(t) + S_3(t) + S_4(t) = N$, the total size of the community.

To develop a model for this process, it is necessary to specify the mechanics of the spread of the disease, the probabilities that an infected person will recover or die, the time spent in various stages, and so on. ♦

This is an example of an *epidemic process*, for which several models are described in *Book 4*.

Activity 2.5 A population model

A colony of bacteria develops by the division (into two) of bacteria and by the death of bacteria. No bacterium joins or leaves the colony.

(a) Identify two random processes to describe the development of this colony, and in each case specify whether the state space and the time domain are discrete or continuous. Write down the state space for each process.

(b) Suppose that the colony starts with two bacteria. Sketch a possible realisation of the size of the colony over time.

This is an example of a *birth and death process*. Such processes are analysed in *Book 4*.

Example 2.7 The price of wheat

In an article published in 1953, Professor Sir Maurice Kendall considers wheat prices in Chicago, measured in cents per bushel at weekly intervals from January 1883 to September 1934 (with a gap during the war years). A portion of these data is shown in Figure 2.1.

Kendall, M.G. (1953) 'The analysis of economic time-series, Part I: Prices', *Journal of the Royal Statistical Society, Series A*, vol. 116, no. 1, pp. 11–34.

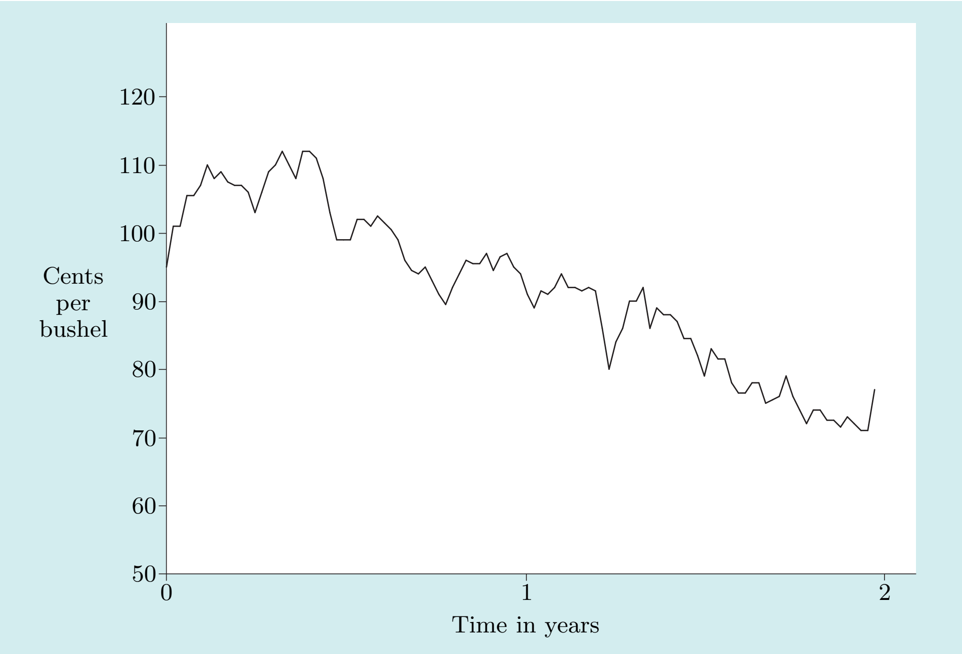

Figure 2.1 The price of wheat in Chicago

There is an overall fall in the price over the two-year period shown in the graph. However, as Kendall reported, first impressions are 'almost as if once a week the Demon of Chance drew a random number ... and added it to the current price to determine the next week's price'. Although observed only once a week, the price of wheat could change at any time, and the price itself varies continuously (though rounded to the nearest cent). The random process $\{Q(t); t \geq 0\}$, where $Q(t)$ is the price of wheat at time t, is therefore an example of a random process where both the time domain and the state space are continuous; it is an example of a *diffusion process*. ♦

This and other diffusion processes are studied in *Book 5*.

Example 2.8 The thickness of wool yarn

The thickness of wool yarn is not uniform along the length of the yarn. Figure 2.2 shows the variation in weight per unit length (essentially, the variation in cross-sectional area) along a strand of wool yarn.

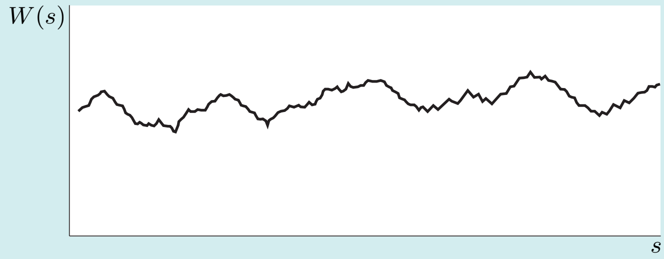

Figure 2.2 Variation of weight per unit length along a strand of wool yarn

The length s along the yarn corresponds to the time variable t that has appeared in previous examples, and is continuous. The random variable $W(s)$, weight per unit length, is also continuous. Thus the 'time' domain and the state space of the random process $\{W(s); s \geq 0\}$ are both continuous. This is an example of a random process developing over distance instead of time. ♦

Random processes in space are discussed in Part III of this book. Random processes in two-dimensional space, where a process develops over an area of land, arise frequently in biology and geology. One such process is described in Example 2.9.

Example 2.9 The bee orchid

Pingle Wood and Cutting is a nature reserve on the route of the now disused Great Eastern Railway line between March and St Ives. It is owned by The Wildlife Trust, and bee orchids can be found there in June. In any year, the distribution of the orchid in the reserve can be thought of as a random process in two-dimensional space. Each point can be identified as (x, y) according to a map reference, and a random variable X is defined by $X(x, y) = 1$ if an orchid grows at that point, and $X(x, y) = 0$ otherwise. Then $\{X(x, y); x \in \mathbb{R}, y \in \mathbb{R}\}$ is a random process where the equivalent of the 'time' variable actually refers to space and is two-dimensional and continuous. The state space is $\{0, 1\}$, which is discrete.

Naturalists might be interested in questions such as the following. Does the orchid grow randomly in the reserve or does it favour certain soil types or south-sloping land? Does it grow singly or in clumps? To answer such questions, a model would have to be set up, and the data collected and compared with the model. ♦

Summary of Section 2

In this section, a variety of examples of random processes have been described briefly. Some of these will be discussed in detail in M343. In most of the examples, the process was developing over time. However, you have seen that it is also possible to have a random process that develops in space.

Modelling events in time

3 The Poisson process

Some data on major earthquakes that occurred in the 20th century are described in Example 3.1.

Example 3.1 Earthquakes

Table 3.1 contains a list of the most serious earthquakes that occurred in the 20th century up to 1977, and includes the date and place of occurrence, the magnitude of the earthquake and the estimated number of fatalities. The magnitude of an earthquake is measured on the Richter scale and relates to the energy released.

Table 3.1 Major earthquakes in the 20th century up to 1977

Date			Magnitude (if known)	Region	Estimated number of fatalities
1902	Dec	16		Turkestan	4 500
1905	Apr	4	8.6	India: Kangra	19 000
	Sept	8		Italy: Calabria	2 500
1906	Jan	31	8.9	Colombia	1 000
	Mar	16		Formosa: Kagi	1 300
	Apr	18	8.3	California: San Francisco	700
	Aug	17	8.6	Chile: Santiago, Valparaiso	20 000
1907	Jan	14		Jamaica: Kingston	1 600
	Oct	21	8.1	Central Asia	12 000
1908	Dec	28	7.5	Italy: Messina, Reggio	83 000
1911	Jan	3	8.7	China: Tien-Shan	450
1912	Aug	9	7.8	Marmara Sea	1 950
1915	Jan	13	7.0	Italy: Avezzano	29 980
	Oct	3	7.6	California, Nevada	0
1920	Dec	16	8.6	China: Kansu, Shansi	100 000
1922	Nov	11	8.4	Peru: Atacama	600
1923	Sept	1	8.3	Japan: Tokyo, Yokohama	143 000
1925	Mar	16	7.1	China: Yunnan	5 000
1927	Mar	7	7.9	Japan: Tango	3 020
	May	22	8.3	China: Nan-Shan	200 000
1929	May	1	7.1	Iran: Shirwan	3 300
	June	16	7.8	New Zealand: Buller	17
1930	July	23	6.5	Italy: Ariano, Melfi	1 430
1931	Feb	2	7.9	New Zealand: Hawke's Bay	255
1933	Mar	2	8.9	Japan: Morioka	2 990
1934	Jan	15	8.4	India: Bihar-Nepal	10 700
1935	Apr	20	7.1	Formosa	3 280
	May	30	7.5	Pakistan: Quetta	30 000
1939	Jan	25	8.3	Chile: Talca	28 000
	Dec	26	7.9	Turkey: Erzincan	30 000
1943	Sept	10	7.4	Japan: Tottori	1 190
1944	Dec	7	8.3	Japan: Tonankai, Nankaido	1 000
1945	Jan	12	7.1	Japan: Mikawa	1 900
1946	Nov	10	7.4	Peru: Ancash	1 400
	Dec	20	8.4	Japan: Tonankai, Nankaido	1 330
1948	June	28	7.3	Japan: Fukui	5 390
	Oct	5	7.6	Turkmenia, Ashkhabad	Unknown
1949	Aug	5	6.8	Ecuador: Ambato	6 000
1950	Aug	15	8.7	India, Assam, Tibet	1 530
1952	Mar	4	8.6	Japan: Tokachi	28
	July	21	7.7	California: Kern County	11
1954	Sept	9	6.8	Algeria: Orléansville	1 250
1955	Mar	31	7.9	Phillipines: Mindanao	430
1956	June	9	7.7	Afghanistan: Kabul	220
	July	9	7.7	Aegean Sea: Santorini	57
1957	July	28	7.8	Mexico: Acapulco	55
	Dec	4	8.0	Mongolia: Altai-Gobi	30
	Dec	13	7.1	Iran: Farsinaj, Hamadan	1 130
1958	July	10	7.8	Alaska, Brit. Columbia, Yukon	5
1960	Feb	29	5.8	Morocco: Agadir	14 000
	May	22	8.5	Chile: Valdivia	5 700
1962	Sept	1	7.3	Iran: Qazvin	12 230
1963	July	26	6.0	Yugoslavia: Skopje	1 200
1964	Mar	28	8.5	Alaska: Anchorage, Seward	178
1968	Aug	31	7.4	Iran: Dasht-e Bayaz	11 600
1970	May	31	7.8	Peru: Nr Lima	66 000
1972	Dec	23	6.2	Nicaragua: Managua	5 000
1974	Dec	28	6.2	Pakistan: Pattan	5 300
1975	Feb	4	7.5	China: Haicheng, Liaoning	Few
1976	Feb	4	7.9	Guatemala	22 000
	May	6	6.5	Italy: Gemona, Friuli	1 000
	July	27	7.6	China: Tangshan	650 000
1977	Mar	4	7.2	Romania: Vrancea	2 000

An earthquake is included if its magnitude was at least 7.5 or if a thousand or more people were killed.

These data are from the discontinued Open University module S237 *The Earth: structure, composition and evolution*, Block 2.

Seismologists would study these data with specific objectives in mind. They might wish to study the structure of the Earth or to predict future earthquakes, for example. In this section, the times of occurrence will be studied. In order to do this, the times (in days) between successive earthquakes have been calculated; these are shown in Table 3.2. (The numbers should be read down the columns.)

Table 3.2 Times between major earthquakes (in days)

840	280	695	402	335	99	436	83	735
157	434	294	194	1354	304	30	832	38
145	736	562	759	454	375	384	328	365
44	584	721	319	36	567	129	246	92
33	887	76	460	667	139	9	1617	82
121	263	710	40	40	780	209	638	220
150	1901	46	1336	556	203	599	937	

These times range from 9 days up to 1901 days (which is over five years), so they have a very large variance. It is difficult to appreciate a pattern from studying a list of figures, so in order to develop some intuition about the pattern, the data are presented in two ways in Figures 3.1 and 3.2.

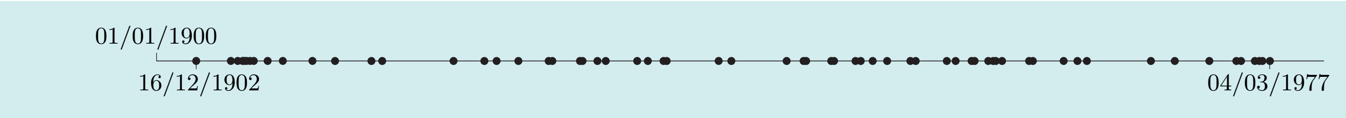

Figure 3.1 Times at which major earthquakes occurred, measured in days

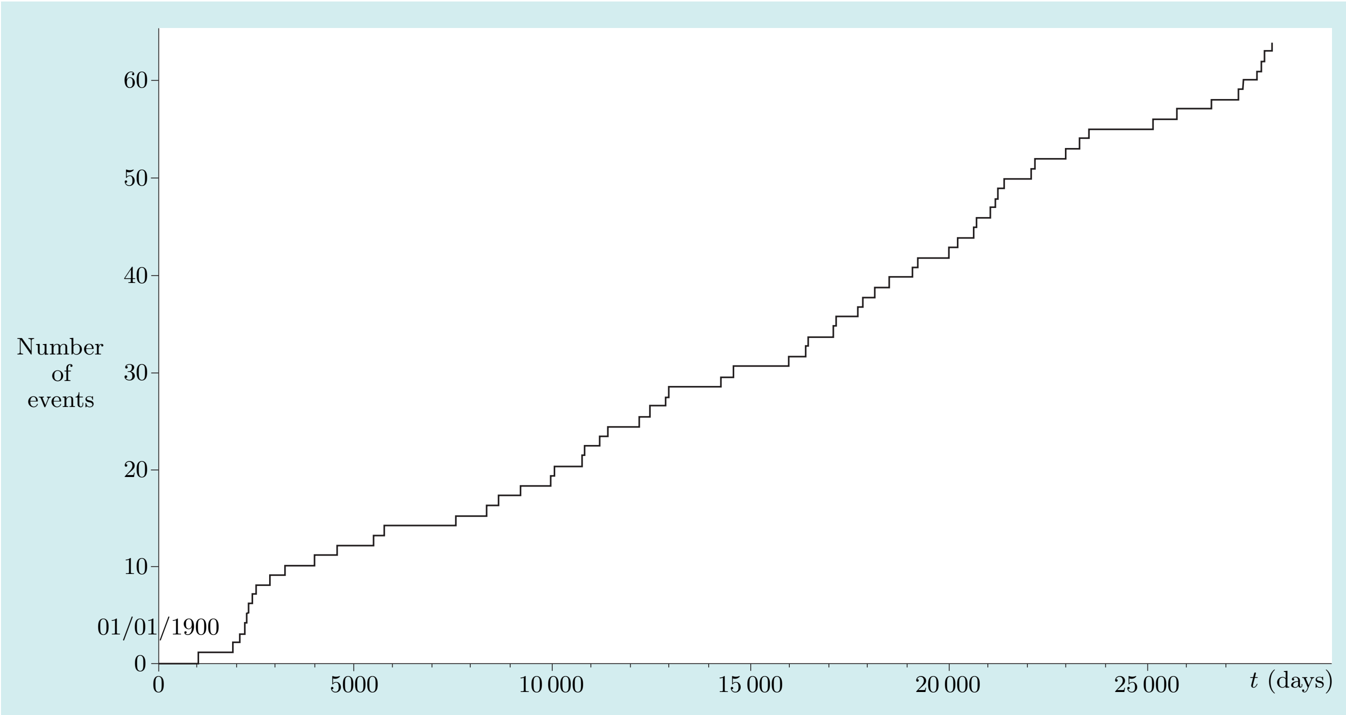

Figure 3.2 Cumulative number of earthquakes over time

In Figure 3.1, dots on a time axis show the incidence of earthquakes over the period of observation. Figure 3.2 gives a cumulative count with passing time.

The trend in Figure 3.2 could be approximated very roughly by a straight line; there is no pronounced curvature. This suggests that the rate of occurrence of earthquakes has remained more or less steady. However, the random element in the sequence of earthquake times is evident from both figures, and this is the element for which a model is required. ♦

In this section, a model that might be suitable for the occurrence of major earthquakes is described. This model is the **Poisson process**. It is a model for events occurring at random for which the rate of occurrence of events remains constant over time. In Subsection 3.1, the Poisson process is defined and some basic ideas and results are discussed. Some standard notation for events occurring in continuous time is introduced in Subsection 3.2. The simulation of events in a Poisson process is discussed briefly in Subsection 3.3.

3.1 Basic ideas and results

The Poisson process is the continuous-time analogue of the Bernoulli process. In a Poisson process, events occur 'at random', but instead of occurring as a result of regular trials, as in the Bernoulli process, they can occur at any time. The Poisson process provides a good model for such varied situations as the decay of radioactive atoms in a lump of material, the times of arrival of customers to join a queue, the instants at which cars pass a point on a road in free-flowing traffic, and the times of goals scored in a soccer match.

The Bernoulli process is a useful model when an event either occurs (a success) or does not occur (a failure) at each of a clearly defined sequence of opportunities (trials). The process is characterised by the following assumptions: the probability of success remains the same from trial to trial, and the outcome at any trial is independent of the outcomes of previous trials. In this sense, the sequence of successes and failures is quite haphazard.

Events may also occur in a random, haphazard kind of a way in continuous time, when there is no notion of a 'trial' or 'opportunity'. Examples of unforecastable random events occurring in continuous time (with an estimate of the rate at which they might happen) include the following:

- machine breakdowns on the factory floor (one every five days);
- light bulb failures in the home (one every three months);
- arrivals at a hospital casualty ward (one every ten minutes at peak time);
- major earthquakes worldwide (one every fourteen months);
- power cuts in the home ('frequently' in winter, 'seldom' in summer).

Typically, a realisation of this sort of random process is represented as a sequence of points plotted on a time axis, as in Figure 3.3, the points giving the times of occurrence.

Figure 3.3 Schematic representation of a random sequence of events in time

The Poisson process is defined in the following box.

Poisson process

The **Poisson process** is a model for the occurrence of events in continuous time in which the following assumptions are made.

- Events occur singly.
- The rate of occurrence of events remains constant.
- The incidence of future events is independent of the past.

Consider, for instance, the occurrence of light bulb failures in the home. Failures might be well (or at least adequately) modelled by a Poisson process: they never (or rarely) happen simultaneously and there is no particular reason why the rate at which they occur should vary with passing time. Perhaps it is just arguable that the incidence of past events provides indicators for the future. But remember that few models are 'right'; most are adequate, at best. And a Poisson process may be an adequate model for failures for the purpose of determining, say, the stock of light bulbs to keep in the home.

On the other hand, the incidence of power cuts in the home would not be well modelled by a Poisson process: the rate is greater in winter than in summer, so the second assumption is not reasonable in this case.

Two random variables are of particular interest for the Poisson process: the number of events that occur over any particular period of observation (for example, breakdowns in a month), which is a discrete random variable; and the time between successive events (or 'waiting time' between successive events, as it is often called), which is a continuous random variable. The distributions of these random variables are stated in the following box.

Poisson process: two main results

Suppose that events occur at random at rate λ in such a way that their occurrence may be modelled as a Poisson process. Then:

- ◇ $N(t)$, the number of events that occur during an interval of length t, has a Poisson distribution with parameter λt: $N(t) \sim \text{Poisson}(\lambda t)$;
- ◇ T, the waiting time between successive events, has an exponential distribution with parameter λ: $T \sim M(\lambda)$.

These two results are derived in Section 4. Their use is illustrated in Example 3.2.

Example 3.2 Arrivals at a casualty department

Suppose that over moderately short intervals, the incidence of patients arriving at a casualty department may usefully be modelled as a Poisson process in time with (on average) 10 minutes between arrivals.

Since the mean time between arrivals is 10 minutes, the rate of the Poisson process is

$$\lambda = \tfrac{1}{10} \text{ per minute.}$$

Therefore the number of arrivals in half an hour has a Poisson distribution with parameter

$$\lambda t = \tfrac{1}{10} \text{ per minute} \times 30 \text{ minutes} = 3.$$

That is,

$$N(30) \sim \text{Poisson}(3).$$

The probability that two patients arrive in half an hour is

$$P(N(30) = 2) = \frac{e^{-3}3^2}{2!} \simeq 0.2240.$$

If $X \sim \text{Poisson}(3)$, then

$$P(X = x) = \frac{e^{-3}3^x}{x!}$$

for $x = 0, 1, \ldots$.

The waiting time between arrivals has an exponential distribution with parameter $\frac{1}{10}$, so the probability that the interval between arrivals exceeds half an hour is given by $P(T > 30)$.

The c.d.f. of T is

$$F(t) = P(T \le t) = 1 - e^{-\lambda t}, \quad t \ge 0,$$

so

$$P(T > t) = e^{-\lambda t}.$$

Therefore

$$P(T > 30) = e^{-30/10} = e^{-3} \simeq 0.0498. \quad ♦$$

Activity 3.1 Nerve impulses

In a psychological experiment, nerve impulses were found to occur at the rate of 458 impulses per second. Assume that a Poisson process is a suitable model for the incidence of impulses.

(a) Calculate the probability that not more than one nerve impulse occurs in an interval of $\frac{1}{100}$ second.

(b) Calculate the probability that the interval between two successive impulses is less than $\frac{1}{1000}$ second.

Activity 3.2 Major earthquakes

Suppose that the incidence of major earthquakes worldwide may be adequately modelled as a Poisson process, and that earthquakes occur at the rate of one every fourteen months.

(a) Calculate the probability that there will be at least three major earthquakes in a period of ten years.

(b) Calculate the probability that the waiting time between successive major earthquakes exceeds two years.

3.2 Notation for continuous-time processes

Events in an interval

For *any* continuous-time process where successive events are counted, the number of events that occur between times t_1 and t_2 is denoted $X(t_1, t_2)$.

A realisation of a continuous-time process is depicted in Figure 3.4.

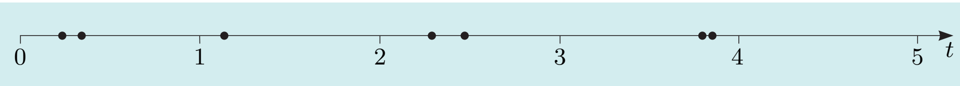

Figure 3.4 Events in a continuous-time process

In this realisation, the observed value of $X(2, 4)$ is 4, and the observed value of $X(0, 4)$ is 7. By convention, the random variable $X(0, t)$ is usually written simply as $X(t)$.

Activity 3.3 Notation

For the realisation of the continuous-time process in Figure 3.4, write down the values of $X(1, 3)$, $X(4, 5)$, $X(0, 2)$, $X(1)$ and $X(3)$.

For a Poisson process with rate λ, the number of events that occur during *any* interval of length t (denoted $N(t)$ in Subsection 3.1) has a Poisson distribution with parameter λt. So, in particular, $X(0, t) = X(t) \sim \text{Poisson}(\lambda t)$. Also, since $X(t_1, t_2)$ is the number of events that occur in the interval $(t_1, t_2]$, which has length $t_2 - t_1$, it has a Poisson distribution with parameter $\lambda(t_2 - t_1)$; that is, $X(t_1, t_2) \sim \text{Poisson}(\lambda(t_2 - t_1))$. Although for a Poisson process this distribution depends only on the length of the interval, note that for many continuous-time processes, the distribution of $X(t_1, t_2)$ depends on the values of both t_1 and t_2. You will meet a process for which this is the case in Section 6.

Waiting times

The waiting time from the start of observation to the first event in any continuous-time process is conventionally denoted T_1, and the waiting time between the first event and the second is denoted T_2. In general, the waiting time between the $(n-1)$th event and the nth event is denoted T_n. The waiting time from the start of observation to the time of the nth event is denoted W_n:
$W_n = T_1 + T_2 + \cdots + T_n$.

This notation is illustrated in Figure 3.5.

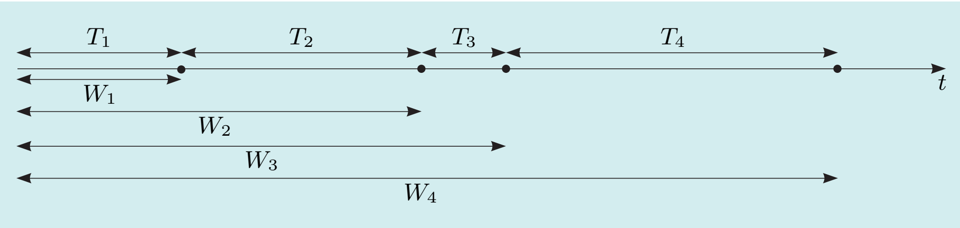

Figure 3.5 Waiting times

For a Poisson process, the random variables $T_1, T_2, \ldots, T_n$ are independent identically distributed exponential random variables with parameter λ.

Activity 3.4 The distribution of W_n

Write down the distribution of W_n, the waiting time from the start of observation to the nth event in a Poisson process with rate λ.

The notation introduced in this subsection, which will be used on many occasions in M343, is summarised in the following box.

Notation for continuous-time processes

- ◇ $X(t_1, t_2)$ is the number of events that occur in the interval $(t_1, t_2]$.
- ◇ $X(t) = X(0, t)$ is the number of events that occur in the interval $(0, t]$.
- ◇ T_1 is the time at which the first event occurs.
- ◇ T_n is the waiting time between the $(n-1)$th event and the nth event.
- ◇ W_n is the time at which the nth event occurs.

3.3 Simulation

The simplest way to simulate the occurrences of events in a Poisson process is to simulate the sequence of waiting times between events. These are independent observations of the random variable $T \sim M(\lambda)$, where λ is the rate of occurrence.

A table of random numbers from the exponential distribution with mean 1 is provided in the *Handbook*. To obtain a sequence of random numbers $t_1, t_2, \ldots$ from the exponential distribution $M(\lambda)$ (which has mean $1/\lambda$), each term in a sequence of random numbers $e_1, e_2, \ldots$ from the table must be multiplied by the mean $1/\lambda$:

Random numbers from $M(1)$ are provided in Table 6 in the *Handbook*.

$$t_j = \frac{1}{\lambda} e_j, \quad j = 1, 2, \ldots .$$

Example 3.3 Simulating events in a Poisson process

Suppose that events in a Poisson process occur at the rate of one every 10 minutes, so that the rate λ is 1/10 per minute, and hence the mean time between events is $1/\lambda = 10$ minutes.

To simulate the sequence of events occurring in the Poisson process, random numbers from $M(1)$ must be multiplied by 10 to obtain the times between successive events in minutes. The details of a simulation of the events from the start of observation for one hour are set out in Table 3.3. The random numbers are taken from the eleventh row of Table 6 in the *Handbook*.

In practice, computers rather than tables are used for simulation.

Table 3.3 A simulation

n	e_n	$t_n = 10e_n$	$w_n = t_1 + \cdots + t_n$
1	0.2316	2.316	$2.316 \simeq 2.32$
2	1.8293	18.293	$20.609 \simeq 20.61$
3	1.1956	11.956	$32.565 \simeq 32.57$
4	3.2992	32.992	$65.557 \simeq 65.56$

The first event occurs after $t_1 \simeq 2.32$ minutes. The time at which the second event occurs is given by $w_2 = t_1 + t_2 \simeq 20.61$ minutes, and the third event occurs at $w_3 = t_1 + t_2 + t_3 \simeq 32.57$ minutes.

The time of the fourth event is $w_4 \simeq 65.56$ minutes, which is more than one hour after the start of observation. So three events occur in the first hour of observation, and the times in minutes at which the first three events occur are

$$w_1 = 2.32, \quad w_2 = 20.61, \quad w_3 = 32.57. \quad \blacklozenge$$

Activity 3.5 Computer failures

In an investigation into computer reliability, a particular unit failed on average every 652 seconds. Assuming that the incidence of failures may be adequately modelled by a Poisson process, use the tenth row of Table 6 in the *Handbook* to simulate one hour's usage after switching the unit on. Give the times of failures to the nearest second.

Musa, J.D., Iannino, A. and Okumoto, K. (1987) *Software Reliability: Measurement, Prediction, Application*, McGraw-Hill.

Summary of Section 3

In this section, the Poisson process has been described. This is the continuous-time analogue of the Bernoulli process. It is a model for events that occur at random in continuous time at a constant rate.

If the rate of occurrence of events is λ, then the number of events that occur in an interval of length t has a Poisson distribution with parameter λt, and the waiting time between successive events has an exponential distribution with parameter λ. You have learned how to simulate the occurrences of events in a Poisson process using random observations from the exponential distribution with mean 1.

Notation has been introduced for the number of events in an interval, for the times between events, and for the times at which events occur. The notation applies to *any* continuous-time process.

Exercises on Section 3

Exercise 3.1 A Poisson process

Events occur according to a Poisson process in time with, on average, sixteen minutes between events.

(a) Write down the probability distribution of the waiting time T between events, where T is measured in hours.

(b) Write down the probability distribution of the number of events in any hour-long interval.

(c) If there were seven events between 2 pm and 3 pm yesterday, calculate the probability that there was at most one event between 3 pm and 4 pm.

(d) Calculate the probability that the waiting time between successive events will exceed half an hour.

Exercise 3.2 Volcanic eruptions

Data collected on major volcanic eruptions in the northern hemisphere give a mean time between eruptions of 29 months. Assume that such eruptions occur as a Poisson process in time.

(a) Calculate the expected number of eruptions during a five-year period.

(b) Calculate the probability that there are exactly two eruptions during a five-year period.

(c) Calculate the probability that at least three years pass after you have read this exercise before the next eruption.

Exercise 3.3 Light bulb failures

In Subsection 3.1, it is suggested that light bulb failures in the home might reasonably be modelled as a Poisson process with a rate of one failure every three months.

(a) Use this model to simulate the incidence of failures during a two-year period, setting out your realisation in a table similar to Table 3.3. Use random numbers e_n from the twelfth row of Table 6 in the *Handbook*.

(b) How many failures were there in your simulation? From which probability distribution is this number an observation?

4 A more formal approach to the Poisson process

In Subsection 3.1, the distribution of the number of events in a Poisson process that occur in an interval of length t, and the distribution of the time between successive events, were stated without proof. In order to derive these results, a more formal approach to the Poisson process is required than has been adopted so far. In this section, the assumptions of a Poisson process are expressed mathematically as three *postulates*. The postulates are then used to derive these results. Although you will not be expected to reproduce the derivations, you should work through this section thoroughly and make sure that you understand the ideas that are introduced here. The approach used here is used again in *Book 4*.

The first two postulates of the Poisson process express mathematically the assumptions that events occur singly and that the rate of occurrence of events remains constant. These postulates state the probability of one event in a short interval and the probability of two or more events in a short interval. The length of this short interval is denoted by δt (read as 'delta t').

The first postulate states that the probability that one event occurs in any interval of length δt is approximately $\lambda\,\delta t$. The fact that this probability is the same for any interval of length δt implies that the rate λ at which events occur remains constant over time. This postulate is written more precisely as follows.

I The probability that (exactly) one event occurs in any small time interval $[t, t + \delta t]$ is equal to $\lambda\,\delta t + o(\delta t)$.

The notation $o(\delta t)$ (read as 'little-oh of δt') is used to represent any function of δt that is of 'smaller order' than δt. Formally, we can write $f(\delta t) = o(\delta t)$ for any function f of δt such that

$$\frac{f(\delta t)}{\delta t} \to 0 \quad \text{as } \delta t \to 0. \tag{4.1}$$

For example, $(\delta t)^2 = o(\delta t)$ since

$$\frac{(\delta t)^2}{\delta t} = \delta t \to 0 \quad \text{as } \delta t \to 0,$$

and $(\delta t)^3 = o(\delta t)$ since

$$\frac{(\delta t)^3}{\delta t} = (\delta t)^2 \to 0 \quad \text{as } \delta t \to 0.$$

Since the notation $o(\delta t)$ is used to represent any function of δt that satisfies (4.1), it follows that

$$\frac{o(\delta t)}{\delta t} \to 0 \quad \text{as } \delta t \to 0.$$

The second postulate states formally the probability that two or more events occur in a short interval of length δt. It is written as follows.

II The probability that two or more events occur in any small time interval $[t, t + \delta t]$ is equal to $o(\delta t)$.

Essentially, this postulate expresses mathematically the assumption that events occur singly in a Poisson process.

The third postulate is a formal statement of the third assumption made in Subsection 3.1.

III The occurrence of events after any time t is independent of the occurrence of events before time t.

These three postulates are summarised in the following box.

Postulates for the Poisson process

A Poisson process is specified by three postulates.

I The probability that (exactly) one event occurs in any small time interval $[t, t + \delta t]$ is equal to $\lambda\,\delta t + o(\delta t)$.

II The probability that two or more events occur in any small time interval $[t, t + \delta t]$ is equal to $o(\delta t)$.

III The occurrence of events after any time t is independent of the occurrence of events before time t.

The distribution of $X(t)$

One sequence of random variables associated with a Poisson process is $\{X(t); t \geq 0\}$, where $X(t)$ denotes the number of events that have occurred by time t. It is assumed here that the process starts at time 0, so $X(0) = 0$. The distribution of $X(t)$ is given by $X(t) \sim \text{Poisson}(\lambda t)$, which may be proved formally as follows.

Suppose that observation of a Poisson process starts at time 0 and continues for a fixed time t. The number of events that occur in the interval $[0, t]$ is a random variable $X(t)$. The probability mass function of $X(t)$ will be denoted $p_x(t)$, so

$$p_x(t) = P(X(t) = x).$$

This is not the standard notation for a p.m.f. It is adopted here to stress that the probability is a function of t as well as of x.

The probability $p_0(t) = P(X(t) = 0)$ will be found first. This is the probability that no event occurs in $[0, t]$. The interval $[0, t]$ and a further short interval $[t, t + \delta t]$ will be considered, and $p_0(t + \delta t)$, the probability that no event has occurred at the end of the second interval, will be derived. Using Postulate III, the occurrence of events after any time t is independent of what happened before time t, so

$$\begin{aligned} p_0(t + \delta t) &= P(\text{no event by time } t + \delta t) \\ &= P(\text{no event in } [0, t]) \times P(\text{no event in } [t, t + \delta t]). \end{aligned}$$

The first of these two probabilities is $p_0(t)$. The second probability is equal to

$$\begin{aligned} &1 - P(\text{one event in } [t, t + \delta t]) - P(\text{two or more events in } [t, t + \delta t]) \\ &= 1 - (\lambda\, \delta t + o(\delta t)) - o(\delta t), \quad \text{using Postulates I and II,} \\ &= 1 - \lambda\, \delta t + o(\delta t). \end{aligned}$$

This illustrates one of the advantages of the notation $o(\delta t)$: expressions involving $o(\delta t)$ can be simplified. Above, $-o(\delta t) - o(\delta t)$ could be replaced by $o(\delta t)$ because adding or subtracting any finite number of functions of order smaller than δt always gives a function of order smaller than δt. Thus

$$p_0(t + \delta t) = p_0(t) \times (1 - \lambda\, \delta t + o(\delta t)),$$

which can be written

$$\frac{p_0(t + \delta t) - p_0(t)}{\delta t} = -\lambda\, p_0(t) + \frac{o(\delta t)}{\delta t}.$$

Since $p_0(t) \leq 1$, $o(\delta t)\, p_0(t) = o(\delta t)$.

Now let δt tend to 0. The left-hand side tends *by definition* to the derivative of $p_0(t)$ with respect to t. On the right-hand side, $o(\delta t)/\delta t$ tends to 0 by definition. This leads to the differential equation

$$\frac{dp_0(t)}{dt} = -\lambda\, p_0(t). \tag{4.2}$$

This equation can be integrated using separation of variables:

The separation of variables method is described in *Book 1* and in the *Handbook*.

$$\int \frac{dp_0(t)}{p_0(t)} = \int -\lambda\, dt.$$

Performing both integrations gives

$$\log p_0(t) = -\lambda t + c. \tag{4.3}$$

When $t = 0$, observation of the process is just starting, so no event has occurred, and hence

$$p_0(0) = P(\text{no event in } [0, 0]) = 1.$$

Putting $t = 0$ in (4.3) gives

$$\log 1 = 0 + c,$$

so $c = 0$. Thus the solution is

$$\log p_0(t) = -\lambda t,$$

which gives

$$p_0(t) = e^{-\lambda t}. \tag{4.4}$$

The differential equations satisfied by $p_x(t)$ $(x = 1, 2, \ldots)$ are derived in a similar way to the differential equation for $p_0(t)$. For example,

$$\begin{aligned} p_1(t+\delta t) &= P(\text{one event has occurred by time } t+\delta t) \\ &= P([\text{one event in } [0,t] \text{ and no event in } [t, t+\delta t]\,] \\ &\qquad \cup [\text{no event in } [0,t] \text{ and one event in } [t, t+\delta t]\,]). \end{aligned}$$

This is the union of two mutually exclusive events, so their separate probabilities can be added:

$$\begin{aligned} p_1(t+\delta t) &= P([\text{one event in } [0,t]\,] \cap [\text{no event in } [t, t+\delta t]\,]) \\ &\quad + P([\text{no event in } [0,t]\,] \cap [\text{one event in } [t, t+\delta t]\,]). \end{aligned}$$

Using the postulates gives

$$p_1(t+\delta t) = p_1(t) \times (1 - \lambda\,\delta t + o(\delta t)) + p_0(t) \times (\lambda\,\delta t + o(\delta t)).$$

Rearranging this equation gives

$$\frac{p_1(t+\delta t) - p_1(t)}{\delta t} = \lambda(p_0(t) - p_1(t)) + \frac{o(\delta t)}{\delta t}.$$

Letting $\delta t \to 0$ leads to the differential equation

$$\frac{dp_1(t)}{dt} = \lambda\, p_0(t) - \lambda\, p_1(t). \tag{4.5}$$

Substituting for $p_0(t)$ using (4.4), this becomes

$$\frac{dp_1(t)}{dt} + \lambda\, p_1(t) = \lambda e^{-\lambda t}.$$

This differential equation can be solved using the integrating factor method. In this case, the integrating factor is $e^{\lambda t}$, so both sides are multiplied by $e^{\lambda t}$:

The integrating factor method is described in *Book 1* and in the *Handbook*.

$$e^{\lambda t}\,\frac{dp_1(t)}{dt} + \lambda e^{\lambda t}\, p_1(t) = \lambda.$$

The left-hand side is the derivative of the product $e^{\lambda t} \times p_1(t)$, so the differential equation can be rewritten as

$$\frac{d}{dt}(e^{\lambda t}\, p_1(t)) = \lambda.$$

Integrating this equation gives

$$e^{\lambda t}\, p_1(t) = \lambda t + c.$$

When $t = 0$, no event has occurred, so $p_1(0) = P(X(0) = 1) = 0$, and hence $c = 0$. Therefore

$$p_1(t) = \lambda t e^{-\lambda t}.$$

For $x = 2, 3, \ldots$,

$$\begin{aligned} p_x(t+\delta t) &= P(x \text{ events have occurred by time } t+\delta t) \\ &= P([x \text{ events in } [0,t] \text{ and no event in } [t, t+\delta t]\,] \\ &\qquad \cup [(x-1) \text{ events in } [0,t] \text{ and one event in } [t, t+\delta t]\,] \\ &\qquad \cup [(x-2) \text{ events in } [0,t] \text{ and two events in } [t, t+\delta t]\,] \\ &\qquad \cup\ \ldots \\ &\qquad \cup [\text{no event in } [0,t] \text{ and } x \text{ events in } [t, t+\delta t]\,]). \end{aligned}$$

This is the union of mutually exclusive events, so their separate probabilities can be added:

$$\begin{aligned} p_x(t+\delta t) &= P([x \text{ events in } [0,t]\,] \cap [\text{no event in } [t, t+\delta t]\,]) \\ &\quad + P([(x-1) \text{ events in } [0,t]\,] \cap [\text{one event in } [t, t, +\delta t]\,]) \\ &\quad + P([(x-2) \text{ events in } [0,t]\,] \cap [\text{two events in } [t, t+\delta t]\,]) \\ &\quad + \cdots + P([\text{no event in } [0,t]\,] \cap [x \text{ events in } [t, t+\delta t]\,]). \end{aligned}$$

Using the postulates gives

$$\begin{aligned} p_x(t+\delta t) &= p_x(t) \times (1 - \lambda\,\delta t + o(\delta t)) + p_{x-1}(t) \times (\lambda\,\delta t + o(\delta t)) + o(\delta t) \\ &= p_x(t) + \lambda(p_{x-1}(t) - p_x(t))\,\delta t + o(\delta t). \end{aligned}$$

Again, the brevity of the interval $[t, t+\delta t]$ has been used to attach probabilities to the possibilities that nothing happens, one thing happens, or more than one thing happens.

Rearranging this equation gives

$$\frac{p_x(t+\delta t) - p_x(t)}{\delta t} = \lambda(p_{x-1}(t) - p_x(t)) + \frac{o(\delta t)}{\delta t}.$$

Letting $\delta t \to 0$ leads to the differential equations

$$\frac{dp_x(t)}{dt} = \lambda(p_{x-1}(t) - p_x(t)), \quad x = 2, 3, \ldots . \tag{4.6}$$

Note that the differential equation (4.5) for $p_1(t)$ is of this form with $x = 1$.

A set of differential equations satisfied by the p.m.f.s $p_x(t)$ has been derived. Each equation contains both $p_x(t)$ and $p_{x-1}(t)$. The equations can be solved recursively.

Activity 4.1 Finding $p_2(t)$

Using the value of $p_1(t)$, the differential equation in (4.6) for $x = 2$ can be solved using the method that was used for $x = 1$. Solve this differential equation to derive an expression for $p_2(t)$.

Continuing in this way, $p_3(t)$ could be found, then $p_4(t)$, and so on. However, if you look at the results so far, you may recognise that $p_0(t)$, $p_1(t)$ and $p_2(t)$ are probabilities from a Poisson distribution. The p.m.f. of the Poisson distribution with parameter μ is

$$p_X(x) = P(X = x) = \frac{e^{-\mu}\mu^x}{x!}, \quad x = 0, 1, 2, \ldots .$$

Hence $p_0(t)$, $p_1(t)$ and $p_2(t)$ are the first three probabilities in a Poisson distribution with parameter λt. That $X(t)$ has a Poisson distribution with parameter λt can be either proved by induction or verified by substitution in the general differential equation. The second of these methods will be used.

If $p_x(t) = e^{-\lambda t}(\lambda t)^x/x!$, then

$$\begin{aligned} \frac{d}{dt}(p_x(t)) &= \frac{-\lambda e^{-\lambda t}(\lambda t)^x}{x!} + \frac{e^{-\lambda t}\lambda^x x t^{x-1}}{x!} \\ &= -\lambda\,p_x(t) + \frac{e^{-\lambda t}(\lambda t)^{x-1}\lambda}{(x-1)!} \\ &= -\lambda\,p_x(t) + \lambda\,p_{x-1}(t). \end{aligned}$$

This is the differential equation (4.6). Therefore the solution of the set of differential equations (4.6) with initial values $p_0(0) = 1$, $p_x(0) = 0$, for $x = 1, 2, \ldots$, is the p.m.f. of the Poisson distribution with parameter λt.

The distribution of $X(t)$

The number of events that occur by time t in a Poisson process is a random variable $X(t)$. In a Poisson process with rate λ, $X(t)$ has a Poisson distribution with parameter λt. That is, $X(t) \sim \text{Poisson}(\lambda t)$ and

$$p_x(t) = P(X(t) = x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, \ldots .$$

Since λ, the rate of occurrence of events, is constant, this result holds for any time interval of length t; it does not matter when observation of the process starts.

Waiting times

The other quantity that is often of interest in a Poisson process, and indeed in many random processes, is the time between successive events. You have seen that the time from the start of the process until the first event is denoted T_1, and the time between the $(n-1)$th event and the nth event is denoted T_n. Each T_n is referred to as an **inter-event time**.

The distribution of T_1 can be derived using the distribution of $X(t)$. Since T_1 exceeds t if and only if no event occurs in $[0, t]$,

$$\begin{aligned} P(T_1 > t) &= P(\text{no event occurs in } [0, t]) \\ &= P(X(t) = 0) \\ &= e^{-\lambda t}, \quad \text{since } X(t) \sim \text{Poisson}(\lambda t). \end{aligned}$$

If T_1 has c.d.f. $F(t)$, then

$$F(t) = P(T_1 \leq t) = 1 - P(T_1 > t),$$

so

$$F(t) = 1 - e^{-\lambda t}, \quad t \geq 0.$$

Hence T_1 has the exponential distribution with parameter λ: $T_1 \sim M(\lambda)$.

By the memoryless property of the exponential distribution, observation can start at any stage of the process: the time at which the previous event occurred is irrelevant. Whether it occurred immediately before observation started or a long time before, the time until the next event always has an exponential distribution. Also, for $n = 1, 2, \ldots$, the distribution of the inter-event time T_n is exponential with parameter λ.

The distribution of T_n

In a Poisson process with rate λ, T_1, the time from the start of observation to the first event, has an exponential distribution with parameter λ.

For $n = 2, 3, \ldots$, T_n, the time between the $(n-1)$th event and the nth event, has an exponential distribution with parameter λ.

That is,

$$T_n \sim M(\lambda), \quad n = 1, 2, \ldots .$$

This result provides the simplest method of simulating observations from a Poisson process. Independent observations from $M(\lambda)$ can be used to simulate the inter-event times.

This result was used in Example 3.3.

Many of the processes studied in M343 are extensions of the Poisson process. Several models for events occurring in time, obtained by modifying the postulates in various ways, are discussed in Sections 5 to **??**. The Poisson process also provides the basis for many of the models discussed in *Book 4*.

Summary of Section 4

In this section, the assumptions of a Poisson process have been expressed mathematically as postulates, and the postulates have been used to derive the distributions of $X(t)$, the number of events that occur by time t, and T_1, the time at which the first event occurs.

5 The multivariate Poisson process

A **multivariate Poisson process** is a Poisson process in which each event may be just one of several different types of event. For instance, in traffic research, an event may correspond to the passage of a vehicle past a point of observation by the side of a road. The vehicles may themselves be classified as private cars, lorries, buses, motorcycles, and so on. So there are several types of event, and each event in the process is of just one type.

A typical realisation of a multivariate Poisson process is shown in Figure 5.1.

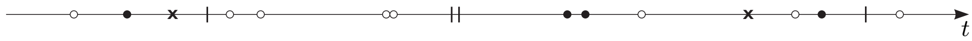

Figure 5.1 Events in a multivariate Poisson process

In general, the multivariate Poisson process may be described as follows.

Multivariate Poisson process

Suppose that events occur as a Poisson process in time at rate λ, and that each event is one of k types. If the probability that an event is of type i is p_i, where $p_1 + \cdots + p_k = 1$, and occurrences of events of different types are independent of each other, then the process is a **multivariate Poisson process**.

Consider the occurrence of events of type i in the small interval $[t, t + \delta t]$:

$$\begin{aligned}
&P(\text{one event of type } i \text{ occurs in } [t, t+\delta t]) \\
&= P(\text{one event occurs in } [t, t+\delta t] \text{ and it is of type } i) \\
&\quad + P(\text{more than one event in } [t, t+\delta t] \text{ and one of them is of type } i) \\
&= P(\text{one event in } [t, t+\delta t]) \times P(\text{an event is of type } i) + o(\delta t) \\
&= (\lambda\,\delta t + o(\delta t)) \times p_i + o(\delta t) \\
&= \lambda p_i\,\delta t + o(\delta t).
\end{aligned}$$

So

$$P(\text{one event of type } i \text{ occurs in } [t, t+\delta t]) = \lambda p_i\,\delta t + o(\delta t).$$

This is Postulate I for a Poisson process with rate λp_i. The other two postulates are also satisfied by the process of events of type i. Therefore events of type i occur as in a Poisson process with rate λp_i.

It follows that the multivariate Poisson process is built up of k independent Poisson processes with rates $\lambda p_1, \ldots, \lambda p_k$. This result can be used to calculate probabilities associated with events of different types.

Example 5.1 Vehicles passing an observer

Vehicles pass an observer standing at the edge of a main road according to a Poisson process at the rate of 100 vehicles per hour. Of these vehicles, the proportions of vehicles that are cars, lorries, coaches and motorcycles are 0.6, 0.3, 0.08 and 0.02, respectively.

What is the probability that more than four lorries pass the observer in a ten-minute interval? And what is the average waiting time between successive motorcycles passing the observer?

The process of vehicles passing the observer is a multivariate Poisson process: the proportions of the four types of vehicle are $p_1 = 0.6$, $p_2 = 0.3$, $p_3 = 0.08$ and $p_4 = 0.02$. Since $\lambda = 100$ per hour, the rates at which vehicles of the different types pass the observer are

$$\lambda_1 = \lambda p_1 = 60 \text{ per hour}, \quad \lambda_2 = \lambda p_2 = 30 \text{ per hour},$$
$$\lambda_3 = \lambda p_3 = 8 \text{ per hour}, \quad \lambda_4 = \lambda p_4 = 2 \text{ per hour}.$$

Thus, for example, lorries pass the observer at the rate $\lambda_2 = 30$ per hour.

If the number of lorries that pass the observer in t hours is denoted $L(t)$, then the number of lorries that pass the observer in ten minutes $\left(\frac{1}{6} \text{ hour}\right)$ is $L\left(\frac{1}{6}\right)$. This has a Poisson distribution with parameter $\lambda_2 t = 30/6 = 5$, so the probability that more than four lorries pass the observer in ten minutes is $P\left(L\left(\frac{1}{6}\right) > 4\right)$, where $L\left(\frac{1}{6}\right) \sim \text{Poisson}(5)$. That is,

$$\begin{aligned} P\left(L\left(\tfrac{1}{6}\right) > 4\right) &= 1 - P\left(L\left(\tfrac{1}{6}\right) \leq 4\right) \\ &= 1 - e^{-5}\left(1 + 5 + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!}\right) \\ &\simeq 0.5595. \end{aligned}$$

Now consider the waiting time between successive motorcycles. This has an exponential distribution with parameter $\lambda_4 = 2$. So the mean waiting time between successive motorcycles is

$$\frac{1}{\lambda_4} = \tfrac{1}{2} \text{ hour} = 30 \text{ minutes.}$$ ♦

Activity 5.1 *Bank customers*

Customers arrive at a bank according to a Poisson process at the rate of ten per minute. The proportion of customers who simply wish to draw out money from a cashpoint machine (type A) is 0.6, the proportion who wish to pay in money (type B) is 0.3, and the proportion who wish to carry out a more complicated transaction at the counter (type C) is 0.1.

(a) Calculate the probability that more than five customers arrive in an interval of length 30 seconds.

(b) Calculate the probability that six customers of type A arrive in one minute.

(c) Calculate the probability that six customers of type A, three of type B and at least one of type C arrive in one minute.

A multivariate Poisson process can alternatively be described as follows.

Suppose that events of type 1 occur as a Poisson process in time at rate λ_1, events of type 2 occur as a Poisson process in time at rate λ_2, …, and events of type k occur as a Poisson process in time at rate λ_k. If the Poisson processes may be assumed to be developing independently of one another, then the sequence of events obtained by superposing the events from the processes on the same time axis occurs as a Poisson process with rate

$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k.$$

In any realisation of this process, the probability that an event is of type i is given by

$$p_i = \frac{\lambda_i}{\lambda} = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \cdots + \lambda_k}.$$

You will need to use this result in Activity 5.2.

Activity 5.2 Telephone calls

A university tutor has noticed that over the course of an evening, telephone calls arrive from students according to a Poisson process at the rate of one every 90 minutes. Independently, calls arrive from members of her family according to a Poisson process at the rate of one every three hours, and calls from friends arrive according to a Poisson process at the rate of one call per hour. She does not receive any other calls.

(a) Calculate the probability that between 7 pm and 9 pm tomorrow evening, the tutor's telephone will not ring.

(b) Calculate the probability that the first call after 9 pm is from a student.

(c) Given that she receives four telephone calls one evening, calculate the probability that exactly two of the calls are from members of her family.

Summary of Section 5

The multivariate Poisson process is a model for events occurring at random in time in which each event may be one of several types. You have seen that it can also be thought of as the process obtained when the events in several independent Poisson processes are superposed on the same time axis.

Exercises on Section 5

Exercise 5.1 Post office customers

Customers arrive at a small post office according to a Poisson process at the rate of eight customers an hour. In general, 70% of customers post letters, 5% post parcels and the remaining 25% make purchases unrelated to the postal service.

(a) At what rate do customers arrive at the post office to post parcels?

(b) Calculate the probability that the interval between successive customers arriving to post parcels is greater than an hour.

(c) Calculate the probability that over a three-hour period, fewer than five customers arrive to post letters.

(d) Calculate the median waiting time between customers arriving to post something (either a letter or a parcel).

Exercise 5.2 Library acquisitions

New acquisitions arrive independently at a local library as follows: new works of fiction according to a Poisson process at the rate of eight a week, biographies according to a Poisson process with rate one a week, works of reference according to a Poisson process at the rate of one every four weeks, and non-text items according to a Poisson process at the rate of five a week. Assume that the library operates for seven days a week.

(a) Calculate the probability that at least two non-text acquisitions will arrive next week.

(b) Calculate the probability that no new work of fiction will arrive tomorrow.

(c) On average, what proportions of new acquisitions are fiction, biography, reference and non-text?

6 The non-homogeneous Poisson process

The Poisson process is a model for events occurring at random in continuous time at a constant rate λ. In Subsection 6.1, several examples are described of situations where the rate at which events occur cannot reasonably be assumed to be constant, and a model for such events is introduced. This model is the non-homogeneous Poisson process. The work in Subsection 6.1 includes a chapter of the computer book. Some basic results for the non-homogeneous Poisson process are derived in Subsection 6.2, and simulation is discussed in Subsection 6.3.

6.1 The model

Examples are used in this subsection to motivate and introduce a model for events occurring at random in time at a rate that changes with time.

Example 6.1 Mining accidents

Data are available on the dates of accidents in coal mines that were due to explosions and in which there were ten or more fatalities. Figure 6.1 shows the cumulative number of such explosions in coal mines in Great Britain for the period 15 March 1851 to 22 March 1962.

Jarrett, R.G. (1979) 'A note on intervals between coal-mining disasters', *Biometrika*, vol. 66, no. 1, pp. 191–3.

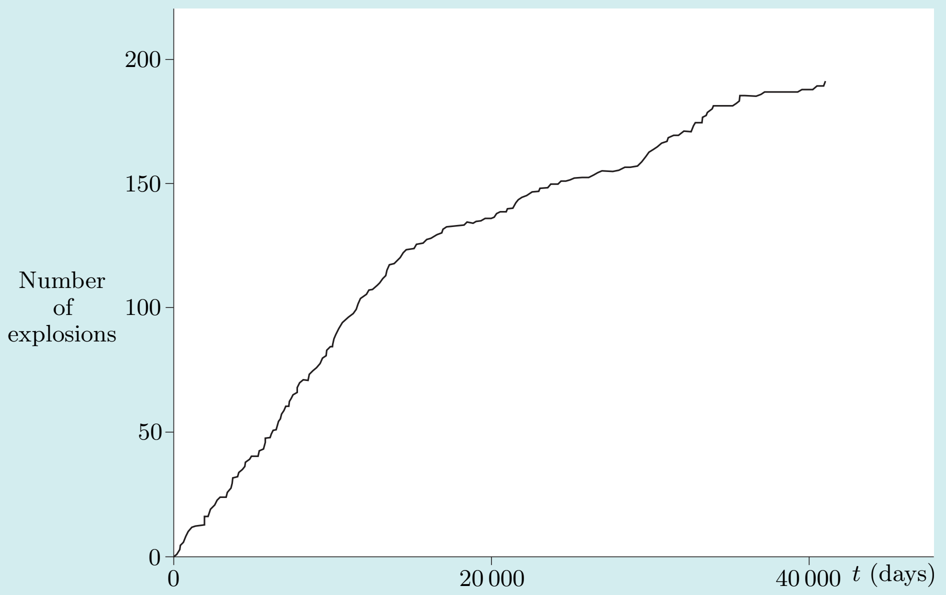

Figure 6.1 Cumulative number of explosions in coal mines in Great Britain, in which ten or more miners were killed, against time

This graph is roughly linear for the first 14 000 days (up to about 1890), but the gradient then becomes less steep. This means that the rate of occurrence of accidents decreased. A possible explanation for this is that safety precautions in coal mines were improved in about 1890. This is an example of a process where the rate of occurrence of events is not constant, but changes with time. ♦

Example 6.2 Road accidents

Another example of an event in time is a fatality in a road accident. The number of such fatalities in Great Britain in each year from 1992 to 2008 is shown in Figure 6.2.

These data were obtained from the website www.statistics.gov.uk/STATBASE in January 2010.

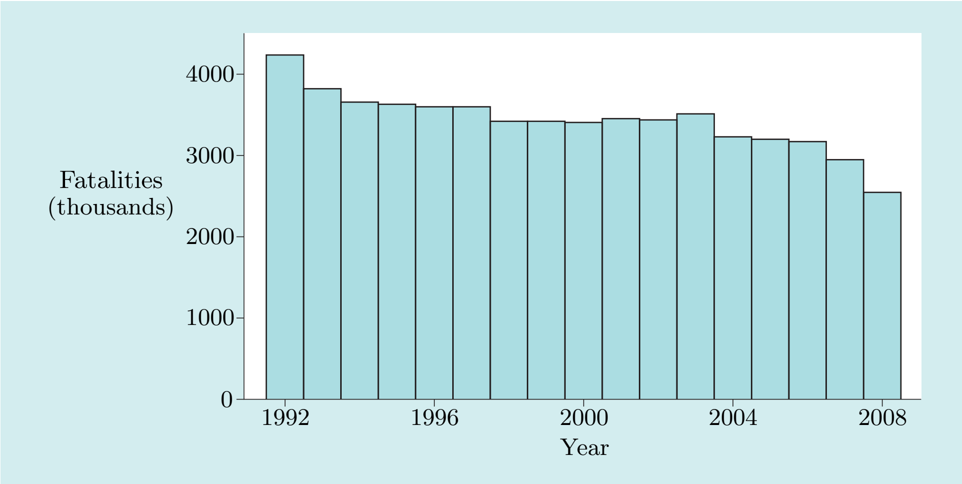

Figure 6.2 Fatalities in road accidents in Great Britain from 1992 to 2008

Figure 6.2 shows that over the period the annual number of deaths from road accidents tended to decline. As with coal-mining disasters, the rate of occurrence of events appears to alter with time. ♦

Example 6.3 Mistakes during learning

In any learning situation (such as a child learning to ride a bicycle, or an adult learning to perform some complex task, such as text-processing or bricklaying) there will be an initial period during which many accidents will happen and mistakes will be made. Then, as the learner becomes more proficient, mistakes still occur haphazardly – that is, at random – but at a rate that is lower than in the initial stages. Figure 6.3 shows a possible realisation of events, their incidence becoming sparser with passing time. (But notice that there is no regular pattern to the events: it is in the nature of accidents and mistakes that they are unexpected and unforecastable.)

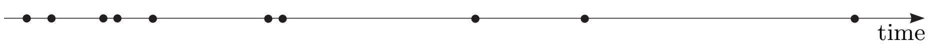

Figure 6.3 Realisation of a random process: mistakes during learning ♦

Example 6.4 Learning to ride

A young girl learning to ride a bicycle has accidents at random. However, as she improves, the rate at which she has accidents decreases, so the Poisson process does not provide an adequate model. A model is needed in which the rate decreases with time.

One possibility is to modify the Poisson process by changing the first postulate, which states that the probability that one event occurs in $[t, t + \delta t]$ is $\lambda\,\delta t + o(\delta t)$. Suppose that the probability that an accident occurs in the interval $[t, t + \delta t]$ is $[24/(2 + t)]\,\delta t + o(\delta t)$, where t is measured in days. The accident rate $24/(2 + t)$ decreases with time: when the girl starts to learn ($t = 0$) she has an accident once every two hours on average, but after ten days ($t = 10$) the rate has dropped to two a day, and after a year it is only about one a fortnight.

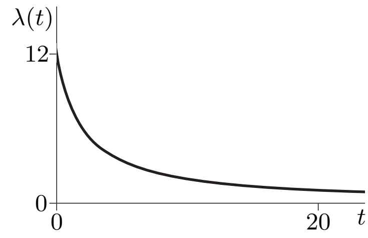

Figure 6.4 Accident rate when learning to ride a bicycle

The second and third postulates remain unchanged: accidents occur independently of each other, and the rate at time t is independent of how many accidents the girl has had before time t.

If $\lambda(t)$ denotes the accident rate at time t, then $\lambda(t) = 24/(2 + t)$, $t \geq 0$. A graph of $\lambda(t)$ against t is shown in Figure 6.4.

Note that this model makes the rather unrealistic assumption that the girl does nothing else during the learning period. ♦

Activity 6.1 Arrivals at an accident and emergency unit

Suppose that a model is required for arrivals at the accident and emergency unit in a hospital. Experience from monitoring arrivals over a long period suggests that there is a daily cycle, and that the unit is busier at some times of day than at others, so a Poisson process is not an adequate model for arrivals. In fact, arrivals may be regarded as occurring at a roughly constant rate from (say) 6 am to 6 pm, but the rate starts to rise after 6 pm until 2 am, after which arrivals are rather sparse until 6 am.

Suppose that $\lambda(t)$ denotes the arrival rate at time t. Draw a rough sketch showing what a graph of $\lambda(t)$ against t might look like over a 24-hour period.

The **non-homogeneous Poisson process**, in which the rate changes with time, is defined by three postulates. As has already been suggested, the second and third postulates are the same as for the Poisson process. The first postulate is a modification of the first postulate of the Poisson process that allows the rate of occurrence of events to vary with time.

I The probability that (exactly) one event occurs in the small time interval $[t, t + \delta t]$ is equal to $\lambda(t)\,\delta t + o(\delta t)$.

The function $\lambda(t)$ can take any form – for example, linear, quadratic or trigonometric – provided that $\lambda(t) > 0$. The behaviour of a non-homogeneous Poisson process can be illustrated by running simulations for several different functions $\lambda(t)$. This is most easily done on a computer. The rest of this subsection consists of working through a chapter of the computer book.

Refer to Chapter 1 of the computer book for the rest of the work in this subsection.

6.2 Basic results

In this subsection, the distributions of the number of events in an interval and the times at which successive events occur in a non-homogeneous Poisson process are discussed. The derivations of these results are similar to those of the corresponding results for a Poisson process that were given in Section 4.

The number of events in an interval

For a non-homogeneous Poisson process, $\lambda(t)$, the rate at which events occur, changes with time, so the distribution of $X(t_1,t_2)$, the number of events in the interval $(t_1,t_2]$, depends on t_1 as well as on the length of the interval.

First consider the interval $(0,t]$. The number of events in $(0,t]$ is denoted $X(0,t)$, or simply $X(t)$. The distribution of $X(t)$ can be derived by modifying the argument used in Section 4 for a Poisson process. The details are very similar, and you will not be expected to derive the result, so they will be omitted. The result is stated in the following box.

The distribution of $X(t)$

If $X(t)$ is the number of events that occur in a non-homogeneous Poisson process with rate $\lambda(t)$ during the interval $(0,t]$, then $X(t)$ has a Poisson distribution with parameter $\mu(t)$,

$$X(t) \sim \text{Poisson}(\mu(t)), \qquad (6.1)$$

where

$$\mu(t) = \int_0^t \lambda(u)\,du. \qquad (6.2)$$

The mean of a Poisson distribution is equal to its parameter, so the expected number of events in the interval $(0,t]$ is

$$E[X(t)] = \mu(t).$$

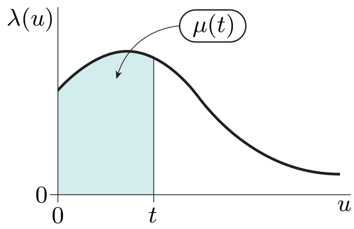

Figure 6.5 The expected number of events in the interval $(0,t]$

Since $\mu(t)$ is found by integrating the rate $\lambda(t)$, it can be represented by an area under the graph of $\lambda(t)$. This is illustrated in Figure 6.5.

Now consider the number of events that occur in the interval $(t_1,t_2]$ (where $t_2 > t_1$). Since the rate $\lambda(t)$ changes with time, the distribution of the number of events in the interval depends not only on the length of the interval but also on the value of t_1. First note that

$$X(t_1,t_2) = X(0,t_2) - X(0,t_1) = X(t_2) - X(t_1).$$

It follows that the mean of $X(t_1,t_2)$, which is denoted $\mu(t_1,t_2)$, is given by

$$\begin{aligned}\mu(t_1,t_2) = E[X(t_1,t_2)] &= E[X(t_2)] - E[X(t_1)]\\ &= \mu(t_2) - \mu(t_1)\\ &= \int_{t_1}^{t_2} \lambda(u)\,du.\end{aligned}$$

This result is illustrated in Figure 6.6.

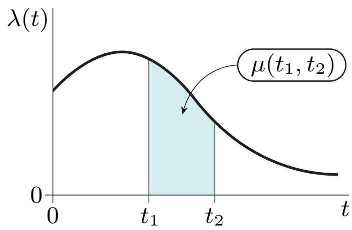

Figure 6.6 The expected number of events in $(t_1,t_2]$

In fact, the number of events occurring in $(t_1,t_2]$ has a Poisson distribution with parameter $\mu(t_1,t_2)$. This result is stated in the following box.

The distribution of $X(t_1,t_2)$

If $X(t_1,t_2)$ is the number of events that occur in a non-homogeneous Poisson process with rate $\lambda(t)$ during the interval $(t_1,t_2]$ (where $t_2 > t_1$), then $X(t_1,t_2)$ has a Poisson distribution with parameter $\mu(t_1,t_2)$:

$$X(t_1,t_2) \sim \text{Poisson}(\mu(t_1,t_2)), \qquad (6.3)$$

where

$$\mu(t_1,t_2) = \mu(t_2) - \mu(t_1). \qquad (6.4)$$

The application of these results is illustrated in Example 6.5.

Example 6.5 Errors in a maze

Suppose that the hourly error rate for a laboratory rat learning its way around a maze is given by

$$\lambda(t) = 8e^{-t}, \quad t \ge 0.$$

This rate is illustrated in Figure 6.7.

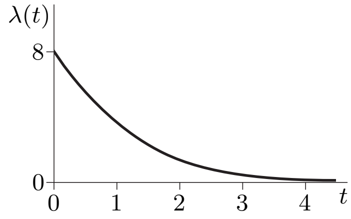

Figure 6.7 The error rate $\lambda(t) = 8e^{-t}$, $t \ge 0$

A good approach in any problem involving a non-homogeneous Poisson process is to begin by finding an expression for $\mu(t)$, whether or not this is asked for explicitly. This can then be used to calculate values of $\mu(t_1)$ and $\mu(t_1, t_2)$ for specific values of t_1 and t_2.

The expected number of errors that the rat makes in its first t hours in the maze – that is, during the interval $(0, t]$ – is given by

$$\mu(t) = \int_0^t \lambda(u)\,du = \int_0^t 8e^{-u}\,du = \left[-8e^{-u}\right]_0^t = 8\left(1 - e^{-t}\right).$$

So, for instance, the expected number of errors in the first hour is

$$\mu(1) = 8\left(1 - e^{-1}\right) \simeq 5.057.$$

This is illustrated in Figure 6.8.

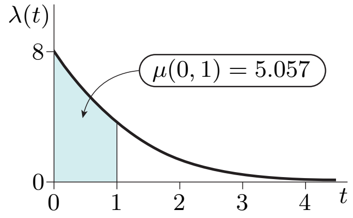

Figure 6.8 The expected number of errors in the first hour, $\mu(0, 1) = \mu(1)$

The expected number of errors that the rat makes during its second hour in the maze is given by

$$\begin{aligned}\mu(1, 2) &= \mu(2) - \mu(1)\\ &= 8\left(1 - e^{-2}\right) - 8\left(1 - e^{-1}\right)\\ &\simeq 1.860.\end{aligned}$$

This is illustrated in Figure 6.9.

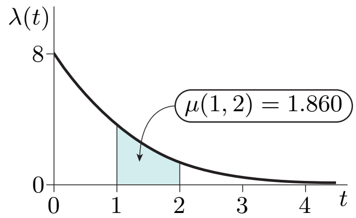

Figure 6.9 The expected number of errors in the second hour, $\mu(1, 2)$

Therefore the number of errors that the rat makes in the second hour, which is denoted $X(1, 2)$, has a Poisson distribution with mean 1.86. Hence, for example, the probability that the rat makes more than two errors in the second hour is

$$\begin{aligned}P(X(1, 2) > 2) &= 1 - P(X(1, 2) \le 2)\\ &= 1 - e^{-1.86}\left(1 + 1.86 + \frac{1.86^2}{2!}\right)\\ &\simeq 0.285. \quad \blacklozenge\end{aligned}$$

Activity 6.2 Learning to ride

In Example 6.4, the accident rate at time t of a young girl learning to ride a bicycle is given by

$$\lambda(t) = \frac{24}{2 + t}, \quad t \ge 0,$$

where t is measured in days.

(a) Find the expected number of accidents in the first t days.

(b) Calculate the expected number of accidents during the first week.

(c) Calculate the expected number of accidents during the third week. What is the probability that the girl has eight accidents in the third week?

(d) Calculate the probability that the fourth week is free of accidents.

Event times

As for the Poisson process, the distribution of $X(t)$ can be used to obtain the distribution of the waiting time T_1 until the first event. The derivation makes use of the fact that the waiting time to the first event exceeds t if and only if there is no event in $(0, t]$: $[T_1 > t]$ and $[X(t) = 0]$ are equivalent events. It follows that

$$P(T_1 > t) = P(X(t) = 0).$$

Since $X(t) \sim \text{Poisson}(\mu(t))$,

$$P(X(t) = 0) = e^{-\mu(t)},$$

and hence

$$P(T_1 > t) = e^{-\mu(t)}.$$

The c.d.f. of T_1 is

$$F_{T_1}(t) = P(T_1 \leq t) = 1 - P(T_1 > t).$$

So the c.d.f. of the waiting time until the first event is given by

$$F_{T_1}(t) = 1 - e^{-\mu(t)}, \quad t > 0.$$

Similarly, the distribution of $X(t_1, t_2)$ can be used to obtain the distribution of the time T from the start of observation until the next event after some given time v. The time T exceeds t if and only if there is no event in the interval $(v, t]$, so

$$P(T > t) = P(X(v, t) = 0).$$

Since $X(v, t) \sim \text{Poisson}(\mu(v, t))$, it follows that

$$P(T > t) = e^{-\mu(v,t)},$$

and hence the c.d.f. of the time T at which the next event after time v occurs is given by

$$F_T(t) = P(T \leq t) = 1 - e^{-\mu(v,t)}, \quad t > v.$$

These results will be used in Subsection 6.3 to simulate the times $W_1, W_2, \ldots$ at which events occur in a non-homogeneous Poisson process. The results are summarised in the following box.

Times of events

The c.d.f. of T_1, the waiting time until the first event occurs, is

$$F_{T_1}(t) = 1 - e^{-\mu(t)}, \quad t > 0. \tag{6.5}$$

The c.d.f. of T, the time from the start of observation until the first event after time v occurs, is

$$F_T(t) = 1 - e^{-\mu(v,t)}, \quad t > v. \tag{6.6}$$

Activity 6.3 *Event times*

Suppose that events in a non-homogeneous Poisson process occur at the rate

$$\lambda(t) = 2t, \quad t \geq 0.$$

(a) Find the expected number of events that occur by time t.

(b) If observation starts at time $t = 0$, find the probability $P(T_1 > t)$, where T_1 is the waiting time until the first event.

(c) Find $E(T_1)$, the expected waiting time until the first event.

(d) If T is the time from the start of observation until the occurrence of the first event after time v, find $P(T > t)$.

Use the following result from the *Handbook*:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx = \sqrt{\frac{\pi}{\alpha}}.$$

6.3 Simulation

In Subsection 3.3, the times of occurrence of events in a Poisson process were simulated by obtaining observations on the inter-event times $T_1, T_2, \ldots$. For a non-homogeneous Poisson process, the times of occurrence of events, $W_1, W_2, \ldots$, can be simulated directly using the probability-integral transformation and Results (6.5) and (6.6). The probability-integral transformation states that an observation x of a continuous random variable X can be simulated by solving the equation $F(x) = u$ for x, where u is a random observation from $U(0,1)$, and $F(x)$ is the c.d.f. of X. The c.d.f. of $W_1 = T_1$, which is given by (6.5), can be used to simulate the time of occurrence of the first event. The times of occurrence of subsequent events, $W_2, W_3, \ldots$, can be simulated using (6.6).

If $F_{W_1}(w)$ is the c.d.f. of W_1, the time at which the first event occurs, then, given a random observation u from $U(0,1)$, a random observation w_1 of W_1 can be obtained by solving the equation $F_{W_1}(w_1) = u$ for w_1. Since $W_1 = T_1$, the c.d.f. of W_1 is given by (6.5), and hence w_1 is found by solving

$$1 - e^{-\mu(w_1)} = u.$$

Rearranging this equation gives

$$e^{-\mu(w_1)} = 1 - u,$$

or equivalently

$$\mu(w_1) = -\log(1-u).$$

The solution w_1 of this equation is the simulated time of occurrence for the first event.

If the jth event occurs at time w_j, then the time at which the $(j+1)$th event occurs can be simulated as follows. Since W_{j+1} is the time at which the next event occurs after time w_j, putting $v = w_j$ in (6.6) gives the c.d.f. of W_{j+1}:

$$F_{W_{j+1}}(t) = 1 - e^{-\mu(w_j,t)} = 1 - e^{-[\mu(t)-\mu(w_j)]}.$$

A random observation w_{j+1} of W_{j+1} can be obtained by solving for w_{j+1} the equation

$$1 - e^{-[\mu(w_{j+1})-\mu(w_j)]} = u,$$

where u is a random observation from $U(0,1)$. Rearranging this equation gives

$$e^{-[\mu(w_{j+1})-\mu(w_j)]} = 1 - u,$$

or equivalently

$$\mu(w_{j+1}) - \mu(w_j) = -\log(1-u),$$

which can be written as

$$\mu(w_{j+1}) = \mu(w_j) - \log(1-u).$$

These results for simulating the times at which events occur in a non-homogeneous Poisson process are summarised in the following box.

Simulation for a non-homogeneous Poisson process

Given a random observation u from $U(0,1)$, the simulated time of occurrence of the first event in a non-homogeneous Poisson process is w_1, where w_1 is the solution of

$$\mu(w_1) = -\log(1-u). \qquad (6.7)$$

Suppose that the jth event occurs at time w_j, $j = 1, 2, \ldots$. Then, given a random observation u from $U(0,1)$, the simulated time at which the $(j+1)$th event occurs is w_{j+1}, where w_{j+1} is obtained by solving

$$\mu(w_{j+1}) = \mu(w_j) - \log(1-u). \qquad (6.8)$$

Notice that if j is set equal to 0 in (6.8) and w_0 is defined to be equal to 0, then (6.8) can be used as the formula to generate the simulated time w_1 as well as all subsequent times $w_2, w_3, \ldots$.

Example 6.6 Simulating errors made by a rat

In Example 6.5, it was shown that the expected number of errors that a laboratory rat makes during its first t hours in a maze is given by

$$\mu(t) = 8\big(1 - e^{-t}\big).$$

The random numbers $u_1 = 0.964\,17$, $u_2 = 0.633\,36$ and $u_3 = 0.884\,91$ will be used to simulate the times at which the rat makes its first three errors.

Using (6.7), the simulated time w_1 at which the first error is made is the solution of

$$\mu(w_1) = -\log(1 - u_1) = -\log(1 - 0.964\,17) \simeq 3.3290.$$

Solving

$$8\big(1 - e^{-w_1}\big) = 3.3290$$

leads to

$$e^{-w_1} = 1 - \frac{3.3290}{8} = 0.583\,875,$$

giving

$$w_1 \simeq 0.538\,07 \text{ hours} \simeq 32 \text{ minutes}.$$

The simulated time w_2 for the rat's second error is found using (6.8):

$$\mu(w_2) = \mu(w_1) - \log(1 - u_2) = 3.3290 - \log(1 - 0.633\,36) \simeq 4.3324.$$

Solving

$$8\big(1 - e^{-w_2}\big) = 4.3324$$

leads to

$$w_2 \simeq 0.779\,90 \text{ hours} \simeq 47 \text{ minutes}.$$

Using (6.8) again, the simulated time w_3 for the rat's third error can be found:

$$\mu(w_3) = \mu(w_2) - \log(1 - u_3) = 4.3324 - \log(1 - 0.884\,91) \simeq 6.4944.$$

Solving

$$8\big(1 - e^{-w_3}\big) = 6.4944$$

leads to

$$w_3 \simeq 1.670\,25 \text{ hours} \simeq 1 \text{ hour } 40 \text{ minutes}.$$

In this simulation the rat makes its first three errors 32 minutes, 47 minutes and 1 hour 40 minutes after entering the maze. ♦

Activity 6.4 Simulating times of events

The rate at which events occur in the non-homogeneous Poisson process of Activity 6.3 is

$$\lambda(t) = 2t, \quad t \geq 0.$$

Simulate the times at which the first three events occur in a realisation of the process. Use the numbers $u_1 = 0.622$, $u_2 = 0.239$, $u_3 = 0.775$, which are random observations from $U(0,1)$.

The calculations involved in simulating the times of events in a non-homogeneous Poisson process can sometimes be simplified by solving Formula (6.8) algebraically to obtain a recurrence relation for the simulated times $w_1, w_2, w_3, \ldots$. This is illustrated in Example 6.7.

Example 6.7 *Simulating times of machine malfunctions*

Occasionally a machine malfunctions, resulting in the production of defective items. The incidence of these malfunctions may be modelled as a non-homogeneous Poisson process with rate

$$\lambda(t) = \frac{2t}{1+t^2}, \qquad t \geq 0.$$

The expected number of events in the interval $(0, t]$ is given by

$$\begin{aligned}\mu(t) = \int_0^t \lambda(v)\,dv &= \int_0^t \frac{2v}{1+v^2}\,dv \\ &= \left[\log(1+v^2)\right]_0^t \\ &= \log(1+t^2) - \log(1) \\ &= \log(1+t^2).\end{aligned}$$

The times of machine malfunctions can be simulated using Formula (6.8):

$$\mu(w_{j+1}) = \mu(w_j) - \log(1-u).$$

In this case, this gives

$$\log(1+w_{j+1}^2) = \log(1+w_j^2) - \log(1-u).$$

This can be rearranged to give w_{j+1} explicitly, as follows. Taking exponentials gives

$$1 + w_{j+1}^2 = \frac{1+w_j^2}{1-u},$$

so that

$$w_{j+1}^2 = \frac{1+w_j^2}{1-u} - 1 = \frac{(1+w_j^2) - (1-u)}{1-u} = \frac{w_j^2+u}{1-u},$$

and hence

$$w_{j+1} = \sqrt{\frac{w_j^2+u}{1-u}}.$$

This recurrence relation can be used to simulate the occurrence of malfunctions during the interval $(0, 2]$. Using numbers from the fourth row of Table 5 in the *Handbook* (25727, 64334, ...) gives the following simulated times:

$$w_1 = \sqrt{\frac{0^2 + 0.257\,27}{1 - 0.257\,27}} \simeq 0.5885,$$

Recall that w_1 can be found by setting $w_0 = 0$ in (6.8).

$$w_2 = \sqrt{\frac{0.5885^2 + 0.643\,34}{1 - 0.643\,34}} \simeq 1.6658,$$

$$w_3 = \sqrt{\frac{1.6658^2 + 0.086\,91}{1 - 0.086\,91}} \simeq 1.7704,$$

$$w_4 = \sqrt{\frac{1.7704^2 + 0.189\,12}{1 - 0.189\,12}} \simeq 2.0245.$$

Retaining full calculator accuracy throughout also leads to $w_4 \simeq 2.0245$.

The fourth event occurs after time 2, so there are three malfunctions in $(0, 2]$ in this simulation, and these occur at times $w_1 \simeq 0.5885$, $w_2 \simeq 1.6658$, $w_3 \simeq 1.7704$. ♦

Activity 6.5 Another simulation

Show that for the non-homogeneous Poisson process of Activities 6.3 and 6.4, which has rate $\lambda(t) = 2t$, $t \geq 0$, the times at which events occur can be simulated using the recurrence relation

$$w_{j+1} = \sqrt{w_j^2 - \log(1-u)}.$$

Hence simulate the times of the first four events in a realisation of the process. Use the numbers $u_1 = 0.927$, $u_2 = 0.098$, $u_3 = 0.397$, $u_4 = 0.604$, which are random observations from $U(0,1)$.

Summary of Section 6

The non-homogeneous Poisson process is a model for events occurring at random in time at a rate that changes with time. The distribution of the number of events in an interval has been used to calculate probabilities, and you have learned how to simulate the times at which events occur.

Exercises on Section 6

Exercise 6.1 A non-homogeneous Poisson process

Events occur according to a non-homogeneous Poisson process with rate

$$\lambda(t) = \tfrac{3}{8}t^2(4-t), \quad 0 \leq t \leq 4.$$

(a) Sketch the function $\lambda(t)$ for t between 0 and 4, and show on your sketch the expected number of events between $t = 1$ and $t = 2$.

(b) Find $\mu(t)$, the expected number of events that occur by time t.

(c) Find the expected number of events between $t = 1$ and $t = 3$, and the expected total number of events (that is, between $t = 0$ and $t = 4$).

(d) Calculate the probability that more than two events occur between $t = 1$ and $t = 3$.

(e) Given that two events occur between $t = 0$ and $t = 1$, calculate the probability that at least four events occur in total (that is, between $t = 0$ and $t = 4$).

Exercise 6.2 Simulation

Simulate the occurrences of events in the interval $(0, 1]$ in a non-homogeneous Poisson process with rate $\lambda(t) = 3t^2$. Use random numbers from the third row of the table of random digits in the *Handbook* (beginning $u_1 = 0.240\,36$, $u_2 = 0.290\,38, \ldots$).

Solutions to Activities

Solution 1.1

The gambler's ruin The time domain of $\{X_n; n = 0, 1, \dots\}$ is $\{0, 1, 2, \dots\}$, which is discrete. The state space is $\{0, 1, \dots, a\}$, which is also discrete.

Replacing light bulbs The time domain of $\{W_n; n = 1, 2, \dots\}$ is $\{1, 2, 3, \dots\}$, which is discrete. The replacement time of a light bulb is a continuous non-negative random variable, so the state space is $\{w : w \geq 0\}$, which is continuous.

Solution 1.2

(a) The time domain is $\{1, 2, \dots\}$, which is discrete. For practical purposes, the amount of money (in £) spent by a customer can be treated as continuous, though it is actually an integral number of pence. In this case the state space is $\{a : a \geq 0\}$, and is continuous. (If A_n were modelled by a discrete random variable, then the state space would be $\{0, 0.01, 0.02, \dots\}$.)

(b) The time domain is $\{t : 0 \leq t \leq 9\}$, which is continuous. The number of items sold by time t is a non-negative integer, so the state space is $\{0, 1, 2, \dots\}$, which is discrete.

Solution 1.3

(a) The random variable Y_n represents the number of successes in n trials, so $Y_n \sim B(n, p)$.

(b) Since $Y_n = X_1 + \dots + X_{n-1} + X_n = Y_{n-1} + X_n$, and it is known that $Y_{n-1} = y$, it follows that $Y_n = y + X_n$. Therefore Y_n will take the value y if $X_n = 0$ or $y + 1$ if $X_n = 1$, and

$$P(Y_n = y \,|\, Y_{n-1} = y) = P(X_n = 0) = 1 - p,$$
$$P(Y_n = y + 1 \,|\, Y_{n-1} = y) = P(X_n = 1) = p.$$

Solution 1.4

Another sequence of random variables associated with a Bernoulli process is $\{A_n; n = 1, 2, \dots\}$, where A_n is the number of trials necessary to achieve the nth success. The random variable A_n has a negative binomial distribution with range $\{n, n + 1, \dots\}$ and parameters n and p.

The random process has a discrete time domain and a discrete state space. The state space is $\{1, 2, \dots\}$.

There are other possible sequences of random variables.

Solution 1.5

The changing weather from day to day may be well modelled by a Bernoulli process if (and only if!) it is reasonable to assume that the weather on any day is independent of the weather on preceding days, and if p, the probability of rain, does not vary from day to day.

Taken over a period of observation as long as a year, these assumptions would appear to fail: there will be 'wet spells' and 'dry spells'; and, in general, rain is more likely at some times of the year than at others. But over a shorter period, where perhaps seasonal variation may be discounted, the idea of a Bernoulli process might provide a useful model. (Actually, if a good model is required for representing the seasonal variation in weather, then it may be necessary to allow p to vary with n in some quite complicated way.)

Solution 2.1

The time domain of $\{Q(t); t \geq 0\}$ is continuous, but $\{L_n; n = 1, 2, \dots\}$ and $\{W_n; n = 1, 2, \dots\}$ have discrete time domains. The state space of $\{W_n; n = 1, 2, \dots\}$ is $\{t : t \geq 0\}$, which is continuous. Both $\{Q(t); t \geq 0\}$ and $\{L_n; n = 1, 2, \dots\}$ have state space $\{0, 1, 2, \dots\}$, which is discrete.

Solution 2.2

Two random processes associated with the model for machine breakdowns are as follows.

$\{T_n; n = 1, 2, \dots\}$, where T_n is the time at which the nth breakdown occurs. The time domain is $\{1, 2, \dots\}$, which is discrete, and the state space is $\{t : t \geq 0\}$, which is continuous.

$\{Y(t); t \geq 0\}$, where $Y(t)$ is the number of breakdowns that have occurred by time t. The time domain is $\{t : t \geq 0\}$, which is continuous, and the state space is $\{0, 1, 2, \dots\}$, which is discrete.

There are many other possibilities.

Solution 2.3

(a) If the customer has i different cards after $n - 1$ purchases, then the probability that he collects a new one at the nth purchase is $(20 - i)/20$. Hence X_n has a Bernoulli distribution with parameter $p = 1 - i/20$.

(b) This is not a Bernoulli process because the probability of 'success' (receiving a new card) changes during the process as new cards are acquired.

(c) Three possible sequences are given below; there are many others.

$\{Y_n; n = 1, 2, \dots\}$, where Y_n denotes the number of different cards collected by the nth purchase. Both the time domain and the state space are discrete. The state space is $\{1, 2, \dots, 20\}$.

$\{T_k; k = 1, 2, \dots, 20\}$, where T_k is the number of purchases after the customer has $k - 1$ different cards until and including the purchase at which he receives the kth different card. Both the time domain and the state space are discrete. The state space is $\{1, 2, \dots\}$.

$\{W_k; k = 1, 2, \dots, 20\}$, where W_k is the total number of purchases required for the customer to obtain k different cards. Both the time domain and the state space are discrete. The state space is $\{1, 2, \dots\}$.

Solution 2.4

The model is not a Bernoulli process because trials on successive days are not independent.

Solution 2.5

(a) Examples of suitable random processes are as follows.

$\{X(t); t \geq 0\}$, where $X(t)$ is the size of the colony at time t. $X(t)$ is discrete, t is continuous; so the state space is discrete and the time domain is continuous. The state space is $\{0, 1, 2, \ldots\}$.

$\{T_n; n = 1, 2, \ldots\}$, where T_n is the time between the $(n-1)$th and nth events, either divisions or deaths. The state space $\{t : t \geq 0\}$ is continuous, and the time domain is discrete.

$\{Y_n; n = 1, 2, \ldots\}$, where Y_n is the time between the $(n-1)$th and nth divisions. The state space $\{y : y \geq 0\}$ is continuous, and the time domain is discrete.

$\{Z_n; n = 1, 2, \ldots\}$, where Z_n is the time between the $(n-1)$th and nth deaths. The state space $\{z : z \geq 0\}$ is continuous, and the time domain is discrete.

Other possible processes include $\{W_n; n = 1, 2, \ldots\}$, where W_n is the time to the nth event, and $\{D_n; n = 1, 2, \ldots\}$, where D_n is the time to the nth death.

(b) Assuming that only one event (that is, one division or one death) occurs at any given time, your sketch should show steps of one unit up or down at random times. If the size $X(t)$ reaches 0, then it remains there, as there are then no bacteria to divide. A possible realisation is shown in Figure S.1.

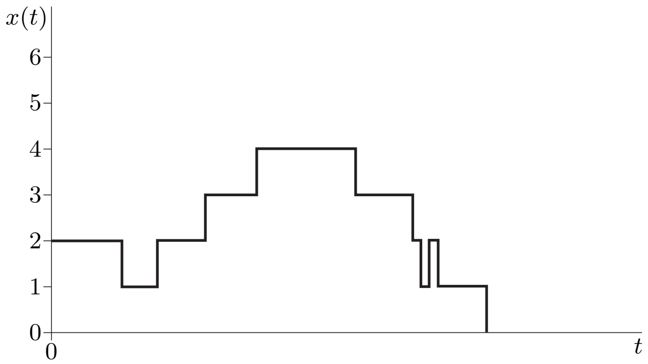

Figure S.1 A realisation of $\{X(t); t \geq 0\}$

Solution 3.1

(a) The rate of occurrence of nerve impulses is $\lambda = 458$ per second, and the length of the interval is $t = 0.01$ second. So $N(0.01)$, the number of impulses in 0.01 second, has a Poisson distribution with parameter

$$\lambda t = 458 \text{ per second} \times 0.01 \text{ second} = 4.58.$$

The probability required is

$$\begin{aligned} P(N(0.01) \leq 1) &= P(N(0.01) = 0) + P(N(0.01) = 1) \\ &= e^{-4.58} + \frac{e^{-4.58} \times 4.58}{1!} \\ &\simeq 0.0572. \end{aligned}$$

(b) The c.d.f. of T, the interval between successive impulses, is $F(t) = 1 - e^{-458t}$, so

$$P\left(T < \tfrac{1}{1000}\right) = F\left(\tfrac{1}{1000}\right) = 1 - e^{-0.458} \simeq 0.3675.$$

Solution 3.2

(a) The rate of the Poisson process is

$$\lambda = \tfrac{1}{14} \text{ per month.}$$

The number of major earthquakes that occur in a ten-year period, or 120 months, is $N(120)$, which has a Poisson distribution with parameter

$$\lambda t = \tfrac{1}{14} \text{ per month} \times 120 \text{ months} \simeq 8.571.$$

Hence the probability that there will be at least three major earthquakes in a ten-year period is

$$\begin{aligned} P(N(120) \geq 3) &= 1 - P(N(120) \leq 2) \\ &\simeq 1 - e^{-8.571}\left(1 + 8.571 + \frac{8.571^2}{2!}\right) \\ &\simeq 0.9912. \end{aligned}$$

(b) The waiting time (in months) between successive major earthquakes has an exponential distribution with parameter $\lambda = 1/14$. The probability that the waiting time exceeds two years, or 24 months, is given by

$$P(T > 24) = 1 - F(24) = e^{-24/14} \simeq 0.1801.$$

Solution 3.3

For the realisation in Figure 3.4, $X(1,3) = 3$, $X(4,5) = 0$, $X(0,2) = 3$, $X(1) = X(0,1) = 2$ and $X(3) = X(0,3) = 5$.

Solution 3.4

Since $W_n = T_1 + T_2 + \cdots + T_n$, it is the sum of n independent exponential random variables $T_1, T_2, \ldots, T_n$, each having parameter λ. Therefore W_n has a gamma distribution $\Gamma(n, \lambda)$.

Solution 3.5

Using the tenth row of the table of random numbers from $M(1)$, the simulation is as shown in Table S.1.

Table S.1 A simulation

n	e_n	$t_n = 652e_n$	$w_n = t_1 + \cdots + t_n$
1	1.5367	1001.9	1001.9
2	0.1395	91.0	1092.9
3	1.3503	880.4	1973.3
4	1.7518	1142.2	3115.5
5	1.6398	1069.1	4184.6

The first four failures occur at times $w_1 = 1001.9$, $w_2 = 1092.9$, $w_3 = 1973.3$, $w_4 = 3115.5$ (in seconds) – that is, after 1002 seconds, 1093 seconds, 1973 seconds and 3116 seconds (to the nearest second). The fifth failure occurs at time 4184.6, which is after the first hour of usage.

Solution 4.1

Setting $x = 2$ in (4.6) gives

$$\frac{dp_2(t)}{dt} = \lambda\, p_1(t) - \lambda\, p_2(t).$$

Substituting the known value of $p_1(t)$ in this differential equation gives

$$\frac{dp_2(t)}{dt} = \lambda^2 t e^{-\lambda t} - \lambda\, p_2(t),$$

or

$$\frac{dp_2(t)}{dt} + \lambda\, p_2(t) = \lambda^2 t e^{-\lambda t}.$$

Next, both sides of this equation are multiplied by the integrating factor, which is again $e^{\lambda t}$:

$$e^{\lambda t}\frac{dp_2(t)}{dt} + \lambda e^{\lambda t}\, p_2(t) = \lambda^2 t.$$

The left-hand side is the derivative of the product $e^{\lambda t} \times p_2(t)$, so the differential equation can be written as

$$\frac{d}{dt}\left(e^{\lambda t}\, p_2(t)\right) = \lambda^2 t.$$

Integrating both sides gives

$$e^{\lambda t}\, p_2(t) = \int \lambda^2 t\, dt = \tfrac{1}{2}\lambda^2 t^2 + c.$$

At time $t = 0$, no event has occurred, so $p_2(0) = P(X(0) = 2) = 0$, and hence $c = 0$. Thus

$$e^{\lambda t}\, p_2(t) = \tfrac{1}{2}\lambda^2 t^2,$$

so

$$p_2(t) = \tfrac{1}{2}\lambda^2 t^2 e^{-\lambda t},$$

which can be rewritten as

$$p_2(t) = \frac{e^{-\lambda t}(\lambda t)^2}{2}.$$

Solution 5.1

The probabilities that an arriving customer is of types A, B and C are $p_{\text{A}} = 0.6$, $p_{\text{B}} = 0.3$ and $p_{\text{C}} = 0.1$, respectively, and $\lambda = 10$ per minute, so

$$\lambda_{\text{A}} = \lambda p_{\text{A}} = 6 \text{ per minute},$$
$$\lambda_{\text{B}} = \lambda p_{\text{B}} = 3 \text{ per minute},$$
$$\lambda_{\text{C}} = \lambda p_{\text{C}} = 1 \text{ per minute}.$$

(a) If $N(t)$ denotes the number of customers who arrive in an interval of length t minutes, then the number of customers who arrive in 30 seconds, or 0.5 minute, is $N(0.5)$, which has a Poisson distribution with parameter

$$\lambda t = (10 \text{ per minute}) \times (0.5 \text{ minute}) = 5.$$

Therefore

$$\begin{aligned} P(N(0.5) > 5) &= 1 - P(N(0.5) \leq 5) \\ &= 1 - e^{-5}\left(1 + 5 + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!} + \frac{5^5}{5!}\right) \\ &\simeq 0.3840. \end{aligned}$$

(b) Let $A(t)$ represent the number of customers of type A who arrive in an interval of length t minutes. Then the number of customers of type A who arrive in one minute is $A(1)$, which has a Poisson distribution with parameter

$$\lambda_{\text{A}} t = (6 \text{ per minute}) \times (1 \text{ minute}) = 6.$$

Therefore

$$P(A(1) = 6) = \frac{e^{-6}6^6}{6!} \simeq 0.1606.$$

(c) Customers of different types arrive independently, so the probability required is given by the product

$$\begin{aligned} &\frac{e^{-6}6^6}{6!} \times \frac{e^{-3}3^3}{3!} \times \left(1 - e^{-1}\right) \\ &\simeq 0.1606 \times 0.2240 \times 0.6321 \\ &\simeq 0.0227. \end{aligned}$$

Solution 5.2

If calls from students are type 1, from family are type 2, and from friends are type 3, then

$$\lambda_1 = 1 \text{ per 90 minutes} = \tfrac{2}{3} \text{ per hour},$$
$$\lambda_2 = 1 \text{ per 3 hours} = \tfrac{1}{3} \text{ per hour},$$
$$\lambda_3 = 1 \text{ per hour}.$$

(a) The rate at which telephone calls arrive is given by

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 2 \text{ per hour}.$$

If $N(t)$ is the number of calls that arrive in t hours, then the number of calls between 7 pm and 9 pm is $N(2)$, which has a Poisson distribution with parameter $\lambda t = 4$. The probability that there will be no calls is given by

$$P(N(2) = 0) = e^{-4} \simeq 0.0183.$$

(b) The proportion of calls that are from students is

$$p_1 = \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} = \tfrac{2}{3}/2 = \tfrac{1}{3}.$$

So the probability that the first call after 9 pm is from a student is $\tfrac{1}{3}$.

(c) The probability that a call received is from a family member is

$$p_2 = \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} = \tfrac{1}{3}/2 = \tfrac{1}{6}.$$

Since calls of different types arrive independently, and the probability that each call is from a family member is $\frac{1}{6}$, N, the number of calls out of 4 that are from family members, has a binomial distribution: $N \sim B(4, \frac{1}{6})$.

The probability required is

$$P(N = 2) = \binom{4}{2} \left(\tfrac{1}{6}\right)^2 \left(\tfrac{5}{6}\right)^2 = \tfrac{25}{216} \simeq 0.1157.$$

Solution 6.1

A possible sketch of the arrival rate $\lambda(t)$ is shown in Figure S.2.

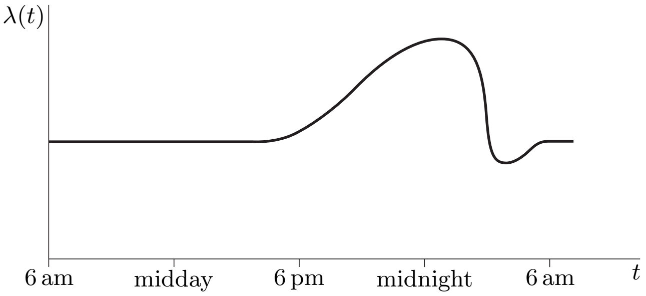

Figure S.2 Arrival rate at an accident and emergency unit over a 24-hour period

Solution 6.2

(a) The expected number of accidents in the first t days is given by

$$\begin{aligned}\mu(t) = \int_0^t \lambda(u)\,du = \int_0^t \frac{24}{2+u}\,du &= \big[24\log(2+u)\big]_0^t \\ &= 24\log(2+t) - 24\log 2 \\ &= 24\log\frac{2+t}{2} \\ &= 24\log\left(1+\frac{t}{2}\right).\end{aligned}$$

(b) The expected number of accidents during the first week ($t = 7$ days) is

$$\mu(7) = 24\log 4.5 \simeq 36.1.$$

Remember that in M343 'log' refers to logarithms to base e and that on most calculators you need to use the key labelled 'ln'.

(c) The expected number of accidents during the third week ($t_1 = 14$ days, $t_2 = 21$ days) is

$$\begin{aligned}\mu(14, 21) &= \mu(21) - \mu(14) \\ &= 24\log 11.5 - 24\log 8 \\ &\simeq 8.71.\end{aligned}$$

The number of accidents has a Poisson distribution with parameter 8.71, so the probability required is

$$P(X(14, 21) = 8) = \frac{e^{-8.71} 8.71^8}{8!} \simeq 0.135.$$

(d) The expected number of accidents during the fourth week is

$$\begin{aligned}\mu(21, 28) &= \mu(28) - \mu(21) \\ &= 24\log 15 - 24\log 11.5 \\ &\simeq 6.38.\end{aligned}$$

Therefore the number of accidents in the fourth week has a Poisson distribution with parameter 6.38.

The probability that the fourth week is free of accidents is

$$P(X(21, 28) = 0) = e^{-6.38} \simeq 0.0017.$$

Solution 6.3

(a) The expected number of events by time t is given by

$$\mu(t) = \int_0^t \lambda(u)\,du = \int_0^t 2u\,du = \big[u^2\big]_0^t = t^2.$$

(b) Using (6.5), the c.d.f. of T_1, the time until the first event, is

$$F_{T_1}(t) = 1 - e^{-\mu(t)} = 1 - e^{-t^2}.$$

Therefore

$$P(T_1 > t) = 1 - F_{T_1}(t) = e^{-t^2}, \quad t \geq 0.$$

(c) The easiest way to find the expected value of T_1 is to use the alternative formula for the mean:

$$\begin{aligned}E(T_1) &= \int_0^\infty (1 - F_{T_1}(t))\,dt \\ &= \int_0^\infty P(T_1 > t)\,dt \\ &= \int_0^\infty e^{-t^2}\,dt.\end{aligned}$$

Setting $\alpha = 1$ in the standard result

$$\int_{-\infty}^\infty e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}$$

gives

$$\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}.$$

Since e^{-x^2} is symmetric about $x = 0$,

$$\int_0^\infty e^{-x^2}\,dx = \tfrac{1}{2}\sqrt{\pi},$$

and hence

$$E(T_1) = \tfrac{1}{2}\sqrt{\pi}.$$

(d) Using (6.6), the c.d.f. of T is

$$\begin{aligned}F_T(t) &= 1 - e^{-\mu(v,t)} \\ &= 1 - e^{-(\mu(t)-\mu(v))} \\ &= 1 - e^{-(t^2-v^2)}.\end{aligned}$$

Therefore

$$P(T > t) = 1 - F_T(t) = e^{-(t^2-v^2)}.$$

Solution 6.4

From the solution to part (a) of Activity 6.3,

$$\mu(t) = t^2, \quad t > 0.$$

Using (6.7), w_1 is the solution of

$$w_1^2 = -\log(1 - u_1) = -\log(1 - 0.622) \simeq 0.972\,86,$$

so

$$w_1 \simeq 0.986.$$

Using (6.8) gives

$$w_2^2 = w_1^2 - \log(1 - u_2),$$

which leads to

$$w_2 \simeq 1.116.$$

Using (6.8) again gives

$$w_3^2 = w_2^2 - \log(1 - u_3),$$

which leads to

$$w_3 \simeq 1.655.$$

Solution 6.5

From the solution to part (a) of Activity 6.3, $\mu(t) = t^2$, so (6.8) reduces to

$$w_{j+1}^2 = w_j^2 - \log(1 - u),$$

and hence

$$w_{j+1} = \sqrt{w_j^2 - \log(1 - u)}.$$

Setting $w_0 = 0$ in this recurrence relation gives

$$w_1 = \sqrt{-\log(1 - 0.927)} \simeq \sqrt{2.6173} \simeq 1.618,$$
$$w_2 = \sqrt{2.6173 - \log(1 - 0.098)} \simeq \sqrt{2.7204} \simeq 1.649,$$
$$w_3 = \sqrt{2.7204 - \log(1 - 0.397)} \simeq \sqrt{3.2263} \simeq 1.796,$$
$$w_4 = \sqrt{3.2263 - \log(1 - 0.604)} \simeq \sqrt{4.1526} \simeq 2.038.$$

(Full calculator accuracy has been retained throughout these calculations.)

Solutions to Exercises

Solution 3.1

The rate at which events occur is

$\lambda = \frac{1}{16}$ per minute $= 3.75$ per hour.

(a) The waiting time between events has an exponential distribution with parameter $\lambda = 3.75$.

(b) The number of events in any hour-long interval, $N(1)$, has a Poisson distribution with mean 3.75.

(c) The time intervals 2 pm–3 pm and 3 pm–4 pm are disjoint, so what happened between 2 pm and 3 pm does not influence what happens in the next hour. The probability of at most one event between 3 pm and 4 pm is

$$P(N(1) \leq 1) = e^{-3.75}(1 + 3.75) \simeq 0.1117.$$

(d) The probability required is

$$\begin{aligned} P(T > 0.5) = 1 - F(0.5) &= e^{-3.75\times 0.5} \\ &= e^{-1.875} \\ &\simeq 0.1534. \end{aligned}$$

Solution 3.2

The eruption rate is

$\lambda = 1$ per 29 months $= \frac{1}{29}$ per month.

(a) The number of eruptions during a five-year period, $N(60)$, has a Poisson distribution with parameter

$$\lambda t = \left(\tfrac{1}{29} \text{ per month}\right) \times (60 \text{ months}) \simeq 2.069,$$

so the expected number of eruptions is 2.069.

(b) The probability of exactly two eruptions is

$$P(N(60) = 2) = \frac{e^{-2.069}(2.069)^2}{2!} \simeq 0.2704.$$

(c) By the memoryless property of the exponential distribution, the time between the start of observation and any previous event can be ignored. If T is the time from the start of observation until the next eruption then, since $P(T \geq t) = e^{-\lambda t}$, where t is in months,

$$P(T \geq 36) = e^{-36/29} \simeq 0.2890.$$

Solution 3.3

(a) The mean time between events is 3 months, so each random number e_n from Table 6 in the *Handbook* must be multiplied by 3 to give a simulated time between events in months. The realisation is shown in Table S.2.

Table S.2

n	e_n	$3e_n$	$w_n = t_1 + \cdots + t_n$
1	3.5677	10.7031	$10.7031 \simeq 10.70$
2	4.1622	12.4866	$23.1897 \simeq 23.19$
3	0.6071	1.8213	$25.0110 \simeq 25.01$

(b) From Table S.2, it is evident that the third event occurred outside the 24-month period being modelled. So two failures were observed: the first occurring after 10.70 months, the second after 23.19 months. The number 2 is a single observation from the Poisson distribution with parameter $24/3 = 8$.

Solution 5.1

The rate at which customers arrive is $\lambda = 8$ per hour. The proportions of customers in the three categories are $p_1 = 0.70$, $p_2 = 0.05$, $p_3 = 0.25$, where the categories are letters, parcels and non-postal, respectively.

(a) The parcel rate is

$$\lambda_2 = \lambda p_2 = 8 \times 0.05 = 0.4 \text{ per hour}.$$

(b) The time T between arrivals of customers posting parcels has an exponential distribution with parameter λ_2, so the probability that the interval between the arrivals of such customers is greater than an hour is

$$P(T > 1) = e^{-\lambda_2 \times 1} = e^{-0.4} \simeq 0.6703.$$

(c) The letter rate is

$$\lambda_1 = \lambda p_1 = 8 \times 0.70 = 5.6 \text{ per hour}.$$

Let $L(t)$ be the number of customers arriving to post letters in t hours. Then $L(3)$, the number of customers posting letters in a three-hour period, has a Poisson distribution with parameter

$$\lambda_1 t = 5.6 \text{ per hour} \times 3 \text{ hours} = 16.8.$$

The probability that there are fewer than five such customers is

$$\begin{aligned} &P(L(3) \leq 4) \\ &= e^{-16.8}\left(1 + 16.8 + \frac{16.8^2}{2!} + \frac{16.8^3}{3!} + \frac{16.8^4}{4!}\right) \\ &\simeq e^{-16.8} \times 4268.33 \\ &\simeq 0.000\,22. \end{aligned}$$

(d) The arrival rate of customers posting letters or parcels is $\lambda(p_1 + p_2) = 8 \times 0.75 = 6$ per hour.

The waiting time between arrivals of such customers is $M(6)$, so the median waiting time (in hours) is the solution of the equation

$$F(t) = 1 - e^{-6t} = \tfrac{1}{2},$$

that is,

$$e^{-6t} = \tfrac{1}{2}.$$

Therefore the median waiting time is

$$\begin{aligned} t &= \tfrac{1}{6}\log 2 \text{ hours} \\ &= 10 \times \log 2 \text{ minutes} \\ &\simeq 6.93 \text{ minutes}. \end{aligned}$$

Solution 5.2

The arrival rates of works of fiction, biographies, works of reference and non-text items are $\lambda_1 = 8$ per week, $\lambda_2 = 1$ per week, $\lambda_3 = 0.25$ per week, $\lambda_4 = 5$ per week, respectively.

(a) Let $N(1)$ be the number of non-text items that arrive in one week; then $N(1) \sim \text{Poisson}(5)$. So the probability that at least two non-text acquisitions will arrive next week is given by

$$\begin{aligned} P(N(1) \geq 2) &= 1 - P(N(1) \leq 1) \\ &= 1 - e^{-5}(1+5) \\ &\simeq 0.9596. \end{aligned}$$

(b) Let $W(t)$ be the number of works of fiction that arrive in t weeks, so the number that arrive in one day is $W\left(\frac{1}{7}\right)$. Since $\lambda_1 = 8$ per week, $W\left(\frac{1}{7}\right) \sim \text{Poisson}\left(\frac{8}{7}\right)$. The probability that no new work of fiction will arrive tomorrow is

$$P\left(W\left(\tfrac{1}{7}\right) = 0\right) = e^{-8/7} \simeq 0.3189.$$

(c) The proportions of new acquisitions in each of the four categories are

$$\begin{aligned} p_{\text{fiction}} &= \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3+\lambda_4} \\ &= \frac{8}{8+1+0.25+5} \\ &= \frac{8}{14.25} \simeq 0.561, \end{aligned}$$

$$p_{\text{biography}} = \frac{1}{14.25} \simeq 0.070,$$

$$p_{\text{reference}} = \frac{0.25}{14.25} \simeq 0.018,$$

$$p_{\text{non-text}} = \frac{5}{14.25} \simeq 0.351.$$

Solution 6.1

(a) A sketch of the function $\lambda(t)$ for $0 \leq t \leq 4$ is shown in Figure S.3.

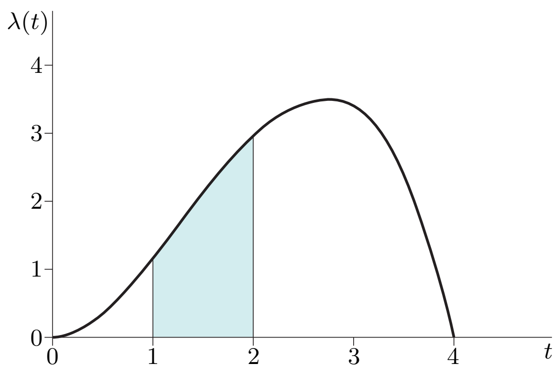

Figure S.3 The event rate $\lambda(t) = \frac{3}{8}t^2(4-t)$, $0 \leq t \leq 4$

The shaded area represents the expected number of events between $t = 1$ and $t = 2$.

(b) The expected number of events that occur by time t is

$$\begin{aligned} \mu(t) &= \int_0^t \lambda(u)\,du = \int_0^t \tfrac{3}{8}u^2(4-u)\,du \\ &= \tfrac{3}{8}\int_0^t \left(4u^2 - u^3\right)du \\ &= \tfrac{3}{8}\left[\tfrac{4}{3}u^3 - \tfrac{1}{4}u^4\right]_0^t \\ &= \tfrac{1}{2}t^3 - \tfrac{3}{32}t^4. \end{aligned}$$

(c) The expected number of events between $t = 1$ and $t = 3$ is

$$\begin{aligned} \mu(1,3) &= \mu(3) - \mu(1) \\ &= \left(\tfrac{1}{2}\times 3^3 - \tfrac{3}{32}\times 3^4\right) - \left(\tfrac{1}{2}\times 1^3 - \tfrac{3}{32}\times 1^4\right) \\ &= 5.906\,25 - 0.406\,25 \\ &= 5.5. \end{aligned}$$

The expected total number of events is

$$\mu(4) = \tfrac{1}{2}\times 4^3 - \tfrac{3}{32}\times 4^4 = 8.$$

(d) The number of events between $t = 1$ and $t = 3$, $X(1,3)$, has a Poisson distribution with parameter 5.5. The probability of more than two events between $t = 1$ and $t = 3$ is given by

$$\begin{aligned} P(X(1,3) > 2) &= 1 - P(X(1,3) \leq 2) \\ &= 1 - e^{-5.5}\left(1 + 5.5 + \frac{5.5^2}{2!}\right) \\ &= 1 - e^{-5.5}(21.625) \\ &\simeq 1 - 0.0884 \\ &= 0.9116. \end{aligned}$$

(e) Given that (exactly) two events occur between $t = 0$ and $t = 1$, the probability that at least four events occur in total is just the probability that at least two events occur between $t = 1$ and $t = 4$, that is,

$$P(X(1,4) \geq 2).$$

The number of events, $X(1,4)$, has a Poisson distribution with parameter

$$\mu(1,4) = \mu(4) - \mu(1) = 8 - 0.406\,25 = 7.593\,75.$$

Therefore

$$\begin{aligned} P(X(1,4) \geq 2) &= 1 - P(X(1,4) \leq 1) \\ &= 1 - e^{-7.593\,75}(1 + 7.593\,75) \\ &\simeq 0.9957. \end{aligned}$$

Solution 6.2

The event rate is $\lambda(t) = 3t^2$, so

$$\mu(t) = \int_0^t \lambda(u)\,du = \int_0^t 3u^2\,du = t^3.$$

The simulated times may be found using (6.8):

$$\mu(w_{j+1}) = \mu(w_j) - \log(1-u).$$

In this case,

$$w_{j+1}^3 = w_j^3 - \log(1-u),$$

so

$$w_{j+1} = \sqrt[3]{w_j^3 - \log(1-u)}.$$

The simulated times are

$$\begin{aligned} w_1 &= \sqrt[3]{w_0^3 - \log(1-u_1)} \\ &= \sqrt[3]{-\log(1-0.240\,36)} \simeq 0.650\,23 \simeq 0.650, \\ w_2 &= \sqrt[3]{0.650\,23^3 - \log(1-0.290\,38)} \\ &\simeq 0.851\,76 \simeq 0.852, \\ w_3 &= \sqrt[3]{0.851\,76^3 - \log(1-0.432\,11)} \simeq 1.057\,85 > 1. \end{aligned}$$

The simulation is required only for $0 \leq t \leq 1$, so two events occur, at $t_1 = 0.650$ and $t_2 = 0.852$.

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