Solutions to the exercises

Section 1

1.1 In describing the formulation of the model, it may be a help to write down some words or phrases first, and then to combine them into a paragraph.

Possible key words or phrases are: pollution has ceased; the water flowing from the lake is polluted; concentration of the pollutant is uniform; small time interval; mass of pollutant; one lake; no change in volume of the lake; input–output principle.

A possible description might read:

> This model considers a polluted lake to which no further pollutant is added. The lake is of constant volume. The water flowing from the lake is polluted, and this is the only way that the pollutant leaves the lake. The input–output principle is applied to the mass of pollutant within the lake, based on the change over a small time interval.

In a modelling report, the description of the formulation should be placed before the actual formulation, in order to guide the reader. However, it should actually be written last, after the formulation has been completed.

1.2 **(a)** The input–output principle is applied to the mass of pollutant within the lake. In the time interval $[t, t + \delta t]$, the input (mass entering the lake) is $q\,\delta t$, while the output is $(r/V)m(t)\delta t$, as shown in the previous text. The accumulation over this time interval is $m(t + \delta t) - m(t)$. Hence the change in the mass of pollutant in this interval is given by

$$m(t + \delta t) - m(t) \simeq q\,\delta t - \frac{r}{V}m(t)\delta t.$$

Putting $k = r/V$, this leads to the differential equation

$$\frac{dm}{dt} = q - km(t). \qquad \text{(S.1)}$$

(b) The integrating factor method gives the general solution of Equation (S.1) as

$$m(t) = \frac{q}{k} + Ce^{-kt},$$

where C is an arbitrary constant. Rearranging the solution, and using the initial condition at $t = 0$ to evaluate C, gives the required result,

$$m(t) = \left(m(0) - \frac{q}{k}\right)e^{-kt} + \frac{q}{k}. \qquad \text{(S.2)}$$

In the long term, the mass of pollutant tends to q/k kg.

(c) If there is, initially, no pollution in the lake, then $m(0) = 0$ and

$$m(t) = \frac{q}{k}\left(1 - e^{-kt}\right).$$

The mass of pollution increases from zero up to the steady-state level q/k kg.

(d) From Equation (S.2), with $m(t) = Vc(t)$, and putting $kV = r$ gives

$$c(t) = \left(c(0) - \frac{q}{r}\right)e^{-kt} + \frac{q}{r}.$$

This is the corresponding mathematical model for pollutant concentration.