Introducing vectors for engineering applications

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Contents

Introduction

Applied mathematics is a key skill for practicing engineers and mathematical modelling is an ever-increasing field within engineering. .

In Section 1 you will explore how vectors are used to model force and motion, and consider how problems involving vectors can be solved using geometry and trigonometry. In Section 2 you explore how to work with vectors represented in component form. Section 3 is concerned with vector algebra, and considers how equations involving vectors can be solved. Finally, Section 4 introduces the scalar product of vectors, a multiplication operation that takes into account direction as well as magnitude.

Solutions to the activities which appear in this course can be found on this page.

This OpenLearn course is an adapted extract from the Open University course T194 *Engineering: mathematics, modelling, applications*

Learning Outcomes

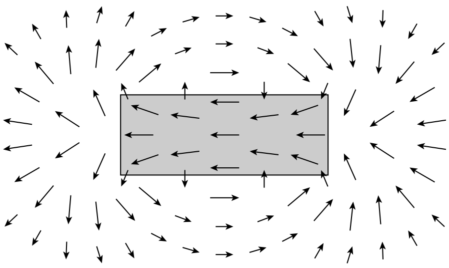
After studying this course, you should be able to:

- identify if a quantity is a vector
- represent vectors from engineering problems in an appropriate form
- model simple engineering systems (such as combining forces) using vectors
- perform simple algebraic procedures using vectors.

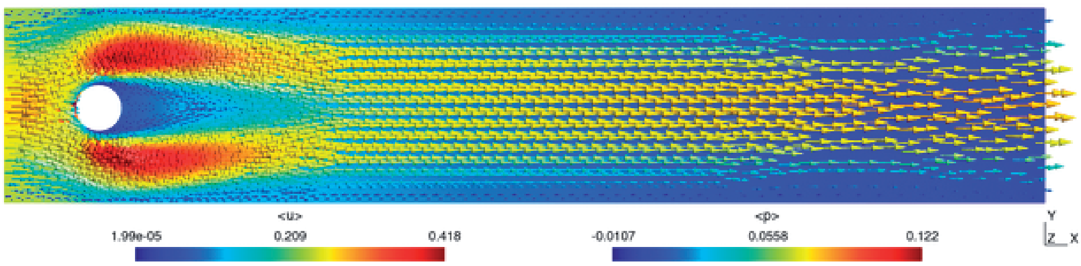
Applications of vectors

Background

It is common in engineering for physical phenomena to be represented as vector fields. A vector field is a mathematical representation of a system that describes how a quantity, such as a force, changes over an interval of time, or an area or volume of space. Figure 1, for example, illustrates vector fields created by magnets (in part (a)) and fluid flow (in part (b)). A vector field can be thought of as a graph where every coordinate has not only a position, but also a magnitude and direction. In the images in Figure 1, these are represented by small arrows, with each individual arrow indicating the direction and magnitude of a force at a specific position.



(a)

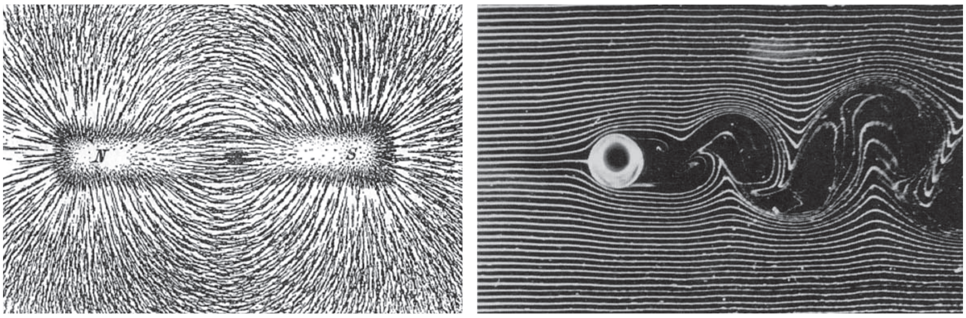


(b)

Figure 1 Examples of models of vector fields: (a) model of a vector field created by a bar magnet; (b) model of a vector field created by flow around a cylinder

When viewed as a collection of individual vectors, vector fields can be very complex and difficult to understand, but when viewed holistically, patterns emerge that give insight into physical phenomena.

The examples in Figure 1 result from mathematical models, but accurately reflect the real-world situations they model. The first image is a diagram showing the magnetic field generated by a bar magnet, and accurately reflects the patterns created physically by iron filings when acted on by such a magnet, as illustrated in Figure 2(a). The image in Figure 1(b) is the result of a computational simulation of the fluid flow around a submerged cylinder, and is a representation of the real-world situation illustrated in Figure 2(b), which was created by smoke filaments in a wind tunnel. This representation is less accurate, with a difference in the flow to the right of the cylinder: in the physical example the smoke filaments become turbulent and messy as a result of the vortices formed in the wake of the cylinder, but in the simulation the flow remains smooth. This difference is a consequence of assumptions made in the mathematical model describing the flow. Assumptions have been made to make the mathematics more manageable by neglecting the complexity that gives rise to the vortices.



(a) (b)

Figure 2 Physical examples of vector fields: (a) magnetic field created by a bar magnet acting on iron filings; (b) turbulence created by flow around a cylinder

Vector fields are beyond the scope of this course, but you are likely to encounter them if you decide to take this area of study further. In preparation for this we will explore the mathematics of vectors.

1 Modelling with vectors

In Background to this course, arrows have been used as representations to introduce the concept of vector quantities. Vector quantities are different from scalar quantities because they describe direction as well as magnitude. For this reason the arrow representation is useful to visually represent vector quantities.

1.1 Modelling motion with perpendicular vectors

Let's consider, where two people, Alice and Bob, are pushing a block of ice, which has a mass of a metric tonne (). Alice and Bob each push on a different face of the block, as illustrated in Figure 3, and the direction in which the block moves is a consequence of the combination of the forces they apply. If Bob applies a force of 130 N to the left face of the block, and Alice applies a force of 110 N to the bottom face, what is the combined force applied to the block, and what is the acceleration of the block?

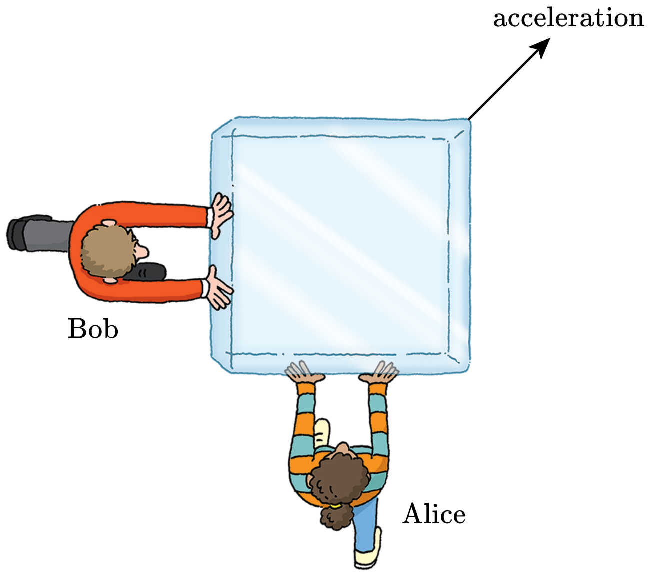


Figure 3 Alice and Bob pushing different sides of a block of ice

1.1.2 Calculating the magnitude of a combined force

The net effect of these two forces by combining them visually using arrows, as illustrated in Figure 4. Figure 4(a) shows an abstraction of the drawing in Figure 3, with Alice and Bob replaced by arrows representing the forces applied by Alice and Bob to the block of ice. Here, represents the force applied by Alice, who is below the block, and represents the force applied by Bob, who is to the left of the block. Notice that the vectors are shown to be acting on the centres of the faces of the block. This is because if forces are applied away from the centres, this can create a rotation, and that is a more complicated situation to model. Such rotational effects are outside the scope of this module.

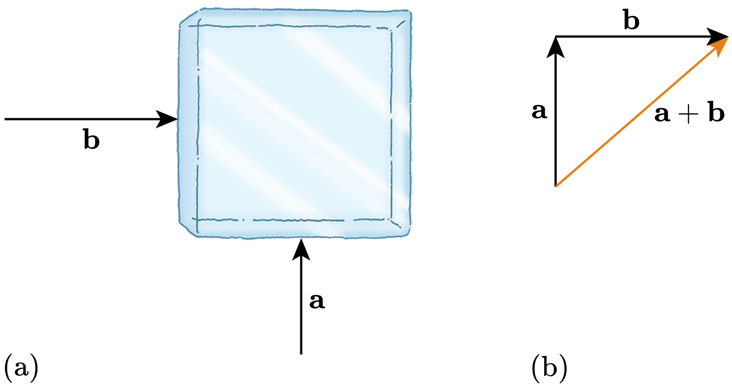
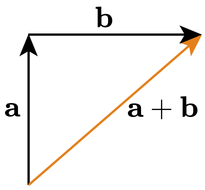


Figure 4 Combining the vectors and

Figure 4(b) shows the result of visually adding the vectors and . . In this example, and are perpendicular, so the triangle formed by , and the resultant is a right-angled triangle, with as the hypotenuse. So we can use Pythagoras' theorem to find the magnitude of , written , from the magnitudes of and , written and .

Activity 1

If is a positive vertical vector that has magnitude 110 N and is a positive horizontal vector that has magnitude 130 N, what is the magnitude of the resultant to two decimal places?



1.1.3 Calculating direction of motion of a combined force

Using the fact that , and the resultant form a right-angled triangle, we can also use trigonometric functions to find the direction of . This is illustrated in Figure 4.5, where the direction of is represented by the angle . But is outside the triangle formed by , and , so we can't directly calculate its size using trigonometry; we also need to use our knowledge of angles and triangles.

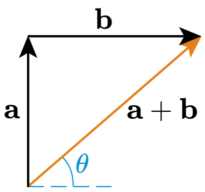
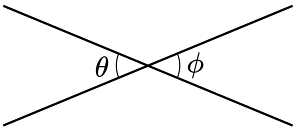


Figure 5 Finding the direction of the resultant

In particular, we can use the properties of angles on lines, as summarised here.

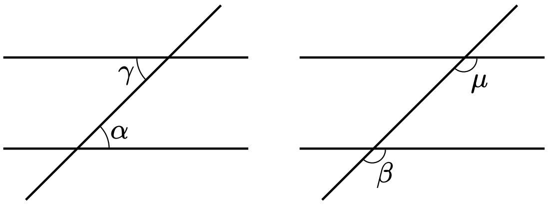
Opposite, corresponding and alternate angles

Where two lines intersect, **opposite angles** are equal. This is commonly referred to as the X-angles rule. For example, in the following diagram, .



Where a line intersects parallel lines:

- **Alternate angles** are equal. This is commonly referred to as the Z-angles rule. For example, in the diagram below, .
- **Corresponding angles** are equal. This is commonly referred to as the F-angles rule. For example, in the diagram below, .



For example, to find the angle in Figure 5, we can use alternate angles (Z-angles). So in Figure 6, angles and are equal, and to find we use trigonometry, e.g. the tangent function.

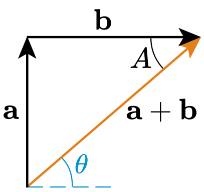


Figure 6 Identifying corresponding angles

Activity 2

In Figure 6, if is a positive vertical vector that has magnitude 110 N and is a positive horizontal vector that has magnitude 130 N, what is the direction of the resultant to one decimal place? Use the fact that and are alternate angles to help you.

1.1.4 Calculating acceleration from a combined force

With the magnitude and direction of the resultant vector now calculated, we can use Newton's second law to determine the acceleration of the block. Recall that Newton's second law states that

The block has a mass of a metric tonne () and because it is made of ice we will ignore any forces due to friction. The force applied by Alice and Bob is , which has the magnitude calculated in Activity 1 and the direction calculated in Activity 2.

Example 1 Finding acceleration from a combined force

A block of ice has a mass of a metric tonne (). If Bob applies a force of 130 N to the left face of the block while Alice applies a force of 110 N to the bottom face, what is the acceleration of the block? Give the magnitude of the acceleration to two decimal places and the angle to one decimal place.

Solution

Newton's second law gives us

so acceleration is given by

In the following calculations we will use the vector to represent acceleration, to avoid confusion with the vector which is the force applied by Alice. The mass of the block of ice is and, when we ignore friction, the only force acting on the block is the resultant force due to Alice and Bob pushing the block, so

and the acceleration of the block is given by

Multiplying a vector by a positive scalar does not change the direction of the vector, so the direction of the acceleration is the same as the direction of , and is given by (to 1 d.p.).

We also know that when multiplying a vector by a positive scalar, its magnitude is changed through multiplication. So, the magnitude of the acceleration is

So, the block accelerates at (to 2 d.p.) in a direction that is (to 1 d.p.) from the positive horizontal direction.

1.2 Models of motion

We have calculated that Alice and Bob's combined force causes the block of ice to accelerate at , but what does this mean? We can put this in context by comparing it to other common magnitudes of acceleration, as shown in Table 1. Our calculated value is less than the magnitude of acceleration of a high-speed train, but within the same order of magnitude, and if we think about how slowly a train accelerates as it initially begins to move, then this comparison sounds about right.

Table 1 Approximate magnitudes of acceleration

Object	Approximate magnitude of acceleration ($\mathrm{m\,s^{-2}}$)
High-speed train	0.25
Executive car	4.3
Sprinter (pulling away from start line)	9.2
Gravity on Earth at sea-level standard	9.8
Saturn V moon rocket (just after launch)	11.2
Mid-engined sports car	15.2
Space shuttle (maximum during launch)	29
Formula One car (maximum under heavy braking)	49
F-16 aircraft (pulling out of dive)	79
Explosive seat ejection from aircraft	147
Automobile crash (100 $\mathrm{km\,h^{-1}}$ into wall)	982
Football struck by foot	2946
Baseball struck by bat	29 460
Closing jaws of a trap-jaw ant	1 000 000
Jellyfish stinger	53 000 000

We can also consider what the calculated acceleration means, by considering how it converts to motion. If we assume that acceleration is constant and in a straight line, then we can calculate speed and distance travelled using the following equations of motion.

Equations of linear motion

For linear motion under constant acceleration, the following equations relate distance (), time (), initial speed (), final speed () and magnitude of acceleration ().

Initial speed:

Final speed:

Finding displacement using initial and final speed:

Finding displacement using initial speed and acceleration:

Example 2 Calculating speed and distance

A stationary block of ice has a mass of a metric ton (). If Bob applies a force of 130 N to the left face of the block, while Alice applies a force of 110 N to the bottom face, use the equations of motion to find how far the block will move and how fast it will be moving after 10 seconds.

Solution

The block of ice is initially stationary so we have an initial speed of

Also, in Example 1 we calculated that the magnitude of acceleration of the block of ice is (to 2 d.p.), so we have

We can calculate the final speed using

and distance travelled using

So after 10 seconds we have

and

The block is travelling at a speed of approximately and has travelled a distance of approximately 8.5 m.

Activity 3

Now use the equations of motion to find how far the block will move and how fast it will be moving after the following times.

a. 30 seconds

b. 1 minute

How good is the model?

Do the answers calculated in Example 2 and Activity 3 seem reasonable to you?

- After 10 seconds we calculated that the block is travelling with a speed of approximately . This is an average walking speed, and sounds pretty reasonable.
- After 30 seconds the block is travelling at a speed approximately three times as fast. This is starting to get quite speedy, and it would be a challenge for Alice and Bob to maintain this pace while pushing a block of ice.
- After one minute the block is travelling at a very fast speed, comparable to the world record speed of reached by Usain Bolt during the 100 m sprint final at the 2009 World Championships in Berlin, and it is clear that Alice and Bob are very unlikely to reach such a speed while pushing a block of ice.

So what has gone wrong? The problem lies in the underlying assumption in the equations of motion that we used. We assumed that acceleration is constant, but this is not realistic. Bob and Alice are unlikely to maintain the same force on the block once it has started moving, and it is much more realistic to assume that they will reduce the force that they apply once the block has reached a comfortable speed. This would result in a reduction in acceleration, and to maintain a constant speed, an acceleration of zero would be required.

For example, look at the velocity and acceleration profiles of a 100 m sprinter in the graphs in Figure 7. Certain assumptions are made that simplify the graphs; in particular, it is assumed that the acceleration at the start of the race is immediately at the maximum value, and it is also assumed that there is no deceleration towards the end of the race. These assumptions are not realistic, but since these graphs are for the purposes of illustration, they are perfectly acceptable. The graphs are similar in shape to what we would expect to see if we plotted the profiles for Alice and Bob pushing the block of ice.

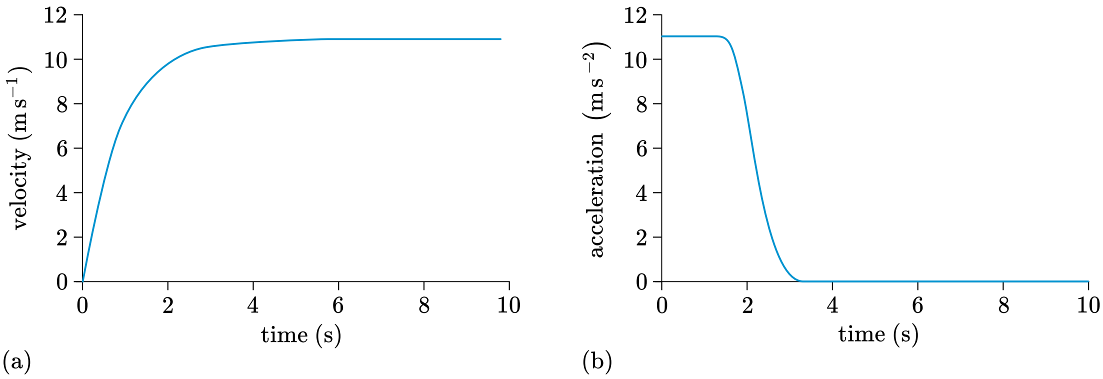


Figure 7 (a) Velocity and (b) acceleration profiles for a 100 m sprint

Consider the velocity profile first. In Figure 7(a), we see that the sprinter quickly increases velocity from 0 to above in the first two seconds of the race. Once maximum velocity is reached, after about 4 seconds, the sprinter maintains this for the rest of the race. The velocity profile is also reflected in the acceleration profile in Figure 7(b). We see a high initial acceleration, above , which after around 2 seconds quickly reduces to zero. There is a strong relationship between these two graphs: where the velocity looks like a straight line, the acceleration is flat, i.e. constant, and where the velocity is flat, acceleration is zero. This is because acceleration is the rate of change of velocity, and the slope of the velocity profile at a specific value of is equal to the acceleration for that value of . This relationship is at the heart of calculus..

Finally, note that for small values of , the graph of the velocity profile is approximately linear, and this gives a flat segment in the acceleration profile, where the acceleration is constant. This means that for small values of an assumption of constant acceleration is not far from wrong, and this is why in Example 2 our calculation for 10 seconds seems reasonable, but for 30 and 60 seconds our calculations seem to be far away from reality. The statistician George Box is quoted as saying ‘all models are wrong, but some are useful’ (Box and Draper, 1987) and this perfectly sums up what we have found here.

1.3 Modelling motion with non-perpendicular vectors

The example of Alice and Bob pushing a block of ice was made simpler by the fact that they were pushing along horizontal and vertical vectors. Let’s look at another example, where the vectors are in arbitrary directions.

Consider the situation in Figure 8, where Alice and Bob have attached ropes to a face of the block of ice and are now pulling it in different directions. If Bob pulls with a force of 130 N at an angle of 47° clockwise from the horizontal, and Alice pulls with a force of 110 N at an angle of 24° anticlockwise from the horizontal, what is the combined force applied to the block, and what is the acceleration of the block?

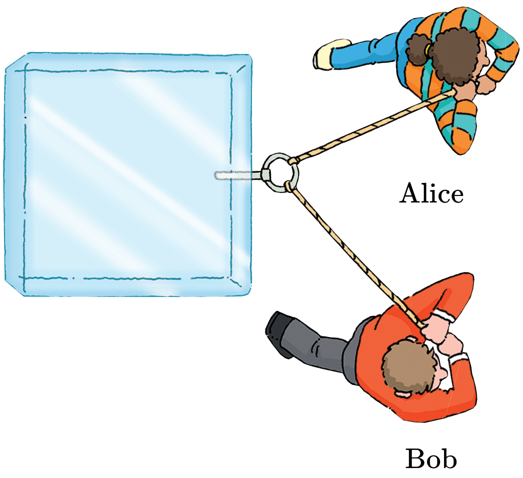


Figure 8 Alice and Bob pulling a block of ice

As before, let's start by creating an abstract drawing, with Alice and Bob replaced by arrows, as illustrated in Figure 9(a). Here, represents the force applied by Alice and represents the force applied by Bob. The vector has a magnitude of 110 N and a direction with an angle of 24° measured anticlockwise from the positive -axis, while has a magnitude of 130 N and a direction with an angle of , measured clockwise from the positive -axis.

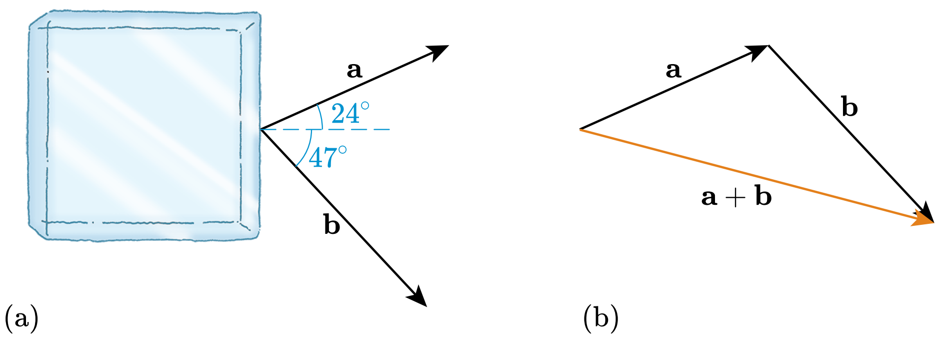


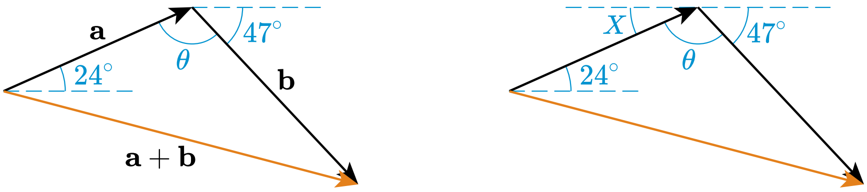
Figure 9 Representing the vectors and

Now, let's calculate the magnitude and direction of the resultant vector . Figure 9(b) shows the result of visually adding the vectors and . Unlike the previous example, and are not perpendicular, so the triangle formed by , and the resultant is not a right-angled triangle. In this case, we cannot use Pythagoras' theorem or the trigonometric functions to calculate the magnitude and direction of , and instead we need to use other properties of triangles, such as the sine or cosine rules.

If we knew one of the interior angles, then we could use the cosine rule,

to calculate the magnitude of from the magnitudes of and , as follows:

where is the interior angle opposite , as illustrated in Figure 10(a).



(a) (b)

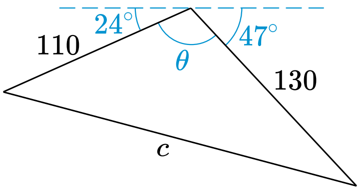
Figure 10 Finding the interior angle θ

To find we can make use of alternate angles (Z-angles). The angle in Figure 10(b) is an alternate angle with the angle indicating the direction of vector , so . Also, , and the angle indicating the direction of vector all lie on a straight line, so

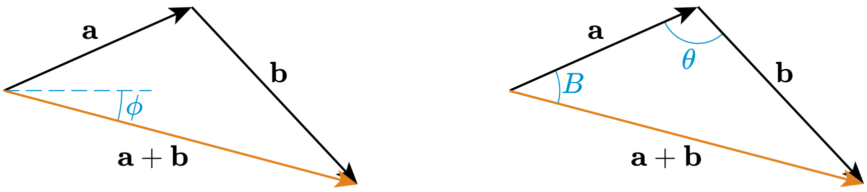
Using this relation, we can now find the size of angle , and use this in the cosine rule to determine the length of .

Activity 4

Use the cosine rule to determine the length of edge in the diagram below, to two decimal places.



If we apply our answer to Activity 4 to the vector diagram in Figure 10, then we have calculated that the magnitude of is approximately 195.73 N. The direction of is given by the angle in Figure 11(a).



(a) (b)

Figure 11 Finding the direction of

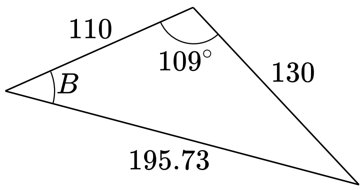
We already know that the direction of , so if we can find the angle in Figure 11(b), then we can calculate from

We can calculate using the sine rule,

as follows:

Activity 5

Use the sine rule to calculate angle in the diagram below, to one decimal place.



From Activities 4 and 5, we can therefore say that, approximately, the magnitude of is 195.73 N and its direction is given by the angle

which is measured clockwise from the horizontal, as illustrated in Figure 12. This is the combined force applied by Alice and Bob to the block of ice.

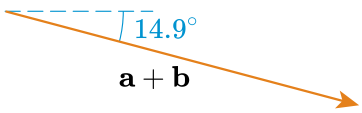


Figure 12

Activity 6

A block of ice has a mass of a metric tonne (), and the resultant force on the block is described by a vector with a magnitude of 195.73 N and a direction of 14.9°, measured clockwise from the positive -axis. Give the magnitude of the acceleration to two decimal places and the direction to one decimal place.

2 Vectors in component form

Carrying out vector calculations using geometric methods is not very efficient, and in complicated engineering systems where there are many vectors acting, calculations can get unwieldy. Alternatively, we can work with vectors algebraically, using their component forms.

2.1 Horizontal and vertical components

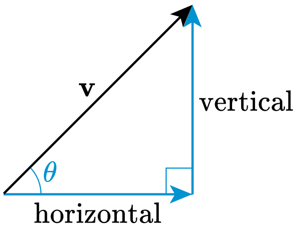


Figure 13 A vector and its components

Vectors are intuitively described using magnitude and direction, but they are more usefully described according to horizontal and vertical components, as illustrated in Figure 13.

A vector that can be described by a magnitude and a direction can also be described by component vectors with:

- the horizontal component pointing in the direction of the positive -axis
- the vertical component pointing in the direction of the positive -axis.

Together, a vector and its component vectors form a right-angled triangle, and the magnitudes of the three vectors define the lengths of the edges of the triangle. So, referring to Figure 13, we can use the sine and cosine functions to determine the magnitudes of the component vectors. The vector has magnitude and direction , and from these its components are calculated as follows.

The sine function is defined as

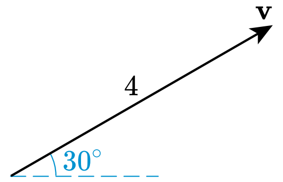
and by recognising that the vertical component is opposite the angle , we get

Similarly, the cosine function is defined as

and by recognising that the horizontal component is adjacent to the angle , we get

Activity 7

The vector has a magnitude of 4 makes an angle of 30° with the positive -axis, as shown in the following diagram. Identify the magnitudes of the horizontal and vertical components of .



2.2 Cartesian unit vectors

In Activity 7 we found that the vector has a vertical component with a magnitude of 2 and a horizontal component with a magnitude of . So we can write

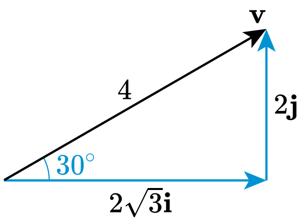


Figure 14 A vector and its components

The vector and its horizontal and vertical components form a triangle, as illustrated in Figure 14, and we discovered in Chapter 1 that vector sums form triangles. So this equation makes sense mathematically, and it is correct to say that a vector is the sum of its horizontal and vertical components.

A shorthand way to write this is

The vectors and are called the **Cartesian unit vectors**. Here, unit means one, so and are vectors with magnitude 1 that point in the directions of the coordinate axes. The unit vector points in the direction of the -axis and, similarly, the unit vector points in the direction of the -axis, as illustrated in Figure 15(a).

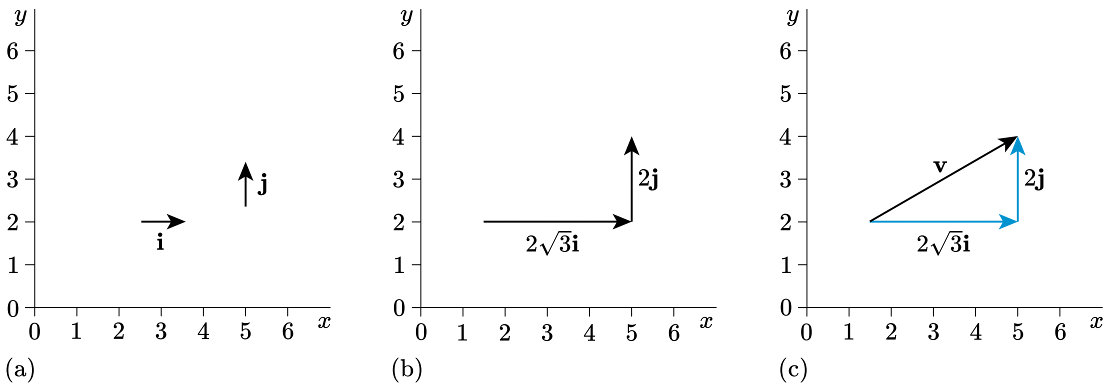


Figure 15 A vector and its components: (a) the Cartesian unit vectors; (b) multiples of the Cartesian unit vectors; (c) describing according to Cartesian unit vectors

Consider what the expression means. We are multiplying the unit vectors and by scalar values and , as illustrated in Figure 15(b). Using the rule for multiplying a vector by a scalar:

- multiplying by gives a vector of magnitude pointing in the direction of the positive -axis
- multiplying by gives a vector of magnitude pointing in the direction of the positive -axis.

So is a quick way to write ‘horizontal component of magnitude ’, and is a quick way to write ‘vertical component of magnitude ’, and the sum of these is the vector , as illustrated in Figure 4.15(c).

Expressing a vector as the sum of scalar multiples of unit vectors is a useful shorthand, and every vector can be described in this way. It works because perpendicular vectors act independently from each other – a change in the horizontal component has no effect on the vertical component, and vice versa. This is also what happens in the physical phenomena that are modelled using vector quantities, such as motion.

Imagine you have two balls, ball A and ball B, and you throw ball A forward at the same time that you drop ball B, as illustrated in Figure 16. Now, consider the velocities of the two balls. For both balls a vertical velocity is produced as a consequence of weight due to gravity. For ball A there is also a horizontal velocity because it has been thrown forward. Which ball do you expect will hit the ground first?

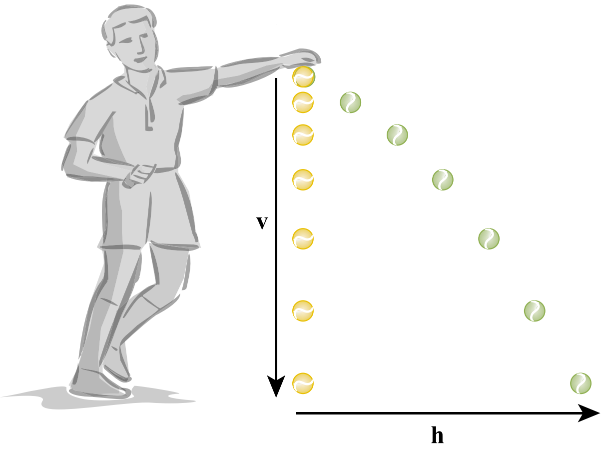


Figure 16 Perpendicular vectors of motion act independently

You may be surprised to hear that both balls will hit the ground at the same time. This is because, regardless of how fast ball A is thrown forward, the horizontal velocity has no effect on the vertical velocity – the vectors are independent because they are perpendicular.

Component form of a vector

If , then the expression is called the component form of . The scalar is called the -component of and the scalar is called the -component of .

Recall, from Chapter 1, that it is convention that if the horizontal component of a vector points in the direction of the negative -axis, then its magnitude is negative, and similarly if the vertical component points in the direction of the negative -axis, then its magnitude is negative. For example, in Figure 17 the component form of vector is , so its -component is and its -component is . Similarly, the component form of the vector is , so its -component is and its
-component is .

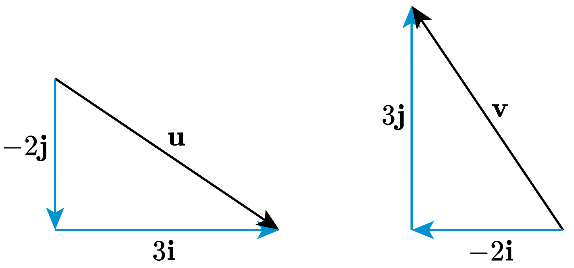
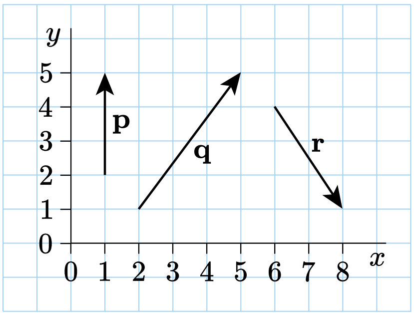


Figure 17 Examples of vectors and their components

Sometimes, the - and -components of a two-dimensional vector are called the - and -components.

Activity 8

Express the following vectors , and in component form.



An alternative way for expressing a vector in component form is the **column vector**, which is common in engineering.

This is a column of numbers surrounded by brackets, where the first number is the -component and the second number is the -component. For example,

Both ways of expressing the vector are equally valid, but column vectors are often preferred because there is no need to explicitly write the and . It is often the convention that the components of a vector are expressed using the same letter as the vector (but not bold or underlined), with subscripts. For example,

Alternative component form of a vector

The vector can be written as .

A vector written in this form is called a column vector.

Activity 9

Use your own grid and draw the following vectors.

-
-

•

2.3 Converting between vector forms

It is useful to be able to convert between the different forms of vectors, so that we can make use of the form that is most appropriate for a given situation. For example, the vector in Figure 18(a) represents the displacement of London from Milton Keynes, and direction and angle are the most intuitive description for this vector. London is approximately 65 km south-east of Milton Keynes. Similarly, the vector in Figure 18(b) represents the motion of an aircraft, and for describing the ground speed of the aircraft, a description of the vector in terms of horizontal and vertical components is required.

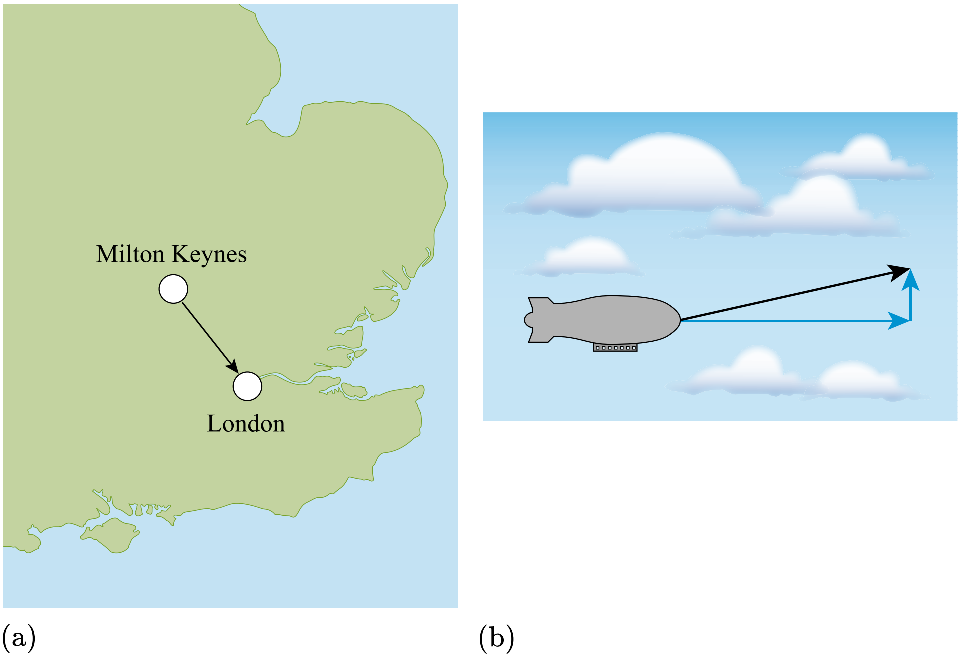


Figure 18 Examples of vectors: (a) displacement of London from Milton Keynes; (b) motion of an aircraft

In both situations it is useful to be able to convert between vector forms. For example, it may be necessary to describe how far east London is from Milton Keynes, and how far south. To describe the direction in which the aircraft is flying, it is useful to use the resultant velocity.

The mathematics that allows us to convert between different forms of vectors.

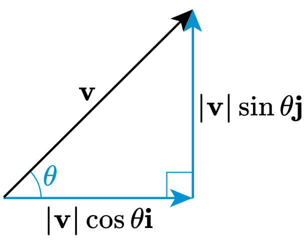


Figure 19 A vector and its components

We have already identified how to calculate the magnitude of the component vectors of a vector. For example, the vector in Figure 19 has magnitude and direction , and its components are defined as and .

Putting these into component form we have the following result.

Component form of a vector in terms of its magnitude and angle with the positive *x*-axis

If the vector makes an angle with the positive -axis, then

Activity 10

Find the component forms of the following vectors. Give your answers to two decimal places.

a. Vector with magnitude 78 and direction given by an angle of 216° with the positive -axis.

b. vector with magnitude 4.4 and direction given by an angle of radians with the positive -axis.

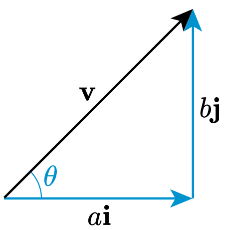


Figure 20 A vector

Going the other way, to find the magnitude and direction of a vector from its component form, the reasoning is the same as how to convert between Cartesian and polar coordinates. For example, the vector illustrated in Figure 20 has horizontal component of magnitude and vertical component of magnitude , and in component form is given by . To calculate the magnitude of , we can use Pythagoras' theorem, and to calculate the angle , we can use the inverse tangent function.

Magnitude and direction of a vector in terms of its components

If the vector has the component form , then its magnitude is given by

and its direction is given by the angle measured anticlockwise from the positive -axis, where

Remember, when using the inverse tan function, care must be taken to ensure that the answer given by a calculator is correct.

Activity 11

Find the magnitude to two decimal places and direction to one decimal place of the following vectors.

a.

b.

c.

2.4 Position vectors

There is a strong similarity between coordinates and vectors, and we have made use of this similarity when converting between vectors of different forms. We can also make use of this similarity when describing the locations of points in space: the location of a point can be described using a vector.

Let be a point in space. Then the **position vector** of is a displacement vector where is the origin. This is illustrated in Figure 21.

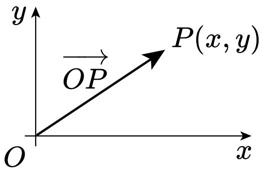


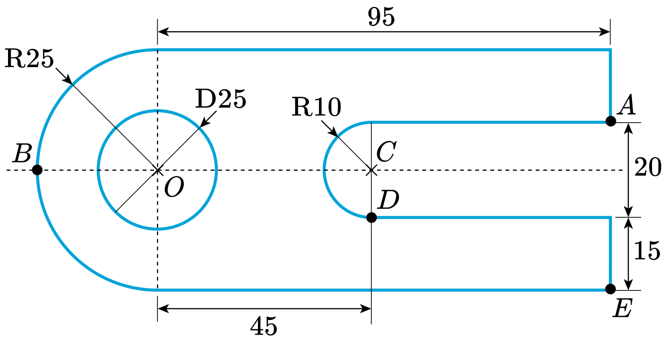
Figure 21 The position vector

The components of the position vector of a point are the same as the coordinates of the point. So the position vector of a point with coordinates is

If a point is denoted by a capital letter, as is the usual convention, then it's often convenient to denote its position vector by the corresponding lowercase, bold (or underlined) letter. For example, we can denote the position vector of the point by , the position vector of point by , and so on.

Activity 12

Identify the position vectors of the points labelled to in the following diagram.



3 Vector algebra with components

So far, we have explored how to visually combine vectors using mathematical operations such as addition and subtraction. We did this with arrows as representations of vectors, but we can also apply the operations algebraically by expressing vectors in component form.

3.1 Vector addition in component form

Let's return to Alice and Bob pushing a block of ice, with each pushing on a different face of the block, as illustrated in Figure 22. If Bob applies a force of 130 N to the left face of the block, and Alice applies a force of 110 N to the bottom face, what is the combined force applied to the block?

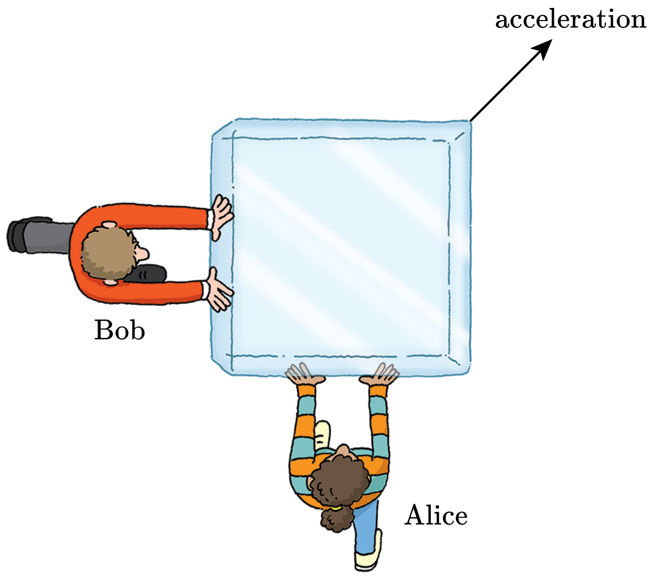


Figure 22 Alice and Bob pushing different sides of a block of ice

Figure 23(a) shows an abstraction of the drawing in Figure 22. Here,
represents the force applied by Alice, who is below the block, and represents the force applied by Bob, who is to the left of the block.

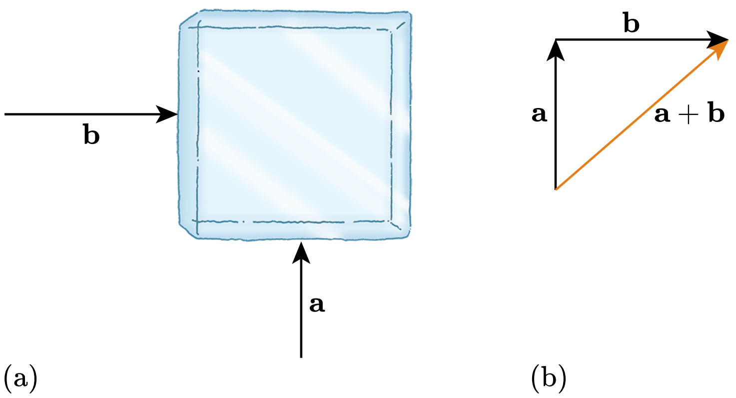


Figure 23 Combining the vectors and

Activity 13

Vector is a vertical vector with magnitude 110 N and is a horizontal vector with magnitude 130 N. Write the vectors and in component form.

Vector is a vertical vector so its -component is zero, and is a horizontal vector so its -component is zero. Together, and are the horizontal and vertical components of the resultant vector , so we can say that

So the net force on the block of ice in Figure 22 is represented by the vector .

This example was straightforward because we were considering perpendicular forces acting vertically and horizontally. But the process we followed is the same for any vector addition. Using the component form of two vectors
and , we add them together algebraically to determine the resultant vector .

Let's consider another example. In Figure 24, Alice and Bob are both pushing the same face of the block of ice. If, as before, Bob applies a force of 130 N and Alice a force of 110 N, what is the combined force applied to the block?

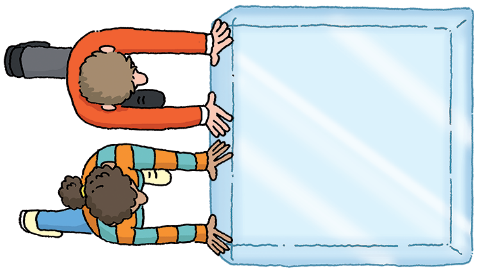


Figure 24 Alice and Bob pushing a block of ice

Again we let the vector represent the force applied by Alice and vector represent the force applied by Bob, so this time and are both horizontal vectors.

Both vectors act horizontally, so are acting in the same direction, the direction of the positive -axis, and we'd expect them to add up with the net result being a stronger force acting in the same direction. If we add the component forms of and , then this is the result we get:

So the resultant vector is , and the net effect is a force of 240 N acting in the direction of the positive -axis. The resultant force on the block of ice is the sum of the forces applied by Alice and Bob, and the block will accelerate faster in the direction they are pushing as we'd intuitively expect.

To summarise, when we add vectors in component form, we add the individual components. This is illustrated visually in Figure 25, where the vector is the sum of the two vectors and , and the components of are the sums of the individual components of and . So expressing in component form gives us the following.

Adding vectors in component form

If and , then

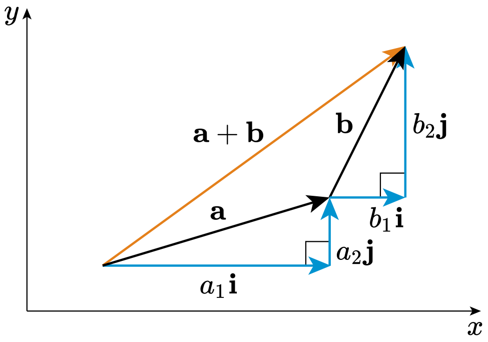


Figure 25 Sum of the vectors and

For example, the sum of the vectors and is given by

We can also add the vectors using column form. For example:

Adding column vectors in component form

If and , then .

This method of adding vectors also extends to sums of more than two vectors. For example, if , and , then

Activity 14

Find the following vector sums.

a.
b.
c.

Using component vectors we can add vector quantities algebraically. Let's reconsider our block of ice example where Alice and Bob are both pulling the block in different directions (see Section 1.2). Using the component form of vectors, we can quickly calculate the combined force applied by Alice and Bob to the block of ice.

Example 3 Calculating the magnitude of combined forces

Alice and Bob have attached ropes to a face of the block of ice and are pulling it in different directions, see Figure 26. Bob pulls with a force of 130 N at an angle of 47° clockwise from the horizontal, and Alice pulls with a force of 110 N at an angle of 24° anticlockwise from the horizontal. Express the forces applied by Alice and Bob in component form, and use these to determine the magnitude of the combined force applied to the block.

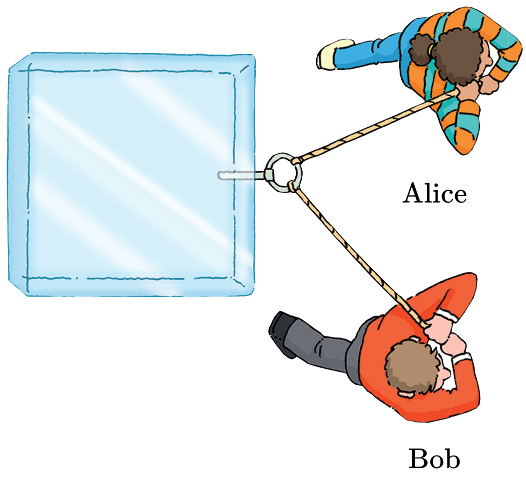


Figure 26 Alice and Bob pulling on a block of ice in different directions

Solution

First let's express the forces applied by Alice and Bob in component form.

Alice applies a force with magnitude 110 N at an angle of 24°. So in component form we have

Bob applies a force with magnitude 130 N at an angle of 360° – 47° = 313°. So in component form we have

To calculate the combined force, add the corresponding components:

So the magnitude of the combined force is

3.2 Scalar multiplication of vectors in component form

In Figure 27 Alice and Bob are both pushing the same face of the block of ice, but this time with the same force.

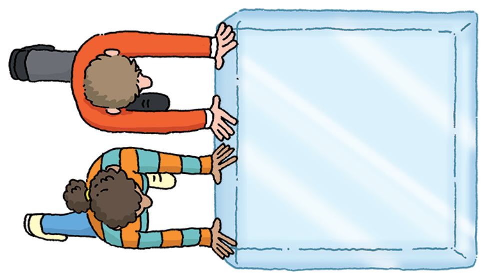


Figure 27 Alice and Bob pushing a block of ice with the same force

Because they are both applying the same force, we can use a single vector to represent this, say , and if the force they apply is 110 N, then

Now, the combined force exerted by both Alice and Bob is

This confirms that when we multiply a vector with a scalar quantity, the magnitude of the vector is multiplied by the scalar; if the scalar is positive its direction stays the same, but if the scalar is negative the direction is reversed.

For the situation of Alice and Bob pushing the block of ice we have

Other examples are illustrated in Figure 28. The vector can be written in component form as , and the scalar multiples of are written in component form as follows:

and

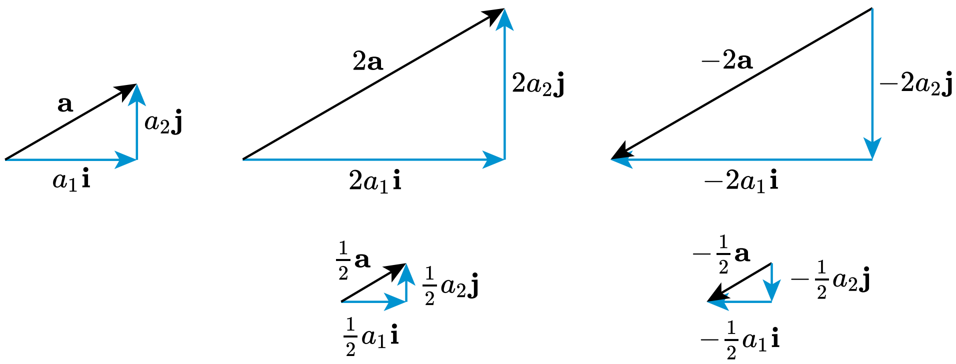


Figure 28 Scalar multiplication of a vector , in component form

Scalar multiplication of a vector in component form

If and is a scalar, then

In column notation, if and is a scalar, then

For example, if , then .

Activity 15

Let and . Find each of the following scalar multiples.

a.

b.

c.

d.

e.

f.

3.3 Vector subtraction in component form

When subtracting a vector visually it is first necessary to find the negative of the vector being subtracted by reversing its direction. Algebraically, a similar process is followed, but if we follow the standard rules of algebra, it is a much more intuitive process. For example, consider the vector expression where the vector is subtracted from the vector . We can make sense of this by writing the expression as

where is the negative of . The negative of a vector has the same magnitude but the opposite direction, and for a vector we say its negative is

With this in mind, we can say that to subtract vectors in component form, we subtract each component of one vector from the corresponding component of the other.

Subtracting vectors in component form

If and , then

In column notation, if and , then

Example 4 !Warning! Calibri not supportedCalculating vector subtraction in component form

Let and . Find .

Solution

Subtracting the components of from the corresponding components of gives

Activity 16

Find the following vectors.

a.

b.

c.

3.4 Combining vector operations

When using vectors to model engineering systems it is often necessary to carry out multiple operations to combine vectors in different ways. When vectors are expressed in component form, combining operations involves following the standard rules of algebra. In the next example, the vector operations of addition, subtraction and scalar multiplication are combined.

Example 5 Simplifying a combination of vectors in component form

Let , and .

Find in component form.

Solution

Substitute in the expressions for , and in terms of and , and then simplify:

Activity 17

Find each of the following vectors in component form.

a. , where , and
b.
c. , where and are any scalars

3.5 Vector algebra

Throughout this course we have explored calculations involving vectors, and from these we can identify general properties of addition, subtraction and scalar multiplication of vectors. For example, using the properties of addition and scalar multiplication we found that for any vectors and :

All the algebraic properties of vectors can be summarised using eight basic algebraic properties. These are mathematical expressions of results that may seem like common sense, but when we express them using this notation, it confirms the ways that we can apply standard rules of algebraic manipulation to vectors. Notice that multiplication and division of vectors are not included in the list; this is because these operations are defined differently for vectors, and we will explore one definition of vector multiplication in the next section.

Properties of vector algebra

The following properties hold for all vectors , and , and all scalars and .

1.
2.
3.
4.
5.
6.
7.
8.

In many ways vector quantities behave in a similar manner to scalar quantities, and these eight properties allow us to perform some operations on vector expressions in a similar way to numbers, or algebraic expressions.

Example 6 Simplifying vector expressions

Simplify the vector expression

Solution

Expand the brackets using property 5:

Collect like terms using property 6:

The properties of vector algebra also allow us to manipulate equations containing vectors, which are known as **vector equations**, in a similar way to ordinary equations. For example, we can add or subtract vectors on both sides of such an equation, and we can multiply or divide both sides by a non-zero scalar. We can use these methods to rearrange a vector equation to make a particular vector the subject, or to solve an equation for an unknown vector.

Activity 18

a. Simplify the vector expression

b. Rearrange the vector equation

   to express in terms of and .

4 Scalar product of vectors

This section explores a way to multiply two vectors, which is called the **scalar product** or the **dot product**. It is called the scalar product, because when using this method of multiplication, the result is a scalar quantity. It is also called the dot product because it is written using the symbol ‘’, for example, the dot product of vectors and is written .

Let’s start by considering what multiplication might mean in the context of vectors. For scalar quantities, multiplication can be thought of as repeated counting. For example, can mean . Alternatively, we can think of multiplication as taking a magnitude and growing it. For example, can mean taking a magnitude of 4 and making it 3 times larger. For the scalar product of vectors, it is useful to think of multiplication in terms of growth.

Vectors have direction as well as magnitude, and if we consider vector operations in terms of growth, then we can describe them as follows.

- Adding vectors: accumulate growth from several vectors.
- Scalar multiplication: make an existing vector grow.
- Scalar product: apply the directed growth of one vector to another vector. The result is how much stronger we have made the original.

For example, if we are talking about force vectors, then the scalar product gives us a measure of how much push one vector can give to another. This cannot be just a matter of multiplying the magnitude of the vectors, because their directions need to be taken into consideration. So the scalar product is a multiplication operation that takes into consideration the directions of vectors. With this concept in mind, let’s look at some examples.

4.1 Scalar product of a vector from components

Consider vectors and in Figure 29. In component form these are written as and . How can we calculate the scalar product ?

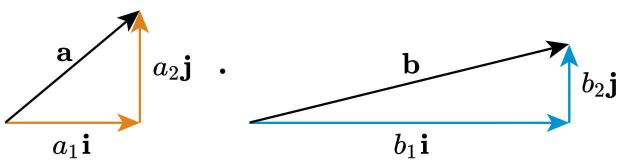


Figure 29 Finding the scalar product of and by comparing components

The scalar product will tell us how much vector will grow vector , and to determine this we want to identify how much the vectors interact. One method is to consider how much the horizontal and vertical components of the vectors interact, as illustrated in Figure 30. There are four possible combinations to consider: horizontal to horizontal, horizontal to vertical, vertical to horizontal, and vertical to vertical.

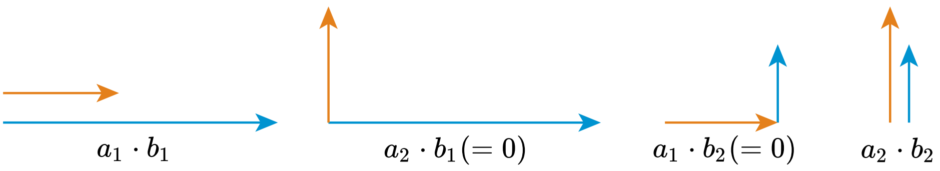


Figure 30 Interacting component vectors in the scalar product of and

Horizontal components do not interact with vertical components (and vice versa) because they are independent of each other, so and , and they do not contribute to the value of scalar product. Horizontal components interact with each other, and vertical components interact with each other, so and both contribute to the value of .

The expression is a measure of how much the scalar quantity grows the scalar quantity , so it is equal to , and similarly is equal to . The scalar product is a combination of these, so

For example, if and , then

Scalar product of vectors in terms of components

If and , or and in column notation, then

Activity 19

Suppose that , and . Find the following.

a.

b.

c.

4.2 Scalar product of a vector from magnitude and direction

Another way to consider the scalar product is to consider how it is defined in terms of the magnitudes and directions of two vectors. Consider again the vectors and in Figure 29. We want to find out how much vector will grow vector . So again we want to identify how much the vectors interact – and one way to do this is to determine how much vector points in the direction of vector .

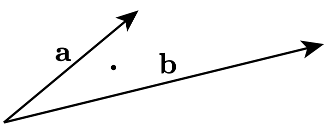


Figure 31 Finding the scalar product of and by comparing magnitudes and directions

In Figure 31, the vectors are arranged so that their tails meet, and this makes it possible to compare their magnitudes and directions. To make this explicit, we can draw the components of , not in terms of horizontal and vertical directions, but in terms of the direction where is pointing, as illustrated in Figure 32(a). Formally, the component of that points in the direction of is called the **projection** of onto , and if the angle between and is , then the length of the projection of onto is , as shown in Figure 32(b).

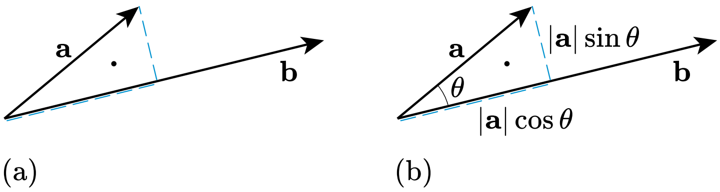


Figure 32 Projecting onto

Comparing the components of with will give us a measure of how much and interact, as illustrated in Figure 33. In the direction of , has a component of magnitude , and has a component of magnitude , so the contribution to the value of is . Perpendicular to , has a component of magnitude , and has a component of magnitude 0, so the contribution to the value of is 0.

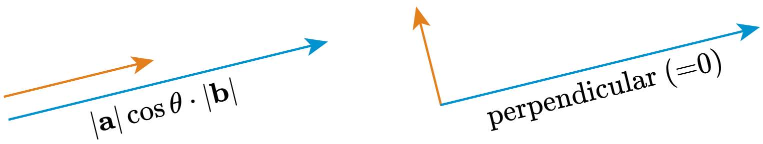


Figure 33 Interacting component vectors in the scalar product of and

So

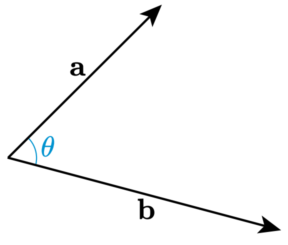
and this is a measure of how much the scalar quantity grows the scalar quantity .
and are parallel, so

and the scalar product is given by

Scalar product of vectors in terms of magnitude and direction

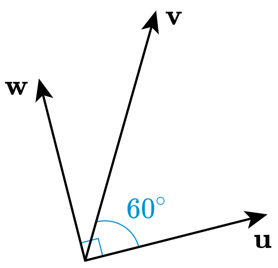
The scalar product of two vectors and is

where is the angle between and .



Activity 20

Suppose that , and are vectors with magnitudes 4, 3 and 2 respectively, and directions as shown in the following figure.



Find the following scalar products.

a.

b.

c.

4.3 Properties of the scalar product

Activity 20 illustrates two important properties of the scalar product. First, if two non-zero vectors are perpendicular, then their scalar product is zero. This is because if and are perpendicular, then

This also works the other way, so that if the scalar product of two non-zero vectors is zero, then the vectors are perpendicular. This is because if and are non-zero vectors, then the only way that can be equal to zero is if . This implies that .

The second property is that the scalar product of a vector with itself is equal to the square of the magnitude of the vector. This is because if is any non-zero vector, then the angle between and itself is 0°, so

These, and other, properties of the scalar product in the following list can all be proved using the definition of the scalar product in a similar way.

Scalar product properties

The following properties hold for all vectors , and , and every scalar .

1. If and are non-zero and perpendicular, then and .
2.
3.
4.
5.

We can use these properties to simplify expressions containing scalar products of vectors.

Example 7 Simplifying an expression containing a scalar product

Expand and simplify the expression , where and are vectors.

Solution

Expand the brackets by using property 4:

Simplify by using property 3, so:

Using property 2, simplify further to get:

Activity 21

Expand and simplify the expression , where and are vectors.

4.4 Finding the angle between two vectors

The scalar product of two vectors has an important application in calculating the angle between two vectors. If we start with the definition of the scalar product in terms of the magnitudes and directions of the vectors, and rearrange it, then we get the following result.

Angle between two vectors

The angle between any two non-zero vectors and is given by

We can use this result to find the angle between two vectors in component form.

Example 8 Calculating the angle between two vectors in component form

Alice and Bob have attached ropes to a face of the block of ice and are pulling it in different directions, see Figure 34. Vector describes the force applied by Alice, and in component form is given by . Vector describes the force applied by Bob, and in component form is given by .

What is the angle between these vectors, to one decimal place?

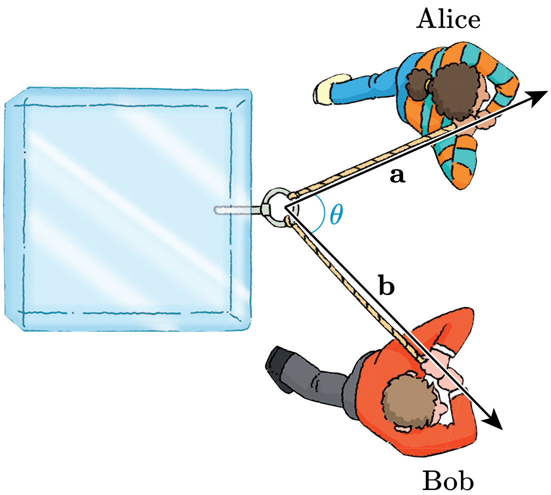


Figure 34 Alice and Bob pulling a block of ice

Solution

First let's use the components of and to find , and . We have

Using these we can calculate :

So

Therefore the angle between the vectors is 64.4° (to 1 d.p.).

Activity 22

Find, to the nearest degree, the angle between the vectors

Conclusion

This course developed techniques that make it easier to work with vectors. Instead of working with vectors geometrically, it is much more efficient to work with them in component form. When vectors are represented according to their components, engineering problems involving vectors can be solved by carrying out standard algebraic operations.

This foundation in manipulating and working with vectors will allow you to start thinking about modelling forces in increasingly complex situations as well as other scenarios, such as modelling movement of gasses, liquids or particles.

This OpenLearn course is an adapted extract from the Open University course T194 *Engineering: mathematics, modelling, applications*.

Solutions to activities

Activity 1

is the hypotenuse of the right-angled triangle formed by , and the resultant , so from Pythagoras' theorem its magnitude is given by

This gives

So the magnitude of is 170.29 (to 2 d.p.).

Activity 2

Angles and are alternate angles, so they are equal. We can find using the tangent function, which is given by

So

therefore

This gives (to 1 d.p.) and this is the direction of the vector .

Activity 3

a. After 30 seconds we have

and

So the block is travelling at a speed of approximately and has travelled a distance of approximately 76.5 m.

b. After 60 seconds we have

and

So the block is travelling at a speed of approximately and has travelled a distance of approximately 306 m.

Activity 4

First we need to determine the size of angle . We can use the angles 24° and 47° to calculate because they sit on the same straight line as , so

Now, using the cosine rule, we can calculate the length of edge :

So

Activity 5

Using the sine rule, we get

so

Using the inverse sine function, we get

Activity 6

Newton's second law gives

so acceleration is given by

therefore

The direction of the acceleration is the same as the direction of , and this is measured clockwise from the positive -axis.

The magnitude of the acceleration is

So the block accelerates at (to 2 d.p.) in a direction that is measured clockwise from the positive -axis.

Activity 7

The magnitude of the vertical component is given by

so

The magnitude of the horizontal component is given by

so

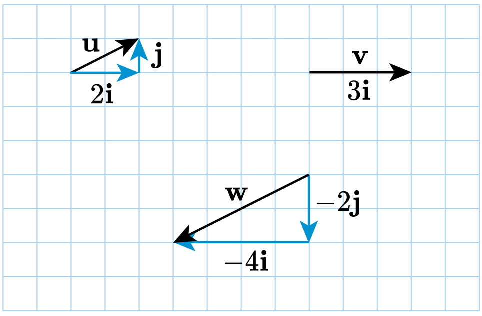
Activity 8

The horizontal displacement is and the vertical displacement is , so .

The horizontal displacement is and the vertical displacement is , so .

The horiztonal displacment is and the vertical displacement is , so .

Activity 9



Activity 10

a. The component form of the vector is given by

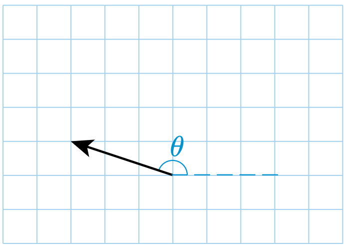
b. The component form of the vector is given by

Activity 11

a. The magnitude of is

and the direction is given by

The calculator value for is , but looking at a drawing of the vector below, shows that this is not the correct angle. Instead, we are looking for a value of that is greater than and less than .



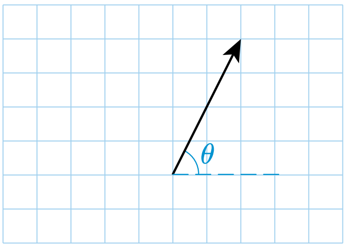
By considering the graph of , we can identify the angles that are within the range, and using the periodicity of the tangent function we can say that the value for is

b. The vector has no vertical component, and a negative horizontal component. So it points in the negative -direction and its magnitude is

c. The magnitude of is

and the direction is given by

The calculator value for is . Looking at a drawing of the vector below shows that this is the correct value for , because it is greater than and less than .



Activity 12

Activity 13

and .

Activity 14

a.

b.

c.

Activity 15

a.

b.

c.

d.

e.

f.

Activity 16

a.

b.

c.

Activity 17

a.

b.

c.

Activity 18

a.

b. Rearranging gives

So .

Activity 19

a.

b.

c.

Activity 20

a.

b.

c.

Activity 21

Expand the brackets by using property 4:

Simplify by using property 3 to give

Simplify further by using property 2:

Activity 22

First let’s use the components of and to find , and . We have

Using these we can calculate :

So

The angle between the vectors is 27° (to the nearest degree).

References

Box, G.E.P. and Draper, N.R. (1987) *Empirical Model-building and Response Surfaces*, Oxford, John Wiley & Sons.

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