

**M337\_2**

**Introduction to complex analysis**

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**Session 1: Differentiation**

## Introduction to differentiation

The derivative of a real function f at a point c is the gradient of the tangent to the graph of f at c. This gradient is calculated by finding the gradient of the chord joining the point left parenthesis c comma f of c right parenthesis to a (nearby) point left parenthesis x comma f of x right parenthesis, and taking the limit as x approaches c (Figure 1).

Start of Figure

Displayed image

Figure 1 A chord between two points on a graph

[View description - Figure 1 A chord between two points on a graph](" \l "Unit1_Session1_Description1)

End of Figure

Now, the gradient of the chord is equal to the ratio

Start of $1

f of x minus f of c divided by x minus c full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session1_Alternative9)

End of $1

This ratio is often called the difference quotient for f at c, and its limit as x tends to c provides a formal definition of the (real) derivative of f at c, denoted by f super prime of c. Thus

Start of $1

f super prime of c equals lim over x right arrow c of f of x minus f of c divided by x minus c full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session1_Alternative17)

End of $1

In the case of complex functions, it is difficult to think about derivatives in terms of gradients of tangents, since the graph of a complex function is not drawn in two dimensions. Instead we define the derivative of a complex function directly in terms of difference quotients, using the notion of complex limits.

Fortunately, the derivatives of many complex functions turn out to have the same form as those of the corresponding real functions. For example, the derivative of the complex sine function is the complex cosine function, and the complex exponential function is its own derivative. On the other hand, the complex modulus function fails to be differentiable at any point of double-struck cap c, even though the real modulus function (Figure 2) is differentiable at every point of double-struck cap r minus zero. This reflects the fact that complex differentiation imposes a much stronger condition on functions than does real differentiation. Indeed, as the course progresses, you will see that differentiable complex functions have remarkably pleasant properties. For example, if a complex function can be differentiated once throughout a region, then it can be differentiated any number of times. There is no equivalent result for real functions.

Start of Figure

Displayed image

Figure 2 Graph of y equals absolute value of x

[View description - Figure 2 Graph of y equals absolute value of x](" \l "Unit1_Session1_Description2)

End of Figure

Section 1 will define complex differentiation and show how the definition can be used to establish whether a function is differentiable. By introducing rules for combining differentiable functions, you will see how complex polynomial and rational functions can be differentiated just as in the real case. The end of Section 1 will give a geometric interpretation of complex differentiation by introducing the idea of a complex scale factor.

Section 2 will introduce the concept of partial differentiation for real functions of two real variables, and use it to establish a relationship between complex differentiation and real differentiation. This relationship sometimes enables us to differentiate a complex function using real derivatives. Indeed, at the end of the section, this approach will be used to show that the complex exponential function is its own derivative.

Start of Box

This OpenLearn course is an extract from the Open University course [M337 Complex analysis](https://www.open.ac.uk/courses/modules/m337).

End of Box

## 1 Derivatives of complex functions

After working through this section, you should be able to:

* use the definition of derivative to show that a given function is differentiable, and to find its derivative
* use the Combination Rules for differentiation to differentiate polynomial and rational functions
* use various strategies to show that a given function is not differentiable at a point
* interpret the derivative of a complex function at a point as a rotation and a scaling of a small disc centred at the point.

## 1.1 Defining differentiable functions

As with limits and continuity, the way in which the derivative of a complex function is defined is similar to the real case. Thus a complex function is said to have a derivative at a point alpha element of double-struck cap c if the **difference quotient**, defined by

Start of $1

f of z minus f of alpha divided by z minus alpha comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative2)

End of $1

tends to a limit as z tends to alpha. Equivalently, it is sometimes more convenient to replace z by alpha plus h, and examine the corresponding limit as h tends to 0. The difference quotient then has the form

Start of $1

f times left parenthesis alpha plus h right parenthesis minus f of alpha divided by h comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative8)

End of $1

where h is a complex number. The equivalence of these two limits can be justified by noting that if z equals alpha plus h, then ‘z right arrow alpha’ is equivalent to ‘h right arrow zero’.

Start of Box

**Definitions**

Let f be a complex function whose domain contains the point alpha. Then the **derivative of** bold-italic f**at** bold-italic alpha is

Start of $1

lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha times left parenthesis or lim over h right arrow zero of f times left parenthesis alpha plus h right parenthesis minus f of alpha divided by h right parenthesis comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative17)

End of $1

provided that this limit exists. If it does exist, then f is **differentiable at** bold-italic alpha. If f is differentiable at every point of a set cap a, then f is **differentiable on** bold-italic cap a. A function is **differentiable** if it is differentiable on its domain.

The derivative of f at alpha is denoted by f super prime of alpha, and the function

Start of $1

f super prime colon z long right arrow from bar f super prime of z

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative27)

End of $1

is called the **derivative of** bold-italic f. The domain of f super prime is the set of all complex numbers at which f is differentiable.

End of Box

The function f super prime is sometimes called the derived function of f.

### Remarks

1. The existence of the limit

Start of $1

lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative33)

End of $1

implicitly requires the domain of f to contain alpha as one of its limit points. This always holds if the domain of f is a region.

1. The derivative f super prime of z is sometimes written as d times f divided by d times z times left parenthesis z right parenthesis or d divided by d times z times left parenthesis f of z right parenthesis.
2. Some other texts use the phrase complex derivative in place of derivative to draw a distinction with the standard real derivative of a function f colon double-struck cap r squared long right arrow double-struck cap r squared (which we will not need).

In certain cases it is easy to find the derivative of a function directly from the definition above.

Start of Example

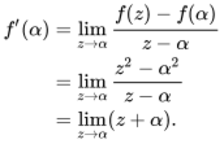
**Example 1**

Use the definition of derivative to find the derivative of the function f of z equals z squared.

**Solution**

The domain of f of z equals z squared is the whole of double-struck cap c, so let alpha be an arbitrary point of double-struck cap c. Then

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative46)

End of $1

Now z long right arrow from bar z plus alpha is a basic continuous function, continuous at alpha, so we see that equation sequence part 1 f super prime of alpha equals part 2 alpha plus alpha equals part 3 two times alpha.

Since alpha is an arbitrary complex number, the derivative of f is the function f super prime of z equals two times z. Its domain is the whole of double-struck cap c.

End of Example

Notice the way in which the troublesome z minus alpha term cancels from the numerator and the denominator in the calculation of f super prime of alpha in the preceding example. This often happens when you calculate derivatives directly from the definition.

Start of Exercise

**Exercise 1**

Start of Question

Use the definition of derivative to find the derivative of

1. the constant function f of z equals one
2. the function f of z equals z.

End of Question

[View answer - Exercise 1](" \l "Unit1_Session2_Answer1)

End of Exercise

Example 1 and Exercise 1 show that the functions f of z equals one, f of z equals z and f of z equals z squared are differentiable on the whole of double-struck cap c. Functions that have this property are given a special name.

Start of Box

**Definition**

A function is **entire** if it is differentiable on the whole of double-struck cap c.

End of Box

Not all functions are entire; indeed, many interesting aspects of complex analysis arise from functions that fail to be differentiable at various points of double-struck cap c.

Start of Exercise

**Exercise 2**

Start of Question

Use the definition of derivative to find the derivative of the function f of z equals one solidus z. Explain why f is not entire.

End of Question

[View answer - Exercise 2](" \l "Unit1_Session2_Answer2)

End of Exercise

Although the function f of z equals one solidus z is not entire, it is differentiable on the whole of its domain double-struck cap c minus zero. This domain is a region because it is obtained by removing the point 0 from double-struck cap c. (The removal of a point from a region leaves a region.) As the course progresses, you will discover that regions provide an excellent setting for analysing the properties of differentiable functions. We therefore make the following definitions.

Start of Box

**Definitions**

A function that is differentiable on a region script cap r is said to be **analytic on** bold-script cap r. If the domain of a function f is a region, and if f is differentiable on its domain, then f is said to be **analytic**. A function is **analytic at a point** bold-italic alpha if it is differentiable on a region containing alpha.

End of Box

It follows immediately from the definition that if a function is analytic on a region script cap r, then it is automatically analytic at each point of script cap r.

Notice that a function can have a derivative at a point without being analytic at the point. For example, in the next section we will ask you to show that the function g of z equals absolute value of z squared has a derivative at 0, but at no other point. This means that there is no region on which g is differentiable, and hence no point at which g is analytic.

By contrast, f of z equals one solidus z is analytic at every point of its domain. It is an analytic function, and it is analytic on any region that does not contain 0. Three such regions are illustrated in Figure 3.

Start of Figure

Displayed image

Figure 3 Three regions on which f of z equals one solidus z is analytic

[View description - Figure 3 Three regions on which f of z equals one solidus z is analytic](" \l "Unit1_Session2_Description1)

End of Figure

An appropriate choice of region can often simplify the analysis of complex functions.

Start of Exercise

**Exercise 3**

Start of Question

Classify each of the following statements as True or False.

1. An entire function is analytic at every point of double-struck cap c.
2. If a function is differentiable at each point of a set, then it is analytic on that set.

End of Question

[View answer - Exercise 3](" \l "Unit1_Session2_Answer3)

End of Exercise

There is a close connection between differentiation and continuity. The function f of z equals one solidus z, for example, is not only differentiable, but also continuous on its domain. This is no accident for, as in real analysis, differentiability implies continuity.

Start of Box

**Theorem 1**

Let f be a complex function that is differentiable at alpha. Then f is continuous at alpha.

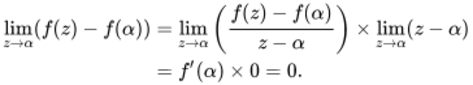
End of Box

Start of Proof

ProofLet f be differentiable at alpha; then

lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha equals f super prime of alpha full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative122)

To prove that f is continuous at alpha, we will show that f of z right arrow f of alpha as z right arrow alpha. We do this by proving the equivalent result that f of z minus f of alpha right arrow zero as z right arrow alpha. By the Product Rule for limits of functions, we have 

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative129)

Hence f of z right arrow f of alpha as z right arrow alpha, so f is continuous at alpha. âˆŽ

End of Proof

In fact, differentiability implies more than continuity. Continuity asserts that for all z close to alpha, f of z is close to f of alpha. For differentiable functions, this ‘closeness’ has the ‘linear’ form described in the following theorem.

Start of Box

**Theorem 2 Linear Approximation Theorem**

Let f be a complex function that is differentiable at alpha. Then f can be approximated near alpha by a linear polynomial. More precisely,

Start of $1

f of z equals sum with 3 summands f of alpha plus left parenthesis z minus alpha right parenthesis times f super prime of alpha plus e of z comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative142)

End of $1

where e is an ‘error function’ satisfying e of z solidus left parenthesis z minus alpha right parenthesis right arrow zero as z right arrow alpha.

End of Box

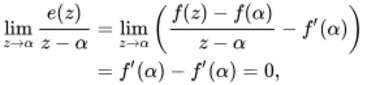
Informally speaking, the statement ‘e of z solidus left parenthesis z minus alpha right parenthesis right arrow zero as z right arrow alpha’ means that ‘e of z tends to zero faster than z minus alpha does’.

Start of Proof

ProofWe have to show that the function e defined by

e of z equals f of z minus f of alpha minus left parenthesis z minus alpha right parenthesis times f super prime of alpha

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative151)

satisfies e of z solidus left parenthesis z minus alpha right parenthesis right arrow zero as z right arrow alpha. Dividing e of z by z minus alpha and letting z tend to alpha, we obtain 

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative158)

as required. âˆŽ

End of Proof

Theorem 1 and Theorem 2 are often used to investigate the properties of differentiable functions. An illustration of this occurs in the next subsection, where Theorem 1 is used in a proof of the Combination Rules for differentiation. Later in this section we use Theorem 2 to give a geometric interpretation of complex differentiation.

## 1.2 Combining differentiable functions

It would be tedious if we had to use the definition of the derivative every time we needed to differentiate a function. Fortunately, once the derivatives of simple functions like z long right arrow from bar one and z long right arrow from bar z are known, we can find the derivatives of other more complicated functions by applying the following theorem.

Start of Box

**Theorem 3 Combination Rules for Differentiation**

Let f and g be complex functions with domains cap a and cap b, respectively, and let alpha be a limit point of cap a intersection cap b. If f and g are differentiable at alpha, then

1. **Sum Rule** f plus g is differentiable at alpha, and

Start of $1

left parenthesis f plus g right parenthesis super prime times left parenthesis alpha right parenthesis equals f super prime of alpha plus g super prime of alpha

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative172)

End of $1

1. **Multiple Rule** lamda times f is differentiable at alpha, for lamda element of double-struck cap c, and

Start of $1

left parenthesis lamda times f right parenthesis super prime times left parenthesis alpha right parenthesis equals lamda times f super prime of alpha

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative176)

End of $1

1. **Product Rule** f times g is differentiable at alpha, and

Start of $1

left parenthesis f times g right parenthesis super prime times left parenthesis alpha right parenthesis equals f super prime of alpha times g of alpha plus f of alpha times g super prime of alpha

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative179)

End of $1

1. **Quotient Rule** f solidus g is differentiable at alpha (provided that g of alpha not equals zero right parenthesis, and

Start of $1

left parenthesis f divided by g right parenthesis super prime times left parenthesis alpha right parenthesis equals g of alpha times f super prime of alpha minus f of alpha times g super prime of alpha divided by left parenthesis g of alpha right parenthesis squared full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative183)

End of $1

End of Box

We remark that if the domains cap a and cap b in Theorem 3 are regions, then every point of cap a intersection cap b is a limit point of cap a and of cap b.

In addition to these rules, there is a corollary to Theorem 3, known as the Reciprocal Rule, which is a special case of the Quotient Rule.

Start of Box

**Corollary Reciprocal Rule for Differentiation**

Let f be a function that is differentiable at alpha. If f of alpha not equals zero, then one solidus f is differentiable at alpha, and

Start of $1

left parenthesis one divided by f right parenthesis super prime times left parenthesis alpha right parenthesis equals negative f super prime of alpha divided by left parenthesis f of alpha right parenthesis squared full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative194)

End of $1

End of Box

The proof of the Combination Rules for differentiation uses the Combination Rules for limits of functions. In the next example we illustrate the method by proving the Product Rule for differentiation. We use the Sum, Product and Multiple Rules for limits of functions, and we also use the fact that if a function g is differentiable at alpha, then it is continuous at alpha, so lim over z right arrow alpha of g of z equals g of alpha.

Start of Example

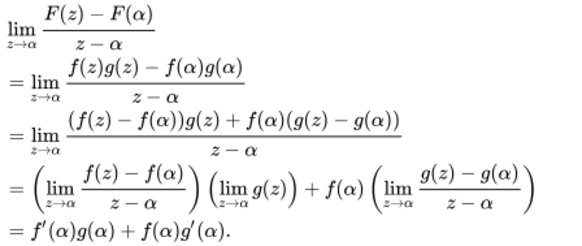
**Example 2**

Prove the Product Rule for differentiation.

**Solution**

Let cap f equals f times g. Then

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative200)

End of $1

End of Example

The proofs of the other Combination Rules are similar. We ask you to prove the Sum and Multiple Rules in Exercise 4, and the Quotient Rule later in Exercise 12.

Start of Exercise

**Exercise 4**

Start of Question

Prove the following rules for differentiation.

1. Sum Rule
2. Multiple Rule

End of Question

[View answer - Exercise 4](" \l "Unit1_Session2_Answer4)

End of Exercise

The Combination Rules enable us to differentiate any polynomial or rational function. (Recall that a rational function is the quotient of two polynomial functions.)

For example, since the function f of z equals z is entire with derivative f super prime of z equals one, we can use the Product Rule repeatedly to show that the function

Start of $1

f of z equals z super n times left parenthesis z element of double-struck cap c right parenthesis

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative208)

End of $1

is entire, and that its derivative is

Start of $1

f super prime of z equals n times z super n minus one times left parenthesis z element of double-struck cap c right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative209)

End of $1

(This result can be proved formally using the Principle of Mathematical Induction.) Next, we can use this fact, together with the Sum and Multiple Rules, to prove that any polynomial function is entire, and that its derivative is obtained by differentiating the polynomial function term by term. For example,

Start of $1

if f of z equals sum with 3 summands z super four minus three times z squared plus two times z plus one comma then f super prime of z equals four times z cubed minus six times z plus two full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative210)

End of $1

In general, we have the following corollary to Theorem 3.

Start of Box

**Corollary Differentiating Polynomial Functions**

Let p be the polynomial function

Start of $1

p of z equals sum with variable number of summands a sub n times z super n plus ellipsis plus a sub two times z squared plus a sub one times z plus a sub zero times left parenthesis z element of double-struck cap c right parenthesis comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative212)

End of $1

where a sub zero comma a sub one comma ellipsis comma a sub n element of double-struck cap c and a sub n not equals zero. Then p is entire with derivative

Start of $1

p super prime of z equals sum with variable number of summands n times a sub n times z super n minus one plus ellipsis plus two times a sub two times z plus a sub one times left parenthesis z element of double-struck cap c right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative216)

End of $1

End of Box

Since a rational function is a quotient of two polynomial functions, it follows from the corollary on differentiating polynomial functions and the Quotient Rule that a rational function is differentiable at all points where its denominator is non-zero; that is, at all points of its domain.

Start of Example

**Example 3**

Find the derivative of

Start of $1

f of z equals two times z squared plus z divided by z squared plus one comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative217)

End of $1

and specify its domain.

**Solution**

By the corollary on differentiating polynomial functions, the derivative of z long right arrow from bar two times z squared plus z is

Start of $1

z long right arrow from bar four times z plus one comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative219)

End of $1

and the derivative of z long right arrow from bar z squared plus one is

Start of $1

z long right arrow from bar two times z full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative221)

End of $1

Provided that z squared plus one is non-zero, we can apply the Quotient Rule to obtain

Start of $1

equation sequence part 1 f super prime of z equals part 2 left parenthesis z squared plus one right parenthesis times left parenthesis four times z plus one right parenthesis minus left parenthesis two times z squared plus z right parenthesis times left parenthesis two times z right parenthesis divided by left parenthesis z squared plus one right parenthesis squared equals part 3 sum with 3 summands negative z squared plus four times z plus one divided by left parenthesis z squared plus one right parenthesis squared full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative223)

End of $1

Since z squared plus one is non-zero everywhere apart from i and negative i, it follows that the domain of f super prime is double-struck cap c minus i comma negative i.

End of Example

Start of Exercise

**Exercise 5**

Start of Question

Find the derivative of each of the following functions. In each case specify the domain of the derivative.

1. f of z equals sum with 3 summands z super four plus three times z cubed minus z squared plus four times z plus two
2. f of z equals z squared minus four times z plus two divided by sum with 3 summands z squared plus z plus one

End of Question

[View answer - Exercise 5](" \l "Unit1_Session2_Answer5)

End of Exercise

So, any rational function is differentiable on the whole of its domain. What is more, this domain must be a region because it is obtained by removing a finite number of points (zeros of the denominator) from double-struck cap c.

Start of Box

**Corollary**

Any rational function is analytic.

End of Box

A particularly simple example of a rational function is f of z equals one solidus z super n, where n is a positive integer. This can be differentiated by means of the Reciprocal Rule:

Start of $1

equation sequence part 1 f super prime of z equals part 2 negative n times z super n minus one divided by left parenthesis z super n right parenthesis squared equals part 3 negative n times z super negative n minus one full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative240)

End of $1

If k is used to denote the negative integer negative n, then we can write f of z equals z super k and f super prime of z equals k times z super k minus one. In this form, it is apparent that the formula for differentiating a negative integer power is the same as the formula for differentiating a positive integer power. The only difference is that for negative powers, 0 is excluded from the domain. We state these observations as a final corollary to Theorem 3.

Start of Box

**Corollary**

Let k element of double-struck cap z minus zero. The function f of z equals z super k has derivative

Start of $1

f super prime of z equals k times z super k minus one full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative247)

End of $1

The domain of f super prime is double-struck cap c if k greater than zero and double-struck cap c minus zero if k less than zero.

End of Box

## 1.3 Non-differentiability

In [Theorem 1](#a4-th1-1) you saw that differentiability implies continuity. An immediate consequence of this is the following test for non-differentiability.

Start of Box

**Strategy A for non-differentiability**

If f is discontinuous at alpha, then f is not differentiable at alpha.

End of Box

Start of Example

**Example 4**

Show that there are no points of the negative real axis at which the function f of z equals Square root of z is differentiable.

**Solution**

The function f of z equals Square root of z is discontinuous at all points of the negative real axis. It follows that there are no points of the negative real axis at which f is differentiable.

End of Example

Start of Exercise

**Exercise 6**

Start of Question

Show that there are no points of the negative real axis at which the principal logarithm function

Start of $1

Log of z equals log of absolute value of z plus i times Arg of z

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative260)

End of $1

is differentiable.

End of Question

[View answer - Exercise 6](" \l "Unit1_Session2_Answer6)

End of Exercise

The converse of [Theorem 1](#a4-th1-1) is not true; if a function is continuous at a point, then it does not follow that it is differentiable at the point. A particularly striking illustration of this is provided by the modulus function f of z equals absolute value of z. This is continuous on the whole of double-struck cap c and yet, as you will see, it fails to be differentiable at any point of double-struck cap c.

Since f of z equals absolute value of z is continuous, Strategy A cannot be used to show that f is not differentiable at a given point alpha. Instead we return to the definition of derivative and show that the difference quotient for f fails to have a limit.

In general, if the domain cap a of a function f contains alpha as one of its limit points, then the existence of the limit

Start of $1

lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative271)

End of $1

means that for each sequence left parenthesis z sub n right parenthesis in cap a minus alpha that converges to alpha,

Start of $1

lim over n right arrow normal infinity of f of z sub n minus f of alpha divided by z sub n minus alpha

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative275)

End of $1

exists, and has a value that is independent of the sequence left parenthesis z sub n right parenthesis.

So, if two such sequences left parenthesis z sub n right parenthesis and left parenthesis z sub n super prime right parenthesis can be found for which

Start of $1

lim over n right arrow normal infinity of f of z sub n minus f of alpha divided by z sub n minus alpha not equals lim over n right arrow normal infinity of f of z sub n super prime minus f of alpha divided by z sub n super prime minus alpha comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative279)

End of $1

then f cannot be differentiable at alpha.

In the next example, you will see that f of z equals absolute value of z is not differentiable at zero. This result should not surprise you because the real modulus function is not differentiable at zero. Indeed, the proof is identical to that of the real case.

Start of Example

**Example 5**

Prove that f of z equals absolute value of z is not differentiable at 0.

**Solution**

We need to find two sequences left parenthesis z sub n right parenthesis and left parenthesis z sub n super prime right parenthesis that converge to 0 which, when substituted into the difference quotient, yield sequences with different limits. A simple choice is to pick sequences left parenthesis z sub n right parenthesis and left parenthesis z sub n super prime right parenthesis that approach 0 along the real axis: one from the right, and one from the left, as shown in Figure 4.

Start of Figure

Displayed image

Figure 4 Sequences converging to zero from the right and left

[View description - Figure 4 Sequences converging to zero from the right and left](" \l "Unit1_Session2_Description2)

End of Figure

There is no point in picking sequences that are more complicated than they need to be, so let z sub n equals one solidus n, n equals one comma two comma ellipsis. Then

Start of $1

equation sequence part 1 lim over n right arrow normal infinity of absolute value of z sub n minus absolute value of zero divided by z sub n minus zero equals part 2 lim over n right arrow normal infinity of one solidus n divided by one solidus n equals part 3 one full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative293)

End of $1

Now let z sub n super prime equals negative one solidus n, n equals one comma two comma ellipsis. Then

Start of $1

equation sequence part 1 lim over n right arrow normal infinity of absolute value of z sub n super prime minus absolute value of zero divided by z sub n super prime minus zero equals part 2 lim over n right arrow normal infinity of one solidus n divided by negative one solidus n equals part 3 negative one full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative296)

End of $1

Since the two limits do not agree, the difference quotient does not have a limit as z tends to 0. It follows that f of z equals absolute value of z is not differentiable at 0.

End of Example

The next exercise asks you to extend the method used in Example 5 to show that f of z equals absolute value of z is not differentiable at any point of double-struck cap c.

Start of Exercise

**Exercise 7**

Start of Question

Let alpha be any non-zero complex number, and consider the circle through alpha centred at the origin. By choosing one sequence left parenthesis z sub n right parenthesis that approaches alpha along the circumference of the circle, and another sequence left parenthesis z sub n super prime right parenthesis that approaches alpha along the ray from zero through alpha, prove that f of z equals absolute value of z is not differentiable at alpha.

Start of Figure

Displayed image

[View description - Uncaptioned Figure](" \l "Unit1_Session2_Description3)

End of Figure

End of Question

[View answer - Exercise 7](" \l "Unit1_Session2_Answer7)

End of Exercise

The modulus function illustrates an important difference between real and complex differentiation. When the modulus function is treated as a real function, the limit of its difference quotient has to be taken along the real line. But when treated as a complex function, the limit of the difference quotient is required to exist however the limit is taken. This explains why the real modulus function is differentiable at all non-zero real points, whereas the complex modulus function fails to be differentiable at any point of double-struck cap c. More generally, it shows that complex differentiability is a much stronger condition than real differentiability.

In Exercise 7 you were asked to prove that the modulus function fails to be differentiable by observing that its behaviour along the circumference of a circle centred at 0 is different from its behaviour along a ray. Similar observations can be applied to other functions. For example, in the next exercise you may find it helpful to notice that directions of paths parallel to the imaginary axis are reversed by the function f of z equals z macron, whereas directions of paths parallel to the real axis are left unchanged (Figure 5).

Start of Figure

Displayed image

Figure 5 Images of horizontal and vertical lines under f of z equals z macron

[View description - Figure 5 Images of horizontal and vertical lines under f of z equals z macron](" \l "Unit1_Session2_Description4)

End of Figure

Start of Exercise

**Exercise 8**

Start of Question

Show that there are no points of double-struck cap c at which the complex conjugate function f of z equals z macron is differentiable.

End of Question

[View answer - Exercise 8](" \l "Unit1_Session2_Answer8)

End of Exercise

For some functions f, you may be able to find a sequence left parenthesis z sub n right parenthesis that converges to alpha for which the sequence

Start of $1

(equation 1)

w sub n equals f of z sub n minus f of alpha divided by z sub n minus alpha comma n equals one comma two comma ellipsis comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative348)

End of $1

is divergent. In such cases, there is no need to look for a second sequence.

Start of Example

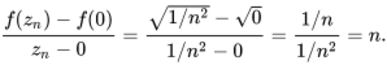
**Example 6**

Show that the function f of z equals Square root of z is not differentiable at 0.

**Solution**

Strategy A cannot be used here, since f is continuous at 0. Instead we look for a sequence left parenthesis z sub n right parenthesis that converges to zero for which the [sequence 1](#a4-got) (above) is divergent. To make the square roots easy to handle, let z sub n equals one solidus n squared, n equals one comma two comma ellipsis. Then

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative355)

End of $1

This sequence tends to infinity, and is therefore divergent. It follows that f is not differentiable at 0.

End of Example

The methods exemplified above for showing that a function is not differentiable at a given point can be summarised as follows.

Start of Box

**Strategy B for non-differentiability**

To prove that a function f is not differentiable at alpha, apply the strategy for proving that a limit does not exist to the difference quotient

Start of $1

f of z minus f of alpha divided by z minus alpha full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative359)

End of $1

End of Box

If you think that a given function is not differentiable, then you should try to apply Strategy A or Strategy B. A third strategy for proving that f is not differentiable at a point will appear in Section 2.1. If, on the other hand, you think that the function is differentiable, then you should try to find the derivative.

Start of Exercise

**Exercise 9**

Start of Question

Decide whether each of the following functions is differentiable at i. If it is, then find its derivative at i.

1. f of z equals Re of z
2. f of z equals sum with 3 summands two times z squared plus three times z plus five
3. f of z equals case statement case 1column 1 comma z comma less than less than of ReRez zero case 2column 1 comma four comma greater than or equals greater than or equals of ReRez zero

End of Question

[View answer - Exercise 9](" \l "Unit1_Session2_Answer9)

End of Exercise

## 1.4 Higher-order derivatives

In [Exercise 2](#a4-prob1-2) you saw that the function f of z equals one solidus z has derivative f super prime of z equals negative one solidus z squared, a result that you can also obtain using the Reciprocal Rule. If you now apply the Reciprocal Rule to the derivative f super prime of z equals negative one solidus z squared, then you obtain a function

Start of $1

left parenthesis f super prime right parenthesis super prime times left parenthesis z right parenthesis equals two divided by z cubed times left parenthesis z not equals zero right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative388)

End of $1

In general, for a differentiable function f, the function left parenthesis f super prime right parenthesis super prime is called the **second derivative of** bold-italic f, and is denoted by f super prime prime. Continued differentiation gives the so-called **higher-order derivatives of** bold-italic f. These are denoted by f super prime prime comma f super prime prime prime comma f super prime prime prime prime comma ellipsis, and the values f super prime prime of alpha comma f super prime prime prime of alpha comma f super prime prime prime prime of alpha comma ellipsis, are called the **higher-order derivatives of** bold-italic f**at** bold-italic alpha.

Since the dashes in this notation can be rather cumbersome, we often indicate the order of the derivative by a number in brackets. Thus f super left parenthesis two right parenthesis comma f super left parenthesis three right parenthesis comma f super left parenthesis four right parenthesis comma ellipsis mean the same as f super prime prime comma f super prime prime prime comma f super prime prime prime prime comma ellipsis, respectively. Here the brackets in f super left parenthesis four right parenthesis are needed to avoid confusion with the fourth power of f.

When we wish to discuss a derivative of general order, we will refer to the bold-italic n**th derivative** bold-italic f super left parenthesis bold-italic n right parenthesis**of** bold-italic f. It is often possible to find a formula for the nth derivative in terms of n. For example, if f of z equals one solidus z, then

Start of $1

f super prime prime of z equals two divided by z cubed comma f super prime prime prime of z equals negative two multiplication three divided by z super four comma f super left parenthesis four right parenthesis of z equals two multiplication three multiplication four divided by z super five comma ellipsis comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative408)

End of $1

so the nth derivative is given by

Start of $1

f super left parenthesis n right parenthesis of z equals left parenthesis negative one right parenthesis super n times n factorial divided by z super n plus one full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative410)

End of $1

(This can be proved formally by the Principle of Mathematical Induction.)

One interesting feature about this formula is that the domain script cap r equals double-struck cap c minus zero remains the same, no matter how often the function f is differentiated. This is a special case of a much more general result which states that: a function that is analytic on a region script cap rhas derivatives of all orders on script cap r. Here we confine our attention to first derivatives, and we continue to do this in the next subsection by giving a geometric interpretation of the first derivative.

## 1.5 A geometric interpretation of derivatives

As we mentioned in this session’s introduction, the derivative of a real function is often pictured geometrically as the gradient of the graph of the function. This interpretation is useful in real analysis, but it is of little use in complex analysis, since the graph of a complex function is not two-dimensional.

Fortunately, there is another way of interpreting derivatives that works for complex functions.

If a complex function f is differentiable at a point alpha, then any point z close to alpha is mapped by f to a point f of z close to f of alpha.

Indeed, by the Linear Approximation Theorem,

Start of $1

f of z equals sum with 3 summands f of alpha plus left parenthesis z minus alpha right parenthesis times f super prime of alpha plus e of z comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative422)

End of $1

where e of z solidus left parenthesis z minus alpha right parenthesis right arrow zero as z right arrow alpha. So if f super prime of alpha not equals zero, then, to a close approximation,

Start of $1

f of z minus f of alpha almost equals f super prime of alpha times left parenthesis z minus alpha right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative426)

End of $1

Multiplication of z minus alpha by f super prime of alpha has the effect of scaling z minus alpha by the factor absolute value of f super prime of alpha and rotating it about 0 through the angle Arg of f super prime of alpha; see Figure 6. We refer to f super prime of alpha as a complex scale factor, because it causes both a scaling and a rotation.

Start of Figure

Displayed image

Figure 6 Scaling and rotating z minus alpha

[View description - Figure 6 Scaling and rotating z minus alpha](" \l "Unit1_Session2_Description5)

End of Figure

We can rewrite the equation above as

Start of $1

(equation 2)

f of z almost equals f of alpha plus f super prime of alpha times left parenthesis z minus alpha right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative434)

End of $1

From this we see that f of z is obtained by scaling and rotating the vector z minus alpha based at f of alpha by the complex scale factor f super prime of alpha, as illustrated in Figure 7.

Start of Figure

Displayed image

Figure 7 Interpreting a derivative as a complex scale factor

[View description - Figure 7 Interpreting a derivative as a complex scale factor](" \l "Unit1_Session2_Description6)

End of Figure

Another useful way to picture how f behaves geometrically is to consider the effect it has on a small disc centred at alpha (still assuming that f super prime of alpha not equals zero). From equation 2 (above), we see that, to a close approximation, a small disc centred at alpha is mapped to a small disc centred at f of alpha. In the process, the disc is rotated through the angle Arg of f super prime of alpha, and it is scaled by the factor absolute value of f super prime of alpha (see Figure 8). As usual, the rotation is anticlockwise if Arg of f super prime of alpha is positive, and clockwise if it is negative.

Start of Figure

Displayed image

Figure 8 The approximate image of a disc centred at a point alpha, where f super prime of alpha not equals zero

[View description - Figure 8 The approximate image of a disc centred at a point alpha, where f super ...](" \l "Unit1_Session2_Description7)

End of Figure

The geometric interpretation of derivatives is more complicated if f super prime of alpha equals zero, and we do not discuss it here.

Start of Example

**Example 7**

Using the notion of a complex scale factor, describe what happens to points close to one plus i under the function f of z equals one solidus z.

**Solution**

To a close approximation, a small disc centred at one plus i is mapped by f to a small disc centred at

Start of $1

equation sequence part 1 f times left parenthesis one plus i right parenthesis equals part 2 one solidus left parenthesis one plus i right parenthesis equals part 3 one divided by two times left parenthesis one minus i right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative454)

End of $1

In the process, the disc is scaled by the factor absolute value of f super prime times left parenthesis one plus i right parenthesis and rotated through the angle Arg of f super prime times left parenthesis one plus i right parenthesis.

Now f super prime of z equals negative one solidus z squared, so

Start of $1

equation sequence part 1 f super prime times left parenthesis one plus i right parenthesis equals part 2 negative one divided by left parenthesis one plus i right parenthesis squared equals part 3 negative one divided by two times i equals part 4 i divided by two comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative458)

End of $1

which has modulus one solidus two and principal argument pi solidus two.

So f scales the disc by the factor one solidus two and rotates it anticlockwise through the angle pi solidus two.

End of Example

Start of Exercise

**Exercise 10**

Start of Question

Using the notion of a complex scale factor, describe what happens to points close to i under the function

Start of $1

f of z equals four times z plus three divided by two times z squared plus one full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative465)

End of $1

End of Question

[View answer - Exercise 10](" \l "Unit1_Session2_Answer10)

End of Exercise

It is important to bear in mind that the complex scale factor interpretation of a derivative is only an approximation, and that it is unlikely to be reliable far from the point under consideration.

## 1.6 Further exercises

Here are some further exercises to end this section.

Start of Exercise

**Exercise 11**

Start of Question

Use the definition of derivative to find the derivative of the function

Start of $1

f of z equals two times z squared plus five full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative478)

End of $1

End of Question

[View answer - Exercise 11](" \l "Unit1_Session2_Answer11)

End of Exercise

Start of Exercise

**Exercise 12**

Start of Question

Prove the Quotient Rule for differentiation.

End of Question

[View answer - Exercise 12](" \l "Unit1_Session2_Answer12)

End of Exercise

Start of Exercise

**Exercise 13**

Start of Question

Find the derivative of each of the following functions f. In each case specify the domain of f super prime.

1. f of z equals sum with 3 summands z squared plus two times z plus one divided by three times z plus one
2. f of z equals z cubed plus one divided by z squared minus z minus six
3. f of z equals one divided by sum with 3 summands z squared plus two times z plus two
4. f of z equals sum with 3 summands z squared plus five times z minus two plus one divided by z plus one divided by z squared

End of Question

[View answer - Exercise 13](" \l "Unit1_Session2_Answer13)

End of Exercise

Start of Exercise

**Exercise 14**

Start of Question

Use Strategy B to show that there are no points of double-struck cap c at which the function

Start of $1

f of z equals Im of z

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative514)

End of $1

is differentiable.

End of Question

[View answer - Exercise 14](" \l "Unit1_Session2_Answer14)

End of Exercise

Start of Exercise

**Exercise 15**

Start of Question

Describe the approximate geometric effect of the function

Start of $1

f of z equals z cubed plus eight divided by z minus six

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative527)

End of $1

on a small disc centred at the point 2.

End of Question

[View answer - Exercise 15](" \l "Unit1_Session2_Answer15)

End of Exercise

## 2 The Cauchy–Riemann equations

After working through this section, you should be able to:

* find the partial derivatives of a function from double-struck cap r squared to double-struck cap r
* use the Cauchy–Riemann equations to show that a function is not differentiable at a given point
* use the Cauchy–Riemann equations to show that a function, such as the exponential function, is differentiable at a given point, and to find the derivative.

This section is challenging, so you may find that you do not appreciate some of the details on a first reading. Most importantly, you should try to understand the definitions, strategies and theorems, and apply them in the examples and exercises.

## 2.1 The Cauchy–Riemann theorems

Here we will explore the relationship between complex differentiation and real differentiation. To do this, we introduce the notion of a partial derivative and use it to derive the Cauchy–Riemann equations (pronounced ‘coh-she ree-man’). These equations are conditions that any differentiable complex function must satisfy, so they can be used to test whether a given complex function is differentiable. In particular, we use them to investigate the differentiability of the complex exponential function. The technique is to split the exponential function

Start of $1

exp of x plus i times y equals e super x times left parenthesis cosine of y plus i times sine of y right parenthesis

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative3)

End of $1

into its real and imaginary parts:

Start of $1

u of x comma y equals e super x times cosine of y and v of x comma y equals e super x times sine of y comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative4)

End of $1

each of which is a real-valued function of the real variables x and y. The derivative of exp is then calculated by using the derivatives of the real trigonometric and exponential functions, which we assume to be known.

Before we deal with the exponential function, however, let us first consider the simpler function f of z equals z cubed. By writing z equals x plus i times y, we see that

Start of $1

equation sequence part 1 f times left parenthesis x plus i times y right parenthesis equals part 2 left parenthesis x plus i times y right parenthesis cubed equals part 3 left parenthesis x cubed minus three times x times y squared right parenthesis plus i times left parenthesis three times x squared times y minus y cubed right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative9)

End of $1

Let us define

Start of $1

u of x comma y equals x cubed minus three times x times y squared and v of x comma y equals three times x squared times y minus y cubed full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative10)

End of $1

Then u and v are the real and imaginary parts of f, respectively; that is, u equals Re of f and v equals Im of f. For the moment we will concentrate on the real part u; part of its graph (given by the equation s equals u of x comma y) is shown in Figure 9. Since u is a function of two real variables, its graph is a surface. The height of the surface above the open x comma y close-plane represents the value of the function at the point open x comma y close. For instance, the point cap p on the surface has coordinates left parenthesis two comma one comma two right parenthesis because equation sequence part 1 u of two comma one equals part 2 two cubed minus three multiplication two multiplication one squared equals part 3 two.

Start of Figure

Displayed image

Figure 9 Graph of u of x comma y equals x cubed minus three times x times y squared

[View description - Figure 9 Graph of u of x comma y equals x cubed minus three times x times y squa ...](" \l "Unit1_Session3_Description1)

End of Figure

Let us now explore the concept of the gradient of the surface at a point such as cap p. We will find that the answer depends on the ‘direction’ from which we approach the point. To make this more precise, consider Figure 10, in which the vertical plane with equation y equals one is shown intersecting the surface in a curve that passes through cap p. By substituting y equals one into u of x comma y equals x cubed minus three times x times y squared, we see that the curve has equation x long right arrow from bar x cubed minus three times x, so we can calculate its gradient at cap p; this is the gradient of the surface in the x-direction at cap p.

Start of Figure

Displayed image

Figure 10 Intersection of the graph of u of x comma y equals x cubed minus three times x times y squared with the vertical plane y equals one

[View description - Figure 10 Intersection of the graph of u of x comma y equals x cubed minus three ...](" \l "Unit1_Session3_Description2)

End of Figure

More generally, whenever we intersect the surface with a vertical plane with equation y equals constant, we obtain a curve on the surface with equation x long right arrow from bar x cubed minus three times x times y squared (where y is considered to be fixed). We can find the gradient at any point left parenthesis a comma b comma u of a comma b right parenthesis on this curve by differentiating with respect to x and then substituting x equals a and y equals b. The resulting expression is called the partial derivative of *u* with respect to *x* at *left parenthesis a comma b right parenthesis*, and it is denoted by

Start of $1

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative46)

End of $1

A curly prefix partial differential of is used rather than a straight d to emphasise that this is a partial derivative, for which we differentiate with respect to one variable and keep the other variable fixed. In our particular case, differentiating u of x comma y equals x cubed minus three times x times y squared with respect to x (and keeping y fixed) gives

Start of $1

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals three times x squared minus three times y squared comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative52)

End of $1

and substituting x equals two and y equals one gives

Start of $1

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis two comma one right parenthesis equals nine full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative55)

End of $1

Hence the gradient of the surface in the x-direction at the point cap p is 9. This is a positive value because near the point cap p, u increases as x increases (with y equals one), as you can see from Figure 10.

Figure 11 shows the vertical plane with equation x equals two intersecting the surface in a different curve that passes through cap p.

Start of Figure

Displayed image

Figure 11 Intersection of the graph of u of x comma y equals x cubed minus three times x times y squared with the vertical plane x equals two

[View description - Figure 11 Intersection of the graph of u of x comma y equals x cubed minus three ...](" \l "Unit1_Session3_Description3)

End of Figure

Reasoning similarly to before, we see that intersecting the surface with a vertical plane with equation x equals constant gives a curve on the surface, and we can obtain the gradient at a point left parenthesis a comma b comma u of a comma b right parenthesis on this curve by differentiating u of x comma y with respect to y while keeping x fixed (and then substituting x equals a and y equals b). The resulting expression is called the partial derivative of *u* with respect to *y* at *left parenthesis a comma b right parenthesis*, and it is denoted by

Start of $1

prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative76)

End of $1

Differentiating u of x comma y equals x cubed minus three times x times y squared with respect to y (and keeping x fixed) gives

Start of $1

prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals negative six times x times y comma so prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis two comma one right parenthesis equals negative 12 semicolon

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative80)

End of $1

this is the gradient of the surface in the y-direction at the point cap p. It is a negative value this time, because when x and y are positive, u decreases as y increases (keeping x fixed), as you can see from Figure 11.

You will need to work with partial derivatives a good deal here, so let us state the definitions formally.

Start of Box

**Definitions**

Let u colon cap a long right arrow double-struck cap r be a function with domain cap a a subset of double-struck cap r squared that contains the point left parenthesis a comma b right parenthesis.

* The **partial derivative of bold-italic u with respect to bold-italic x at bold left parenthesis bold-italic a bold comma bold-italic b bold right parenthesis**, denoted prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis, is the derivative of the function x long right arrow from bar u of x comma b at x equals a, provided that this derivative exists.
* The **partial derivative of bold-italic u with respect to bold-italic y at bold left parenthesis bold-italic a bold comma bold-italic b bold right parenthesis**, denoted prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis, is the derivative of the function y long right arrow from bar u of a comma y at y equals b, provided that this derivative exists.

End of Box

Partial derivatives are real derivatives, not complex derivatives.

The next exercise asks you to work out the partial derivatives of the imaginary part of the complex function f of z equals z cubed.

Start of Exercise

**Exercise 16**

Start of Question

1. Calculate the partial derivatives of v of x comma y equals three times x squared times y minus y cubed.
2. Evaluate these partial derivatives at open two comma one close.

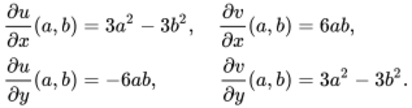
End of Question

[View answer - Exercise 16](" \l "Unit1_Session3_Answer1)

End of Exercise

Let us collect together the partial derivatives of the real and imaginary parts u and v of the function f of z equals z cubed:

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative120)

End of $1

As you can see, we have

Start of $1

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis and prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis equals negative prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative121)

End of $1

This pair of equations is called the Cauchy–Riemann equations, and they hold true for the real and imaginary parts of any differentiable complex function, as the following important theorem testifies.

Start of Box

**Theorem 4 Cauchy–Riemann Theorem**

Let f times left parenthesis x plus i times y right parenthesis equals u of x comma y plus i times v of x comma y be defined on a region script cap r containing a plus i times b.

If f is differentiable at a plus i times b, then prefix partial differential of of u divided by prefix partial differential of of x, prefix partial differential of of u divided by prefix partial differential of of y, prefix partial differential of of v divided by prefix partial differential of of x, prefix partial differential of of v divided by prefix partial differential of of y exist at left parenthesis a comma b right parenthesis and satisfy the **Cauchy–Riemann equations**

Start of $1

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis and prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis equals negative prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative132)

End of $1

End of Box

Start of Proof

Proof Let alpha equals a plus i times b. Suppose that left parenthesis z sub n right parenthesis is any sequence in script cap r minus alpha that converges to alpha. Let us write z sub n equals x sub n plus i times y sub n. According to the definition of a derivative, we have

equation sequence part 1 f super prime of alpha equals part 2 lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha equals part 3 lim over n right arrow normal infinity of f of z sub n minus f of alpha divided by z sub n minus alpha full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative138)

Observe that, by expressing f in terms of its real and imaginary parts, we can write f of z sub n minus f of alpha divided by z sub n minus alpha equals left parenthesis u of x sub n comma y sub n minus u of a comma b divided by left parenthesis x sub n minus a right parenthesis plus i times left parenthesis y sub n minus b right parenthesis right parenthesis plus i of v of x sub n comma y sub n minus v of a comma b divided by left parenthesis x sub n minus a right parenthesis plus i times left parenthesis y sub n minus b right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative140)

(equation 3)We proceed by choosing two different types of sequences left parenthesis z sub n right parenthesis, and observing the behaviour of the expressions in large brackets in equation 3 (above) in each case. For our first choice, let us begin by defining left parenthesis x sub n right parenthesis to be any sequence in double-struck cap r minus a that converges to a. Let z sub n equals x sub n plus i times b, so the sequence left parenthesis z sub n right parenthesis converges to alpha equals a plus i times b. By removing a finite number of terms from left parenthesis x sub n right parenthesis, if need be, we can assume that each point z sub n belongs to the open set script cap r minus alpha. Substituting z sub n equals x sub n plus i times b into equation 3 gives f of z sub n minus f of alpha divided by z sub n minus alpha equals left parenthesis u of x sub n comma b minus u of a comma b divided by x sub n minus a right parenthesis plus i of v of x sub n comma b minus v of a comma b divided by x sub n minus a full stop

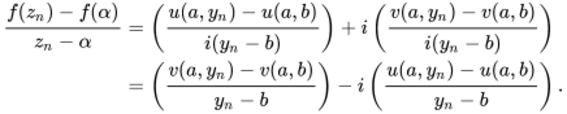
[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative152)

We know that the expression on the left-hand side converges (to f super prime of alpha), so its real and imaginary parts (indicated by the bracketed expressions on the right-hand side) converge too. Since left parenthesis x sub n right parenthesis was chosen to be any sequence in double-struck cap r minus a that converges to a, we see from the definition of partial derivatives that prefix partial differential of of u divided by prefix partial differential of of x and prefix partial differential of of v divided by prefix partial differential of of x exist at left parenthesis a comma b right parenthesis and u of x sub n comma b minus u of a comma b divided by x sub n minus a right arrow prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis and v of x sub n comma b minus v of a comma b divided by x sub n minus a right arrow prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative160)

In summary, we have f super prime of alpha equals prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative161)

(equation 4)Next let left parenthesis y sub n right parenthesis be any sequence in double-struck cap r minus b that converges to b, and define z sub n equals a plus i times y sub n, so z sub n right arrow alpha. Again, by omitting a finite number of terms from left parenthesis y sub n right parenthesis, if need be, we can assume that z sub n element of script cap r minus alpha for all n. Substituting z sub n equals a plus i times y sub n into equation 3 gives 

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative171)

Reasoning as before, we see that prefix partial differential of of u divided by prefix partial differential of of y and prefix partial differential of of v divided by prefix partial differential of of y exist at left parenthesis a comma b right parenthesis and f super prime of alpha equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis minus i times prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative175)

(equation 5)Comparing equation 4 and equation 5 (both above), and equating real and imaginary parts, we obtain the Cauchy–Riemann equations, as required.âˆŽ

End of Proof

Start of Box

**Origin of the Cauchy–Riemann equations**

The Cauchy–Riemann equations are named after the mathematicians Augustin-Louis Cauchy and Bernhard Riemann (1826–1866), who were among the first to recognise the importance of these equations in complex analysis.

The Cauchy–Riemann equations first appeared in the work of another mathematician, however: the Frenchman Jean le Rond d’Alembert (1717–1783), who is perhaps best remembered for his work in classical mechanics. Indeed, the Cauchy–Riemann equations were written down by d’Alembert in an essay on fluid dynamics in 1752 to describe the velocity components of a two-dimensional irrotational fluid flow.

Start of Figure



Jean le Rond d’Alembert (1717–1783)

[View description - Jean le Rond d’Alembert (1717–1783)](" \l "Unit1_Session3_Description4)

End of Figure

End of Box

The Cauchy–Riemann Theorem gives us another strategy for proving the non-differentiability of a complex function. (Two other strategies were described earlier in Section 1.3.) If a complex function is differentiable, then it must satisfy the Cauchy–Riemann equations. So if those equations do not hold, then the function cannot be differentiable.

Start of Box

**Strategy C for non-differentiability**

Let f times left parenthesis x plus i times y right parenthesis equals u of x comma y plus i times v of x comma y. If either

Start of $1

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis not equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis or prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis not equals negative prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative177)

End of $1

then f is not differentiable at a plus i times b.

End of Box

To illustrate this strategy, consider the function

Start of $1

f times left parenthesis x plus i times y right parenthesis equals left parenthesis x squared plus y squared right parenthesis plus i times left parenthesis two times x plus four times y right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative180)

End of $1

The real part u and imaginary part v of this function are given by

Start of $1

u of x comma y equals x squared plus y squared and v of x comma y equals two times x plus four times y full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative183)

End of $1

Hence

Start of Figure

Displayed image

[View description - Uncaptioned Figure](" \l "Unit1_Session3_Description5)

End of Figure

As you can see, the partial derivatives have been grouped into two pairs according to the Cauchy–Riemann equations.

Start of $1

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis and prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals negative prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative184)

End of $1

In this case, these equations are two times x equals four and two equals negative two times y, which are satisfied only when x equals two and y equals negative one; that is, they are satisfied only when z equals two minus i. If z not equals two minus i, then the Cauchy–Riemann equations fail, so Strategy C tells us that f is not differentiable at z.

Notice that the Cauchy–Riemann Theorem and Strategy C do not tell us whether f is differentiable at the point two minus i at which the Cauchy–Riemann equations are satisfied. To deal with points of this type we need another theorem, which we will come to shortly. First, however, try the following exercise, to practise applying Strategy C.

Start of Exercise

**Exercise 17**

Start of Question

Show that each of the following functions fails to be differentiable at all points of double-struck cap c.

1. f times left parenthesis x plus i times y right parenthesis equals e super x minus i times e super y
2. f of z equals z macron

End of Question

[View answer - Exercise 17](" \l "Unit1_Session3_Answer2)

End of Exercise

We have seen that if the Cauchy–Riemann equations are not satisfied, then the function is not differentiable. Let us now describe an example to show that even if the Cauchy–Riemann equations are satisfied, then the function may still not be differentiable.

Consider the function f times left parenthesis x plus i times y right parenthesis equals u of x comma y plus i times v of x comma y, where v of x comma y equals zero for all x and y, and

Start of $1

u of x comma y equals case statement case 1column 1 comma of minmin left curly bracket right curly bracket comma x comma y comma comma comma x comma greater than greater than y zero comma case 2column 1 comma zero comma full stop otherwise full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative218)

End of $1

The graph of u is shown in Figure 12.

Start of Figure

Displayed image

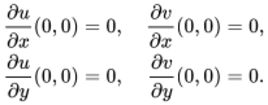
Figure 12 Graph of u

[View description - Figure 12 Graph of u](" \l "Unit1_Session3_Description6)

End of Figure

Since u and v take the value zero at all points on the x- and y-axes, we see that all the partial derivatives vanish at open zero comma zero close; that is,

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative227)

End of $1

However, even though the Cauchy–Riemann equations are satisfied at the origin, f is not differentiable there. To see this, observe that if z sub n equals one solidus n, n equals one comma two comma ellipsis, then

Start of $1

multirelation f of z sub n minus f of zero divided by z sub n minus zero equals u of one solidus n comma zero minus zero divided by one solidus n minus zero equals zero right arrow zero comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative231)

End of $1

whereas if z sub n equals one solidus n plus i solidus n, n equals one comma two comma ellipsis, then

Start of $1

multirelation f of z sub n minus f of zero divided by z sub n minus zero equals u of one solidus n comma one solidus n minus zero divided by one solidus n plus i solidus n minus zero equals one solidus n divided by one solidus n plus i solidus n equals one divided by one plus i right arrow one divided by one plus i full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative234)

End of $1

The two limits zero and one solidus left parenthesis one plus i right parenthesis differ, so f is not differentiable at zero.

This example demonstrates that the differentiability of a complex function does not follow from the Cauchy–Riemann equations alone. However, if certain extra conditions are satisfied, then f is differentiable, as the following theorem reveals.

Start of Box

**Theorem 5 Cauchy–Riemann Converse Theorem**

Let f times left parenthesis x plus i times y right parenthesis equals u of x comma y plus i times v of x comma y be defined on a region script cap r containing a plus i times b. If the partial derivatives prefix partial differential of of u divided by prefix partial differential of of x, prefix partial differential of of u divided by prefix partial differential of of y, prefix partial differential of of v divided by prefix partial differential of of x, prefix partial differential of of v divided by prefix partial differential of of y

* exist at open x comma y close for each x plus i times y element of script cap r
* are continuous at left parenthesis a comma b right parenthesis
* satisfy the Cauchy–Riemann equations at left parenthesis a comma b right parenthesis,

then f is differentiable at a plus i times b and

Start of $1

f super prime times left parenthesis a plus i times b right parenthesis equals prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative253)

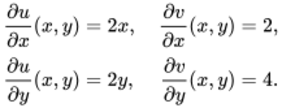
End of $1

End of Box

The proof of this theorem is postponed until the next subsection.

Let us now return to the function f times left parenthesis x plus i times y right parenthesis equals left parenthesis x squared plus y squared right parenthesis plus i times left parenthesis two times x plus four times y right parenthesis, considered earlier, which satisfies the Cauchy–Riemann equations at the point z equals two minus i only, and is therefore not differentiable at any other point. You saw earlier that the partial derivatives exist for every point open x comma y close (so we can choose script cap r equals double-struck cap c in applying Theorem 5) and they satisfy

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative258)

End of $1

Each of these functions is continuous at left parenthesis two comma negative one right parenthesis because each of them is either constant or a multiple of one of the basic continuous functions Re of z or Im of z. For example, the function left parenthesis x comma y right parenthesis long right arrow from bar two times x can be thought of as z long right arrow from bar two times Re of z.

It follows, then, from the Cauchy–Riemann Converse Theorem that f is differentiable at two minus i. In fact, the theorem even tells us the value of f super prime times left parenthesis two minus i right parenthesis, namely

Start of $1

equation sequence part 1 f super prime times left parenthesis two minus i right parenthesis equals part 2 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis two comma negative one right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis two comma negative one right parenthesis equals part 3 two multiplication two plus i multiplication two equals part 4 four plus two times i full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative267)

End of $1

Now, we investigate the differentiability of the complex exponential function, as promised earlier.

Start of Example

**Example 8**

Prove that the complex exponential function f of z equals e super z is entire, and find its derivative.

**Solution**

The real part u and the imaginary part v of f are given by

Start of $1

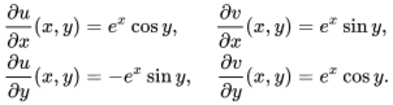
u of x comma y equals e super x times cosine of y and v of x comma y equals e super x times sine of y full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative272)

End of $1

Hence the partial derivatives of u and v exist for every point open x comma y close and satisfy

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative276)

End of $1

Since the real exponential and trigonometric functions are continuous, and the real and imaginary part functions Re of z and Im of z are basic continuous functions, we see from the Combination Rules and Composition Rule for continuous functions that each partial derivative is continuous at every point open x comma y close.

The Cauchy–Riemann equations are satisfied at all points open x comma y close, so the Cauchy–Riemann Converse Theorem tells us that f is differentiable at every point of the complex plane (it is entire) and

Start of $1

equation sequence part 1 f super prime of z equals part 2 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals part 3 e super x times cosine of y plus i times e super x times sine of y equals part 4 e super z full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative282)

End of $1

End of Example

Start of Exercise

**Exercise 18**

Start of Question

Use the Cauchy–Riemann theorems to find the derivatives of the following functions. In each case specify the domain of the derivative.

1. f of z equals sine of z
2. f of z equals absolute value of z squared

(Hint: For part (a), write sine of z equals sine of x plus i times y and use a trigonometric addition identity to find the real and imaginary parts of sine of z.)

End of Question

[View answer - Exercise 18](" \l "Unit1_Session3_Answer3)

End of Exercise

## 2.2 Proof of the Cauchy–Riemann Converse Theorem

The proof of the Cauchy–Riemann Converse Theorem is rather involved and may require more than one reading.

We will need two results from real analysis. The first result is known as the Mean Value Theorem.

Start of Box

**Theorem 6 Mean Value Theorem**

Let f be a real function that is continuous on the closed interval left square bracket a comma x right square bracket and differentiable on the open interval left parenthesis a comma x right parenthesis. Then there is a number c element of left parenthesis a comma x right parenthesis such that

Start of $1

(equation 6)

f of x equals f of a plus left parenthesis x minus a right parenthesis times f super prime of c full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative319)

End of $1

End of Box

To appreciate why this theorem is true, imagine pushing the chord between left parenthesis a comma f of a right parenthesis and left parenthesis x comma f of x right parenthesis in Figure 13 parallel to itself until it becomes a tangent to the graph of f at a point left parenthesis c comma f of c right parenthesis, where c lies somewhere between a and x. Clearly, the gradient of the original chord must be equal to the gradient of the tangent, so

Start of $1

f of x minus f of a divided by x minus a equals f super prime of c full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative327)

End of $1

Multiplication by x minus a gives f of x equals f of a plus left parenthesis x minus a right parenthesis times f super prime of c. Notice that this equation is also true if equation sequence part 1 x equals part 2 c equals part 3 a.

Start of Figure

Displayed image

Figure 13 Graph of the real function f

[View description - Figure 13 Graph of the real function f](" \l "Unit1_Session3_Description7)

End of Figure

The second result that we will need is a Linear Approximation Theorem, which asserts that if u is a real-valued function of two real variables x and y, then for open x comma y close near left parenthesis a comma b right parenthesis, the value of u of x comma y can be approximated by the value of the linear function t defined by

Start of $1

t of x comma y equals sum with 3 summands u of a comma b plus left parenthesis x minus a right parenthesis times prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus left parenthesis y minus b right parenthesis times prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative339)

End of $1

Now, the graph of t is a plane passing through the point cap p equals left parenthesis a comma b comma u of a comma b right parenthesis on the graph of u (Figure 14). Moreover, the partial x- and y-derivatives of t coincide with the partial x- and y-derivatives of u at left parenthesis a comma b right parenthesis. This means that both have the same gradient in the x- and y-directions, so you can think of the plane as the tangent plane to the graph of u at cap p.

Start of Figure

Displayed image

Figure 14 Tangent plane to the graph of u at the point cap p

[View description - Figure 14 Tangent plane to the graph of u at the point cap p](" \l "Unit1_Session3_Description8)

End of Figure

The accuracy with which this tangent plane approximates the graph of u depends on the smoothness of the graph of u. If the graph exhibits the kind of kink shown in Figure 12, then the approximation is not as good as for a function with continuous partial derivatives.

Start of Box

**Theorem 7 Linear Approximation Theorem (double-struck cap r squared to double-struck cap r)**

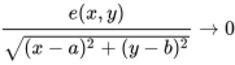
Let u be a real-valued function of two real variables, defined on a region script cap r in double-struck cap r squared containing left parenthesis a comma b right parenthesis. If the partial x- and y-derivatives of u exist on script cap r and are continuous at left parenthesis a comma b right parenthesis, then there is an ‘error function’ e such that

Start of $1

u of x comma y equals sum with 4 summands u of a comma b plus left parenthesis x minus a right parenthesis times prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus left parenthesis y minus b right parenthesis times prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis plus e of x comma y comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative370)

End of $1

where  as left parenthesis x comma y right parenthesis right arrow left parenthesis a comma b right parenthesis.

End of Box

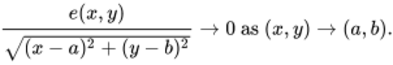
Since Square root of left parenthesis x minus a right parenthesis squared plus left parenthesis y minus b right parenthesis squared is the distance from left parenthesis a comma b right parenthesis to open x comma y close, the theorem asserts that the error function tends to zero ‘faster’ than this distance. Theorem 7 is the real-valued function analogue of [Theorem 2](#a4-th1-2).

Start of Proof

ProofWe have to show that the function e defined by

e of x comma y equals u of x comma y minus u of a comma b minus left parenthesis x minus a right parenthesis times prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis minus left parenthesis y minus b right parenthesis times prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative377)

satisfies 

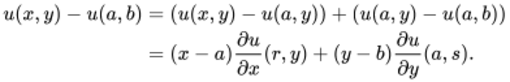
[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative378)

Since the partial derivatives exist on script cap r, they must be defined on some disc centred at left parenthesis a comma b right parenthesis. Let us begin by finding an expression for u of x comma y minus u of a comma b on this disc. If we apply the Mean Value Theorem to the real functions x long right arrow from bar u of x comma y (where y is kept constant) and y long right arrow from bar u of a comma y, then we obtain u of x comma y equals u of a comma y plus left parenthesis x minus a right parenthesis times prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis r comma y right parenthesis comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative385)

where r is between a and x, and u of a comma y equals u of a comma b plus left parenthesis y minus b right parenthesis times prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma s right parenthesis comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative389)

where s is between b and y (see Figure 15). Hence 

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative393)

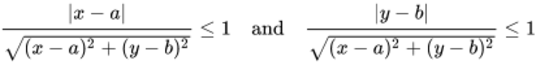
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Figure 15 Vertical and horizontal line segments from left parenthesis a comma b right parenthesis to open x comma y close

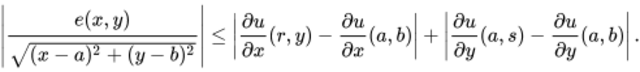
[View description - Figure 15 Vertical and horizontal line segments from left parenthesis a comma b right ...](" \l "Unit1_Session3_Description9)

Substituting this expression for u of x comma y minus u of a comma b into the definition of e, we obtain e of x comma y equals left parenthesis x minus a right parenthesis times left parenthesis prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis r comma y right parenthesis minus prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis right parenthesis plus left parenthesis y minus b right parenthesis times left parenthesis prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma s right parenthesis minus prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative398)

Dividing both sides by Square root of left parenthesis x minus a right parenthesis squared plus left parenthesis y minus b right parenthesis squared, and noting that 

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative400)

(because both left parenthesis x minus a right parenthesis squared and left parenthesis y minus b right parenthesis squared do not exceed left parenthesis x minus a right parenthesis squared plus left parenthesis y minus b right parenthesis squared), we see that 

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative404)

Figure 15 illustrates that as open x comma y close tends to left parenthesis a comma b right parenthesis, so do left parenthesis a comma s right parenthesis and left parenthesis r comma y right parenthesis. So, by the continuity of the partial x- and y-derivatives at left parenthesis a comma b right parenthesis, the two terms on the right of the inequality above must both tend to 0 as open x comma y close tends to left parenthesis a comma b right parenthesis. It follows that e of x comma y solidus Square root of left parenthesis x minus a right parenthesis squared plus left parenthesis y minus b right parenthesis squared tends to 0 as open x comma y close tends to left parenthesis a comma b right parenthesis. âˆŽ

End of Proof

We are now in a position to prove the Cauchy–Riemann Converse Theorem.

Start of Box

**Theorem 5 Cauchy–Riemann Converse Theorem (revisited)**

Let f times left parenthesis x plus i times y right parenthesis equals u of x comma y plus i times v of x comma y be defined on a region script cap r containing a plus i times b. If the partial derivatives prefix partial differential of of u divided by prefix partial differential of of x, prefix partial differential of of u divided by prefix partial differential of of y, prefix partial differential of of v divided by prefix partial differential of of x, prefix partial differential of of v divided by prefix partial differential of of y

* exist at open x comma y close for each x plus i times y element of script cap r
* are continuous at left parenthesis a comma b right parenthesis
* satisfy the Cauchy–Riemann equations at left parenthesis a comma b right parenthesis,

then f is differentiable at a plus i times b and

Start of $1

f super prime times left parenthesis a plus i times b right parenthesis equals prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis full stop

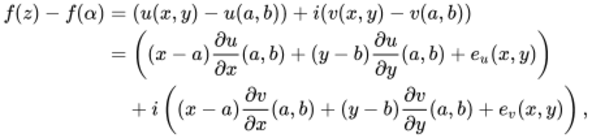
[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative430)

End of $1

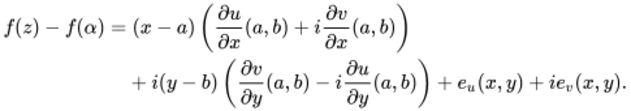
End of Box

Start of Proof

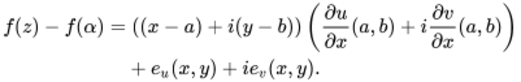
ProofWe need to show that the limit of the difference quotient for f at alpha equals a plus i times b exists and has the value indicated in the theorem. In order to calculate the difference quotient for f at alpha, we find an expression for f of z minus f of alpha. Since u and v fulfil the conditions of Theorem 7, it follows that



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative438)

where e sub u and e sub v are the error functions associated with u and v, respectively. Collecting together terms, we see that 

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative443)

Since u and v satisfy the Cauchy–Riemann equations, both expressions in the large brackets must be equal, so 

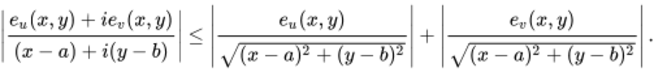
[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative446)

Dividing by z minus alpha equals left parenthesis x minus a right parenthesis plus i times left parenthesis y minus b right parenthesis gives f of z minus f of alpha divided by z minus alpha equals left parenthesis prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis right parenthesis plus left parenthesis e sub u of x comma y plus i times e sub v of x comma y divided by left parenthesis x minus a right parenthesis plus i times left parenthesis y minus b right parenthesis right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative448)

The limit f super prime of alpha of this difference quotient exists, and has the required value prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative450)

provided that we can show that the expression involving the error functions e sub u and e sub v tends to 0 as z equals x plus i times y tends to alpha. To this end, notice that absolute value of left parenthesis x minus a right parenthesis plus i times left parenthesis y minus b right parenthesis is equal to Square root of left parenthesis x minus a right parenthesis squared plus left parenthesis y minus b right parenthesis squared and so, by the Triangle Inequality, 

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative457)

By Theorem 7, both expressions on the right tend to 0 as x plus i times y tends to alpha. Consequently, the expression on the left must also tend to 0, and the theorem follows. âˆŽ

End of Proof

## 2.3 Further exercises

Here are some further exercises to end this section.

Start of Exercise

**Exercise 19**

Start of Question

Calculate the partial derivatives prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis and prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis of each of the following functions.

1. u of x comma y equals sum with 3 summands three times x plus x times y plus two times x squared times y squared
2. u of x comma y equals x times cosine of y plus exp of x times y
3. u of x comma y equals left parenthesis x plus y right parenthesis cubed

End of Question

[View answer - Exercise 19](" \l "Unit1_Session3_Answer4)

End of Exercise

Start of Exercise

**Exercise 20**

Start of Question

Calculate the partial derivatives prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis and prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis of each of the following functions, and evaluate these partial derivatives at left parenthesis one comma zero right parenthesis.

1. u of x comma y equals x cubed times y minus y times cosine of y
2. u of x comma y equals y times e super x minus x times y cubed

End of Question

[View answer - Exercise 20](" \l "Unit1_Session3_Answer5)

End of Exercise

Start of Exercise

**Exercise 21**

Start of Question

Find the gradient of the graph of u of x comma y equals x squared plus two times x times y at the point left parenthesis one comma two comma five right parenthesis in the x-direction and in the y-direction.

End of Question

[View answer - Exercise 21](" \l "Unit1_Session3_Answer6)

End of Exercise

Start of Exercise

**Exercise 22**

Start of Question

Use the Cauchy–Riemann equations to show that there is no point of double-struck cap c at which the function

Start of $1

f times left parenthesis x plus i times y right parenthesis equals e super x times left parenthesis sine of y plus i times cosine of y right parenthesis

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative502)

End of $1

is differentiable.

End of Question

[View answer - Exercise 22](" \l "Unit1_Session3_Answer7)

End of Exercise

Start of Exercise

**Exercise 23**

Start of Question

Use the Cauchy–Riemann equations to show that the function

Start of $1

f times left parenthesis x plus i times y right parenthesis equals left parenthesis x squared plus x minus y squared right parenthesis plus i times left parenthesis two times x times y plus y right parenthesis

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative515)

End of $1

is entire, and find its derivative.

End of Question

[View answer - Exercise 23](" \l "Unit1_Session3_Answer8)

End of Exercise

Start of Exercise

**Exercise 24**

Start of Question

Use the Cauchy–Riemann equations to find all the points at which the following functions are differentiable, and calculate their derivatives.

1. f times left parenthesis x plus i times y right parenthesis equals left parenthesis x squared plus y squared right parenthesis plus i times left parenthesis x squared minus y squared right parenthesis
2. f times left parenthesis x plus i times y right parenthesis equals x times y

End of Question

[View answer - Exercise 24](" \l "Unit1_Session3_Answer9)

End of Exercise

## 2.4 Laplace’s equation and electrostatics

The Cauchy–Riemann equations for a differentiable function f times left parenthesis x plus i times y right parenthesis equals u of x comma y plus i times v of x comma y tell us that

Start of $1

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis and prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals negative prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative547)

End of $1

These partial derivatives are themselves functions of x and y, so, provided that they are suitably well behaved, we can partially differentiate both sides of the first of the two equations with respect to x, and partially differentiate both sides of the second equation with respect to y, to obtain

Start of $1

prefix partial differential of squared of u divided by prefix partial differential of of x squared equals prefix partial differential of squared of v divided by prefix partial differential of of x times prefix partial differential of of y and prefix partial differential of squared of v divided by prefix partial differential of of y times prefix partial differential of of x equals negative prefix partial differential of squared of u divided by prefix partial differential of of y squared full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative552)

End of $1

(Here we have omitted the variables open x comma y close after each derivative, for simplicity.) For sufficiently well-behaved functions, the two partial derivatives

Start of $1

prefix partial differential of squared of v divided by prefix partial differential of of x times prefix partial differential of of y and prefix partial differential of squared of v divided by prefix partial differential of of y times prefix partial differential of of x

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative554)

End of $1

are equal; the order in which you partially differentiate with respect to x and y does not matter. Hence

Start of $1

equation sequence part 1 prefix partial differential of squared of u divided by prefix partial differential of of x squared equals part 2 prefix partial differential of squared of v divided by prefix partial differential of of x times prefix partial differential of of y equals part 3 prefix partial differential of squared of v divided by prefix partial differential of of y times prefix partial differential of of x equals part 4 negative prefix partial differential of squared of u divided by prefix partial differential of of y squared comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative557)

End of $1

which implies that

Start of $1

prefix partial differential of squared of u divided by prefix partial differential of of x squared plus prefix partial differential of squared of u divided by prefix partial differential of of y squared equals zero full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative558)

End of $1

This equation for u is called **Laplace’s equation**. (The imaginary part v of f satisfies Laplace’s equation too.) It is named after the distinguished French mathematician and scientist Pierre-Simon Laplace (1749–1827), who studied the equation in his work on gravitational potentials.

Start of Figure



Pierre-Simon Laplace (1749–1827)

[View description - Pierre-Simon Laplace (1749–1827)](" \l "Unit1_Session3_Description10)

End of Figure

Laplace’s equation has proved to have huge importance to physics, with particular significance in fluid mechanics. It also has a key role in the subject of electrostatics. In that theory, it is known that the electrostatic potential cap v of x comma y at a point open x comma y close of a region without charge satisfies Laplace’s equation. It can be shown that cap v is the real part of some differentiable function f. Using these observations allows one to move between complex analysis and electrostatics: many of the theorems of complex analysis have important physical interpretations in electrostatics.

## 3 Summary of Session 1

In this session you have seen how we can define differentiation for complex functions, check whether such a function is differentiable, and seen how to differentiate complex rational and polynomial functions. You have learnt how this can be extended to the partial derivatives of complex functions of more than one variable, and studied the Cauchy-Riemann equations that link the first partial derivatives of the real and imaginary parts of a differentiable complex function of two variables.

You can now move on to [Session 2: Integration](https://www.open.edu/openlearn/mod/oucontent/view.php?id=139280).

**Session 2: Integration**

## Introduction to integration

This session introduces complex integration, an important concept which gives complex analysis its special flavour. We spend most of this session setting up the complex integral, deriving its main properties, and illustrating various techniques for evaluating it.

To define the integral of a complex function, it is instructive to first consider real integrals, such as

Start of $1

integral over a under b x squared d x equals one divided by three times left parenthesis b cubed minus a cubed right parenthesis comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session1_Alternative1)

End of $1

where a less than b, which represents the area of the shaded part of Figure 1 (for a greater than zero). We can express this equation in words by saying that

Start of Extract

the integral of the function f of x equals x squared over the interval open a comma b close is one divided by three times left parenthesis b cubed minus a cubed right parenthesis.

End of Extract

Start of Figure

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Figure 1 Area under the graph of y equals x squared between a and b

[View description - Figure 1 Area under the graph of y equals x squared between a and b](" \l "Unit2_Session1_Description1)

End of Figure

Suppose now that we wish to integrate the complex function f of z equals z squared between two points alpha and beta in the complex plane. To do this, we first need to specify exactly how to get from alpha to beta. We could, for example, choose the line segment normal cap gamma from alpha to beta, as shown in Figure 2. It turns out (as you will see later) that if we make this choice, then

Start of Extract

the integral of the function f of z equals z squared along the line segment from alpha to beta is one divided by three times left parenthesis beta times cubed minus alpha cubed right parenthesis.

End of Extract

We write this as

Start of $1

integral over normal cap gamma z squared d z equals one divided by three times left parenthesis beta times cubed minus alpha cubed right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session1_Alternative22)

End of $1

Start of Figure

Displayed image

Figure 2 Line segment normal cap gamma from alpha to beta

[View description - Figure 2 Line segment normal cap gamma from alpha to beta](" \l "Unit2_Session1_Description2)

End of Figure

But there are many other paths in the complex plane from alpha to beta, which raises the following question. Do we get the same answer if we integrate the function f of z equals z squared along a different path from alpha to beta?

In order to address this question, we first need to explain exactly what it means to ‘integrate a function along a path’. This is one of the objectives of Section 1, where we briefly review the Riemann integral from real analysis, and then use similar ideas to construct the integral of a complex function along a path in the complex plane. We will see that if f is a complex function that is continuous on a smooth path normal cap gamma colon gamma of t left parenthesis t element of left square bracket a comma b right square bracket right parenthesis in the complex plane, then the integral of f along normal cap gamma, denoted by integral over normal cap gamma f of z d z, is given by the formula

Start of $1

integral over normal cap gamma f of z d z equals integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session1_Alternative36)

End of $1

We can evaluate this integral by splitting f of gamma of t times gamma times super prime times left parenthesis t right parenthesis into its real and imaginary parts u of t and v of t, and evaluating the resulting pair of real integrals:

Start of $1

integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t equals integral over a under b u of t d t plus i times integral over a under b v of t d t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session1_Alternative40)

End of $1

Section 2 begins with this definition of the integral of a complex function along a smooth path, and then extends the idea to allow integration along a contour – a finite sequence of smooth paths laid end to end.

In Section 3 we prove the Fundamental Theorem of Calculus, which shows that integration and differentiation are essentially inverse processes. From this result it follows that the integral of f of z equals z squared along any contour from alpha to beta is one divided by three times left parenthesis beta times cubed minus alpha cubed right parenthesis.

We will need to be careful about how we apply results such as the Fundamental Theorem of Calculus. For example, suppose that the endpoints alpha and beta of normal cap gamma coincide, as illustrated in Figure 3. Then

Start of $1

equation sequence part 1 integral over normal cap gamma z squared d z equals part 2 one divided by three times left parenthesis beta times cubed minus alpha cubed right parenthesis equals part 3 zero full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session1_Alternative48)

End of $1

In this case, the integral of the function f of z equals z squared along normal cap gamma is zero.

Start of Figure

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Figure 3 Contour with initial point alpha and final point beta coinciding

[View description - Figure 3 Contour with initial point alpha and final point beta coinciding](" \l "Unit2_Session1_Description3)

End of Figure

Consider now the function f of z equals one solidus z. We will see later in Example 4 that if we integrate f along the smooth paths normal cap gamma sub one and normal cap gamma sub two shown in Figure 4, where normal cap gamma sub one and normal cap gamma sub two are circles traversed once anticlockwise, then

Start of $1

integral over normal cap gamma sub one one divided by z d z equals zero comma but integral over normal cap gamma sub two one divided by z d z equals two times pi times i full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session1_Alternative60)

End of $1

The reason for this difference will become apparent in Section 3.

Start of Figure

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Figure 4 Circular paths normal cap gamma sub one and normal cap gamma sub two

[View description - Figure 4 Circular paths normal cap gamma sub one and normal cap gamma sub two](" \l "Unit2_Session1_Description4)

End of Figure

Start of Box

This OpenLearn course is an extract from the Open University course [M337 Complex analysis](https://www.open.ac.uk/courses/modules/m337).

End of Box

## 1 Integrating real functions

After working through this section, you should be able to:

* appreciate how the Riemann integral is defined
* state the main properties of the Riemann integral
* appreciate how complex integrals can be defined.

In this section we define the Riemann integral of a continuous real function (named after Bernhard Riemann, whom we met in Session 1 for the Cauchy–Riemann equations) and outline its main properties. We then discuss complex integrals.

## 1.1 Areas under curves

One of the uses of real integration is to determine the area under a curve. For example, the integral of a continuous function f that takes only positive values between the real numbers a and b, where a less than b, is the area bounded by the graph of y equals f of x, the x-axis, and the two vertical lines x equals a and x equals b, as illustrated by the shaded part of Figure 5.

Start of Figure

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Figure 5 Area under the graph of y equals f of x between a and b

[View description - Figure 5 Area under the graph of y equals f of x between a and b](" \l "Unit2_Session2_Description1)

End of Figure

We can estimate this area by first splitting the interval open a comma b close into a finite number of subintervals, such as those shown in Figure 6.

Start of Figure

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Figure 6 Interval open a comma b close split into subintervals

[View description - Figure 6 Interval open a comma b close split into subintervals](" \l "Unit2_Session2_Description2)

End of Figure

We can then underestimate the area under the graph of y equals f of x between a and b by summing the areas of those rectangles that have the various subintervals as bases and for which the top edge of each rectangle touches the graph from below, as shown in Figure 7(a). Similarly, we can overestimate the area under y equals f of x between a and b by summing the areas of those rectangles that have the various subintervals as bases and for which the top edge of each rectangle touches the graph from above, as shown in Figure 7(b).

Start of Figure

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Figure 7 (a) An underestimate (b) An overestimate

[View description - Figure 7 (a) An underestimate (b) An overestimate](" \l "Unit2_Session2_Description3)

End of Figure

We now let the number of subintervals tend to infinity, in such a way that the lengths of the subintervals tend to zero. It can be shown that the underestimates and overestimates of the area tend to a common limit cap a, written as

Start of $1

cap a equals integral over a under b f of x d x full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative21)

End of $1

We call cap a the area under the graph of y equals f of xbetween aand b.

This underestimating and overestimating approach is often how Riemann integration is first introduced, and you may have seen it before. However, we encounter a problem if we try to generalise this particular approach to complex functions. Inequalities between complex numbers have no meaning, so it makes no sense to try to estimate complex numbers from ‘below’ or ‘above’. To get round this problem, we now outline a different approach to defining the integral of a real function – one that does generalise to complex functions.

Rather than underestimating and overestimating the area under the curve with rectangles, we choose a single point inside each subinterval and use this to construct a rectangle whose base is the subinterval, and whose height is the value of the function at the chosen point. The sum of the areas of these rectangles should then be an approximation to the area under the graph. As long as our function f is continuous on the interval open a comma b close, then this modified approach (which does generalise to complex integrals) agrees with the underestimating and overestimating approach.

In this section we use this modified approach to give a formal definition of the Riemann integral, and then we summarise the main properties of the Riemann integral. We omit all proofs, which can be found in texts on real analysis.

## 1.2 Integration on the real line

We wish to define the Riemann integral of a continuous real function f in such a way that if f is positive on some interval open a comma b close, then the integral of f from a to b is the area under the graph of y equals f of x between a and b. This is illustrated by the shaded part of Figure 8. To do this, we first split the interval open a comma b close into a collection of subintervals called a partition.

Start of Figure

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Figure 8 Area under the graph of y equals f of x between a and b

[View description - Figure 8 Area under the graph of y equals f of x between a and b](" \l "Unit2_Session2_Description4)

End of Figure

Start of Box

**Definitions**

A **partition** cap p of the interval open a comma b close is a finite collection of subintervals of open a comma b close,

Start of $1

cap p equals left square bracket x sub zero comma x sub one right square bracket comma left square bracket x sub one comma x sub two right square bracket comma ellipsis comma left square bracket x sub n minus one comma x sub n right square bracket comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative44)

End of $1

for which

Start of $1

multirelation a equals x sub zero less than or equals x sub one less than or equals x sub two less than or equals ellipsis less than or equals x sub n equals b full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative45)

End of $1

The **length** of the subinterval left square bracket x sub k minus one comma x sub k right square bracket is delta times x sub k equals x sub k minus x sub k minus one.

We use absolute value of cap p to denote the maximum length of all the subintervals, so

Start of $1

absolute value of cap p equals max of delta times x sub one comma delta times x sub two comma ellipsis comma delta times x sub n full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative49)

End of $1

End of Box

Given a partition cap p equals left square bracket x sub zero comma x sub one right square bracket comma left square bracket x sub one comma x sub two right square bracket comma ellipsis comma left square bracket x sub n minus one comma x sub n right square bracket of open a comma b close, we can approximate the area under the graph of y equals f of x between a and b by constructing a sequence of rectangles, as shown in Figure 9.

Start of Figure

Displayed image

Figure 9 Approximating the area under a graph using a sequence of rectangles

[View description - Figure 9 Approximating the area under a graph using a sequence of rectangles](" \l "Unit2_Session2_Description5)

End of Figure

Here the kth rectangle has base left square bracket x sub k minus one comma x sub k right square bracket and height f of x sub k (so the top-right corner of the rectangle touches the curve). The area of the rectangle is f of x sub k times left parenthesis x sub k minus x sub k minus one right parenthesis (see Figure 10). Note that we could equally have chosen the rectangle to be of height f of c sub k for any point c sub k in left square bracket x sub k minus one comma x sub k right square bracket, and the theory would still work. This is because, for a continuous function f, the difference between one set of choices of values for c sub k, k equals one comma two comma ellipsis comma n, and another disappears when we take limits of partitions. We have chosen f of x sub k merely for convenience.

Start of Figure

Displayed image

Figure 10 Rectangle of height f of x sub k and width delta times x sub k

[View description - Figure 10 Rectangle of height f of x sub k and width delta times x sub k](" \l "Unit2_Session2_Description6)

End of Figure

Summing the areas of all the rectangles gives an approximation to the area under the graph. This sum is called the Riemann sum for f, with respect to this particular partition. (You may have seen upper Riemann sum and lower Riemann sum defined slightly differently elsewhere.)

Start of Box

**Definition**

The **Riemann sum** for f with respect to the partition

Start of $1

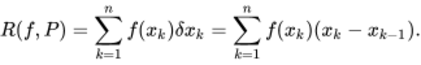
cap p equals left square bracket x sub zero comma x sub one right square bracket comma left square bracket x sub one comma x sub two right square bracket comma ellipsis comma left square bracket x sub n minus one comma x sub n right square bracket

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative70)

End of $1

is the sum

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative71)

End of $1

End of Box

We now calculate the Riemann sum for a particular choice of function and partition, and then ask you to do the same for a second function.

Start of Example

**Example 1**

Let f of x equals x squared, where x element of left square bracket zero comma one right square bracket. Show that for

Start of $1

cap p sub n equals left square bracket zero comma one solidus n right square bracket comma left square bracket one solidus n comma two solidus n right square bracket comma ellipsis comma left square bracket left parenthesis n minus one right parenthesis solidus n comma one right square bracket comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative74)

End of $1

we have

Start of $1

cap r of f comma cap p sub n equals one divided by six times left parenthesis one plus one solidus n right parenthesis times left parenthesis two plus one solidus n right parenthesis comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative75)

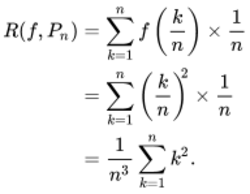
End of $1

and determine lim over n right arrow normal infinity of cap r of f comma cap p sub n.

**Solution**

Each of the n subintervals of cap p sub n has length one solidus n. Therefore

Start of $1

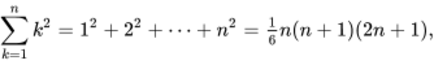


[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative80)

End of $1

Using the identity

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative81)

End of $1

we obtain

Start of $1

equation sequence part 1 cap r of f comma cap p sub n equals part 2 one divided by n cubed multiplication one divided by six times n times left parenthesis n plus one right parenthesis times left parenthesis two times n plus one right parenthesis equals part 3 one divided by six times left parenthesis one plus one solidus n right parenthesis times left parenthesis two plus one solidus n right parenthesis comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative82)

End of $1

as required.

Finally, since left parenthesis one solidus n right parenthesis is a basic null sequence, we see that

Start of $1

equation sequence part 1 lim over n right arrow normal infinity of cap r of f comma cap p sub n equals part 2 one divided by six times left parenthesis one plus zero right parenthesis times left parenthesis two plus zero right parenthesis equals part 3 one divided by three full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative84)

End of $1

End of Example

Now try the following exercise, making use of the identity

Start of $1

sum with variable number of summands one cubed plus two cubed plus ellipsis plus n cubed equals one divided by four times n squared times left parenthesis n plus one right parenthesis squared full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative85)

End of $1

Start of Exercise

**Exercise 1**

Start of Question

Let f of x equals x cubed, where x element of left square bracket zero comma one right square bracket. Show that for

Start of $1

cap p sub n equals left square bracket zero comma one solidus n right square bracket comma left square bracket one solidus n comma two solidus n right square bracket comma ellipsis comma left square bracket left parenthesis n minus one right parenthesis solidus n comma one right square bracket comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative88)

End of $1

we have

Start of $1

cap r of f comma cap p sub n equals one divided by four times left parenthesis one plus one solidus n right parenthesis squared comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative89)

End of $1

and determine lim over n right arrow normal infinity of cap r of f comma cap p sub n.

End of Question

[View answer - Exercise 1](" \l "Unit2_Session2_Answer1)

End of Exercise

The Riemann sums cap r of f comma cap p sub n of Example 1 approximate the area under the graph of y equals x squared between zero and one. The approximation improves as n increases, and we expect the limiting value one divided by three to actually be the area under the graph. However, to be sure that this limit gives us a sensible value, we should check that cap r of f comma cap p sub n right arrow one divided by three for any sequence left parenthesis cap p sub n right parenthesis of partitions of left square bracket zero comma one right square bracket such that absolute value of cap p sub n right arrow zero. The following important theorem, for which we omit the proof, provides this check.

Start of Box

**Theorem 1**

Let f colon left square bracket a comma b right square bracket long right arrow double-struck cap r be a continuous function. Then there is a real number cap a such that

Start of $1

lim over n right arrow normal infinity of cap r of f comma cap p sub n equals cap a comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative109)

End of $1

for any sequence left parenthesis cap p sub n right parenthesis of partitions of open a comma b close such that absolute value of cap p sub n right arrow zero.

End of Box

We can now define the Riemann integral of a continuous function.

Start of Box

**Definition**

Let f colon left square bracket a comma b right square bracket long right arrow double-struck cap r be a continuous function, where a less than b. The value cap a determined by Theorem 1 is called the **Riemann integral** of f over left square bracket a comma b right square bracket, and it is denoted by

Start of $1

integral over a under b f of x d x full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative118)

End of $1

End of Box

The theorem tells us that to calculate the Riemann integral of f over open a comma b close, we can make any choice of partitions left parenthesis cap p sub n right parenthesis for which absolute value of cap p sub n right arrow zero and calculate lim over n right arrow normal infinity of cap r of f comma cap p sub n. Thus the calculation of Example 1 really does demonstrate that

Start of $1

integral over zero under one x squared d x equals one divided by three full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative124)

End of $1

We define the Riemann integral integral over a under b f of x d x when a greater than or equals b, as follows.

Start of Box

**Definitions**

Let f be a continuous real function.

If a greater than b, and left square bracket b comma a right square bracket is contained in the domain of f, then we define

Start of $1

integral over a under b f of x d x equals negative integral over b under a f of x d x full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative131)

End of $1

Also, for values of a in the domain of f, we define

Start of $1

integral over a under a f of x d x equals zero full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative134)

End of $1

End of Box

As we have discussed, for a continuous real function f that takes only positive values on open a comma b close, where a less than b, the Riemann integral

Start of $1

integral over a under b f of x d x

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative138)

End of $1

measures the area under the graph of y equals f of x between a and b. If we no longer require f to be positive, then the integral still has a geometric meaning: it measures the signed area of the set between the curve y equals f of x, the x-axis and the vertical lines x equals a and x equals b, where we count parts of the set above the x-axis as having positive area, and parts of the set below the x-axis as having negative area, as illustrated in Figure 11.

Start of Figure

Displayed image

Figure 11 Signed area determined by the graph of y equals f of x between a and b

[View description - Figure 11 Signed area determined by the graph of y equals f of x between a and b](" \l "Unit2_Session2_Description7)

End of Figure

## 1.3 Properties of the Riemann integral

In practice we do not usually calculate integrals by looking at partitions, but instead use a powerful theorem known as the Fundamental Theorem of Calculus, which allows us to think of integration and differentiation as inverse processes.

To state the theorem, we need the notion of a **primitive** of a continuous real function f colon left square bracket a comma b right square bracket long right arrow double-struck cap r; this is a real function cap f that is differentiable on open a comma b close with derivative equal to f, that is, the function cap f satisfies cap f super prime of x equals f of x, for all x element of left square bracket a comma b right square bracket. A primitive of a function is not unique, because if cap f is a primitive of f, then so is the function with rule cap f of x plus c, for any constant c.

Start of Box

**Theorem 2 Fundamental Theorem of Calculus**

Let f colon left square bracket a comma b right square bracket long right arrow double-struck cap r be a continuous function. If cap f is a primitive of f, then the Riemann integral of f over open a comma b close exists and is given by

Start of $1

integral over a under b f of x d x equals cap f of b minus cap f of a full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative168)

End of $1

End of Box

We denote cap f of b minus cap f of a by left square bracket cap f of x right square bracket sub a super b.

For example, a primitive of f of x equals x squared is cap f of x equals x cubed solidus three, so

Start of $1

equation sequence part 1 integral over zero under one x squared d x equals part 2 left square bracket x cubed divided by three right square bracket sub zero super one equals part 3 one cubed divided by three minus zero cubed divided by three equals part 4 one divided by three comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative173)

End of $1

which agrees with our earlier calculation using Riemann sums.

The Riemann integral has a number of useful properties.

Start of Box

**Theorem 3 Properties of the Riemann integral**

Let f and g be real functions that are continuous on the interval open a comma b close.

1. **Sum Rule** integral over a under b left parenthesis f of x plus g of x right parenthesis d x equals integral over a under b f of x d x plus integral over a under b g of x d x.
2. **Multiple Rule** integral over a under b lamda times f of x d x equals lamda times integral over a under b f of x d x, for lamda element of double-struck cap r.
3. **Additivity Rule**

Start of $1

integral over a under b f of x d x equals integral over a under c f of x d x plus integral over c under b f of x d x comma for a less than or equals c less than or equals b full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative180)

End of $1

1. **Substitution Rule** If g is differentiable on open a comma b close and its derivative g super prime is continuous on open a comma b close, and if f is continuous on g of x colon a less than or equals x less than or equals b, then

Start of $1

integral over a under b f of g of x times g super prime of x d x equals integral over g of a under g of b f of t d t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative187)

End of $1

1. **Integration by Parts** If f and g are differentiable on open a comma b close and their derivatives f super prime and g super prime are continuous on open a comma b close, then

Start of $1

integral over a under b f super prime of x times g of x d x equals left square bracket f of x times g of x right square bracket sub a super b minus integral over a under b f of x times g super prime of x d x full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative194)

End of $1

1. **Monotonicity Inequality** If f of x less than or equals g of x for each x element of left square bracket a comma b right square bracket, then

Start of $1

integral over a under b f of x d x less than or equals integral over a under b g of x d x full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative197)

End of $1

1. **Modulus Inequality** absolute value of integral over a under b f of x d x less than or equals integral over a under b absolute value of f of x d x.

End of Box

The first five properties are probably familiar to you and we have stated them only for reference. The last two inequalities may be less familiar. The Monotonicity Inequality, illustrated in Figure 12, states that if you replace f by a greater function g, then the integral increases.

Start of Figure

Displayed image

Figure 12 Monotonicity Inequality

[View description - Figure 12 Monotonicity Inequality](" \l "Unit2_Session2_Description8)

End of Figure

The Modulus Inequality, illustrated in Figure 13, says that the modulus of the integral of f over open a comma b close (a non-negative number) is less than or equal to the integral of the modulus of f over open a comma b close (another non-negative number). If f is positive, then these two numbers are equal, but if f takes negative values, then at least part of the signed area between y equals f of x and the x-axis is negative, so the first number is less than the second.

Start of Figure

Displayed image

Figure 13 Modulus Inequality

[View description - Figure 13 Modulus Inequality](" \l "Unit2_Session2_Description9)

End of Figure

Start of Exercise

**Exercise 2**

Start of Question

Use the Monotonicity Inequality and the fact that

Start of $1

e super negative x less than or equals e super negative x squared less than or equals one divided by one plus x squared comma for zero less than or equals x less than or equals one comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative209)

End of $1

to estimate integral over zero under one e super negative x squared d x from above and below.

End of Question

[View answer - Exercise 2](" \l "Unit2_Session2_Answer2)

End of Exercise

## 1.4 Introducing complex integration

We come now to the central theme of this course – integrating complex functions. Informed by the discussion in the introduction, we should expect that the integral of a continuous complex function f from one point alpha to another point beta in the complex plane may depend on the path that we choose to take from alpha to beta. So it is necessary to first choose a smooth path normal cap gamma colon gamma of tleft parenthesis t element of left square bracket a comma b right square bracket right parenthesis such that gamma of a equals alpha and gamma of b equals beta (see Figure 14), and then we will define the integral of f along this smooth path, denoting the resulting quantity by

Start of $1

integral over normal cap gamma f of z d z full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative229)

End of $1

Start of Figure

Displayed image

Figure 14 A smooth path from alpha to beta

[View description - Figure 14 A smooth path from alpha to beta](" \l "Unit2_Session2_Description10)

End of Figure

There are two ways to achieve this goal.

One method is to imitate the approach of [Section 1.2](#def-b1-riemann-sum), as follows.

* Choose a partition of the path normal cap gamma into subpaths

Start of $1

cap p equals normal cap gamma sub one comma normal cap gamma sub two comma ellipsis comma normal cap gamma sub n comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative233)

End of $1

determined by points alpha equals z sub zero, z sub one, …, z sub n equals beta, such as those illustrated in Figure 15.

Start of Figure

Displayed image

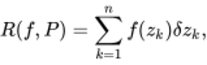
Figure 15 (a) A partition of normal cap gamma (b) A partition of the line segment from zero to one plus i

[View description - Figure 15 (a) A partition of normal cap gamma (b) A partition of the line segment ...](" \l "Unit2_Session2_Description11)

End of Figure

* Define a complex Riemann sum

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative240)

End of $1

where delta times z sub k equals z sub k minus z sub k minus one, for k equals one comma two comma ellipsis comma n, and define

Start of $1

absolute value of cap p equals max of absolute value of delta times z sub one comma absolute value of delta times z sub two comma ellipsis comma absolute value of delta times z sub n full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative243)

End of $1

* Define the complex integral integral over normal cap gamma f of z d z to be

Start of $1

lim over n right arrow normal infinity of cap r of f comma cap p sub n comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative245)

End of $1

where left parenthesis cap p sub n right parenthesis is any sequence of partitions of normal cap gamma for which absolute value of cap p sub n right arrow zero as n right arrow normal infinity.

It can be shown (although it is quite hard to do so) that this limit exists when f is continuous, and that it is independent of the choice of partitions of normal cap gamma. Thus we have defined the integral of a continuous complex function. We can then develop the standard properties of integrals, such as the Additivity Rule and the Combination Rules, by imitating the discussion of the real Riemann integral.

The second, quicker method is to define a complex integral in terms of two real integrals. To do this, we use a parametrisation gamma colon left square bracket a comma b right square bracket long right arrow double-struck cap c of the smooth path normal cap gamma, where gamma of a equals alpha and gamma of b equals beta. Any set of parameter values

Start of $1

t sub zero comma t sub one comma ellipsis comma t sub n colon multirelation a equals t sub zero less than t sub one less than ellipsis less than t sub n equals b

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative256)

End of $1

yields a partition

Start of $1

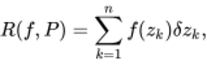
cap p equals normal cap gamma sub one comma normal cap gamma sub two comma ellipsis comma normal cap gamma sub n

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative257)

End of $1

of normal cap gamma, where normal cap gamma sub k is the subpath of normal cap gamma that joins z sub k minus one equals gamma of t sub k minus one to z sub k equals gamma of t sub k, for k equals one comma two comma ellipsis comma n. We can then define the complex Riemann sum

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative264)

End of $1

where delta times z sub k equals z sub k minus z sub k minus one, for k equals one comma two comma ellipsis comma n; see Figure 16.

Start of Figure

Displayed image

Figure 16 A partition of normal cap gamma induced by the parameter values t sub zero comma t sub one comma ellipsis comma t sub n

[View description - Figure 16 A partition of normal cap gamma induced by the parameter values t sub zero ...](" \l "Unit2_Session2_Description12)

End of Figure

Notice that

Start of $1

equation sequence part 1 delta times z sub k equals part 2 z sub k minus z sub k minus one equals part 3 gamma of t sub k minus gamma of t sub k minus one full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative269)

End of $1

Hence, if t sub k is close to t sub k minus one, then, to a good approximation,

Start of $1

multirelation gamma times super prime times left parenthesis t sub k right parenthesis almost equals gamma of t sub k minus gamma of t sub k minus one divided by t sub k minus t sub k minus one equals delta times z sub k divided by delta times t sub k comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative272)

End of $1

where delta times t sub k equals t sub k minus t sub k minus one, so

Start of $1

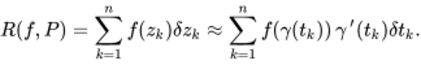
delta times z sub k almost equals gamma times super prime times left parenthesis t sub k right parenthesis times delta times t sub k full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative274)

End of $1

Thus if max of delta times t sub one comma delta times t sub two comma ellipsis comma delta times t sub n is small, then, to a good approximation,

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative276)

End of $1

The expression on the right has the form of a Riemann sum for the integral

Start of $1

(integral 1)

integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative277)

End of $1

Here the integrand

Start of $1

t long right arrow from bar f of gamma of t times gamma times super prime times left parenthesis t right parenthesis times left parenthesis t element of left square bracket a comma b right square bracket right parenthesis

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative278)

End of $1

is a complex-valued function of a real variable. We have defined integrals of only real functions so far, but if we split f of gamma of t times gamma times super prime times left parenthesis t right parenthesis into its real and imaginary parts u of t plus i times v of t, then the integral (1) above can be written as

Start of $1

integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t equals integral over a under b u of t d t plus i times integral over a under b v of t d t comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative281)

End of $1

which is a combination of two real integrals. We then define the integral of f along normal cap gamma by the formula

Start of $1

(formula 2)

integral over normal cap gamma f of z d z equals integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative284)

End of $1

It can be shown that both of these methods for defining the integral of a continuous complex function f along a smooth path normal cap gamma give the same value for

Start of $1

integral over normal cap gamma f of z d z full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative287)

End of $1

In the next section we will develop properties of complex integrals, and there we will use the formula (2) above for the definition of the integral of a complex function f along a path normal cap gamma.

Start of Box

**History of complex integration**

The first significant steps in the development of real integration came in the seventeenth century with the work of a number of European mathematicians. Notable among this group was the French lawyer and mathematician Pierre de Fermat (1601–1665), who found areas under curves of the form y equals a times x super n, for n an integer (possibly negative), using partitions and arguments involving infinitesimals.

A major breakthrough was the discovery of calculus made independently by the English mathematician and scientist Isaac Newton (1642–1727) and the German philosopher and mathematician Gottfried Wilhelm Leibniz (1646–1716). They observed that differentiation and integration are inverse processes, a fact encapsulated in the Fundamental Theorem of Calculus.

Towards the end of the eighteenth century, mathematicians began to consider integrating complex functions.

Two pioneers in this endeavour were Leonhard Euler and Pierre-Simon Laplace. They were mainly concerned with manipulating complex integrals in order to evaluate difficult real integrals such as

Start of $1

integral over negative normal infinity under normal infinity sine of x divided by x d x equals pi and integral over negative normal infinity under normal infinity e super negative x squared d x equals Square root of pi full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative292)

End of $1

However, it was through the work of Augustin-Louis Cauchy that complex integration began to assume the form that is now used in complex analysis. Cauchy’s first paper on complex integrals in 1814 treated complex integrals as purely algebraic objects; it was only much later that he came to properly appreciate their geometric significance.

By the mid to late nineteenth century, mathematicians began to consider how to expand the theory of integration to deal with functions that are not continuous. The first rigorous theory of integration to do this was put forward by Riemann in 1854. The Riemann integral was followed by a number of other formal definitions of integration, some equivalent to Riemann’s, and some more general, such as Lebesgue integration, named after the French mathematician Henri Lebesgue (1875–1941).

End of Box

## 2 Integrating complex functions

After working through this section, you should be able to:

* define the integral of a continuous function along a smooth path, and evaluate such integrals
* explain what is meant by a contour, define the (contour) integral of a continuous function along a contour, and evaluate such integrals
* define the reverse contour of a given contour, and state and use the Reverse Contour Theorem.

## 2.1 Integration along a smooth path

Motivated by the discussion of the preceding section, we make the following definition of the integral of a complex function.

Start of Box

**Definition**

Let normal cap gamma colon gamma of t left parenthesis t element of left square bracket a comma b right square bracket right parenthesis be a smooth path in double-struck cap c, and let f be a function that is continuous on normal cap gamma. Then the **integral of** bold-italic f**along the path** bold cap gamma, denoted by integral over normal cap gamma f of z d z, is

Start of $1

integral over normal cap gamma f of z d z equals integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative8)

End of $1

The integral is evaluated by splitting f of gamma of t times gamma times super prime times left parenthesis t right parenthesis into its real and imaginary parts u of t equals Re of f of gamma of t times gamma times super prime times left parenthesis t right parenthesis and v of t equals Im of f of gamma of t times gamma times super prime times left parenthesis t right parenthesis, and evaluating the resulting pair of real integrals,

Start of $1

integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t equals integral over a under b u of t d t plus i times integral over a under b v of t d t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative12)

End of $1

End of Box

### Remarks

1. Since f is continuous on normal cap gamma and gamma is a smooth parametrisation, the functions t long right arrow from bar f of gamma of t and t long right arrow from bar gamma times super prime times left parenthesis t right parenthesis are both continuous on open a comma b close, so the function t long right arrow from bar f of gamma of t times gamma times super prime times left parenthesis t right parenthesis is continuous on open a comma b close. It follows that the real functions u and v are continuous on open a comma b close, and hence

Start of $1

integral over a under b u of t d t and integral over a under b v of t d t

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative24)

End of $1

exist, so integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t also exists.

1. An important special case is when gamma of t equals tleft parenthesis t element of left square bracket a comma b right square bracket right parenthesis, so normal cap gamma is the real line segment from a to b. Since gamma times super prime times left parenthesis t right parenthesis equals one, we see that integral over normal cap gamma f of z d z equals

Start of $1

integral over a under b f of t d t equals integral over a under b u of t d t plus i times integral over a under b v of t d t comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative33)

End of $1

where u equals Re of f and v equals Im of f. This equation is a formula for the integral of a complex function over a real interval.

1. An alternative notation for integral over normal cap gamma f of z d z is integral over normal cap gamma f.
2. If the path of integration normal cap gamma has a standard parametrisation gamma, then, unless otherwise stated, we use gamma in the evaluation of the integral of f along normal cap gamma.
3. To help to remember the formula used to define integral over normal cap gamma f of z d z, notice that it can be obtained by ‘substituting’

Start of $1

z equals gamma of t comma d times z equals gamma times super prime times left parenthesis t right parenthesis times d times t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative44)

End of $1

We consider d times z equals gamma times super prime times left parenthesis t right parenthesis times d times t to be a shorthand for d times z divided by d times t equals gamma times super prime times left parenthesis t right parenthesis.

The following examples demonstrate how to evaluate integrals along paths. In each case, we follow the convention of Remark 3 and use the standard parametrisation of the path.

Start of Example

**Example 2**

Evaluate

Start of $1

integral over normal cap gamma z squared d z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative47)

End of $1

where normal cap gamma is the line segment from zero to one plus i.

**Solution**

Here f of z equals z squared, and we use the standard parametrisation

Start of $1

gamma of t equals left parenthesis one plus i right parenthesis times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis

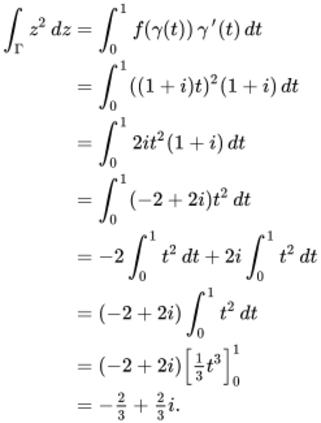
[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative52)

End of $1

of normal cap gamma, which satisfies gamma times super prime times left parenthesis t right parenthesis equals one plus i.

Then f of gamma of t equals left parenthesis left parenthesis one plus i right parenthesis times t right parenthesis squared, so

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative56)

End of $1

End of Example

You need not include every line of working of Example 2 if you do not need to. Here is another example.

Start of Example

**Example 3**

Evaluate

Start of $1

integral over normal cap gamma z macron d z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative57)

End of $1

where normal cap gamma is the line segment from 0 to one plus i.

**Solution**

Here f of z equals z macron, and again we use the standard parametrisation

Start of $1

gamma of t equals left parenthesis one plus i right parenthesis times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative61)

End of $1

of normal cap gamma, which satisfies gamma times super prime times left parenthesis t right parenthesis equals one plus i. Then

Start of $1

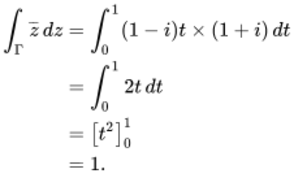
equation sequence part 1 f of gamma of t equals part 2 times times left parenthesis right parenthesis plus plus one it macron equals part 3 left parenthesis one minus i right parenthesis times t comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative64)

End of $1

so

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative65)

End of $1

End of Example

We set out our solution to the next example using the observation and notation of Remark 4.

Start of Example

**Example 4**

Evaluate

Start of $1

integral over normal cap gamma one divided by z d z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative66)

End of $1

where normal cap gamma is the unit circle z colon absolute value of z equals one.

**Solution**

Here f of z equals one solidus z, and we use the standard parametrisation

Start of $1

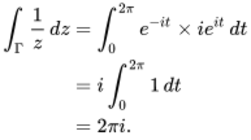
gamma of t equals e super i times t times left parenthesis t element of left square bracket zero comma two times pi right square bracket right parenthesis

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative70)

End of $1

of normal cap gamma. Then z equals e super i times t, one solidus z equals e super negative i times t and d times z equals i times e super i times t times d times t. Hence

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative75)

End of $1

End of Example

Sometimes when evaluating integrals we will use the alternative notation of Example 4 instead of the notation of Example 2 and Example 3; both notations are commonly used in complex analysis.

In the examples above, we used the standard parametrisation in each case. The following exercise suggests that the value of the integral is not affected by the choice of parametrisation.

Start of Exercise

**Exercise 3**

Start of Question

1. Verify that the result of Example 3 is unchanged if we use the smooth parametrisation

Start of $1

gamma of t equals two times left parenthesis one plus i right parenthesis times t times left parenthesis t element of left square bracket zero comma one divided by two right square bracket right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative76)

End of $1

1. Verify that the result of Example 4 is unchanged if we use the smooth parametrisation

Start of $1

gamma of t equals e super three times i times t times left parenthesis t element of left square bracket zero comma two times pi solidus three right square bracket right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative77)

End of $1

End of Question

[View answer - Exercise 3](" \l "Unit2_Session3_Answer1)

End of Exercise

The reason why we have obtained the same values in Exercise 3 as those in Example 3 and Example 4 is because of the following theorem.

Start of Box

**Theorem 4**

Let gamma sub one colon left square bracket a sub one comma b sub one right square bracket long right arrow double-struck cap c and gamma sub two colon left square bracket a sub two comma b sub two right square bracket long right arrow double-struck cap c be two smooth parametrisations of paths with the same image set normal cap gamma, and let f be a function that is continuous on normal cap gamma. Then

Start of $1

integral over normal cap gamma f of z d z

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative91)

End of $1

does not depend on which parametrisation gamma sub one or gamma sub two is used.

End of Box

The proof of Theorem 4 uses the Inverse Function rule and the Chain rule for the derivatives of complex functions, which are not covered within this course. So we shall omit the details of this proof.

In practical terms, this theorem allows you to choose any convenient smooth parametrisation when evaluating a complex integral along a given path. We will see how this can be helpful in the next subsection.

For further practice in integration, try the following exercise.

Start of Exercise

**Exercise 4**

Start of Question

Evaluate the following integrals.

1. integral over normal cap gamma Re of z times d times z, where normal cap gamma is the line segment from 0 to one plus two times i.
2. integral over normal cap gamma one divided by left parenthesis z minus alpha right parenthesis squared d z, where normal cap gamma is the circle with centre alpha and radius r.

End of Question

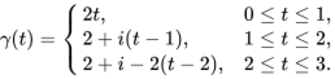
[View answer - Exercise 4](" \l "Unit2_Session3_Answer2)

End of Exercise

## 2.2 Integration along a contour

Consider the path normal cap gamma from 0 to i in Figure 17, with parametrisation gamma colon left square bracket zero comma three right square bracket long right arrow double-struck cap c given by

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative112)

End of $1

This path is not smooth, because gamma is not differentiable at t equals one or t equals two. However, normal cap gamma can be split into three smooth straight-line paths, joined end to end. This leads to the idea of a contour: it is simply what we get when we place a finite number of smooth paths end to end.

Start of Figure

Displayed image

Figure 17 A path normal cap gamma from zero to i

[View description - Figure 17 A path normal cap gamma from zero to i](" \l "Unit2_Session3_Description1)

End of Figure

Start of Box

**Definitions**

A **contour** normal cap gamma is a path that can be subdivided into a finite number of smooth paths normal cap gamma sub one comma normal cap gamma sub two comma ellipsis comma normal cap gamma sub n joined end to end. The order of these constituent smooth paths is indicated by writing

Start of $1

normal cap gamma equals sum with variable number of summands normal cap gamma sub one plus normal cap gamma sub two plus ellipsis plus normal cap gamma sub n full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative122)

End of $1

The **initial point** of normal cap gamma is the initial point of normal cap gamma sub one, and the **final point** of normal cap gamma is the final point of normal cap gamma sub n.

End of Box

The definition of a contour is illustrated in Figure 18.

Start of Figure

Displayed image

Figure 18 The contour normal cap gamma equals sum with 4 summands normal cap gamma sub one plus normal cap gamma sub two plus normal cap gamma sub three plus normal cap gamma sub four

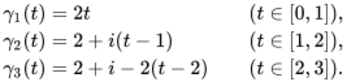
[View description - Figure 18 The contour normal cap gamma equals sum with 4 summands normal cap gamma ...](" \l "Unit2_Session3_Description2)

End of Figure

As an example, the contour normal cap gamma in Figure 17 can be written as sum with 3 summands normal cap gamma sub one plus normal cap gamma sub two plus normal cap gamma sub three, where normal cap gamma sub one, normal cap gamma sub two and normal cap gamma sub three are smooth paths with smooth parametrisations

Start of $1

(equation 3)



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative133)

End of $1

Now, we have seen how to integrate a continuous function along a smooth path. It is natural to extend this definition to contours, by splitting the contour into smooth paths and integrating along each in turn. We formalise this idea in the following definition.

Start of Box

**Definition**

Let normal cap gamma equals sum with variable number of summands normal cap gamma sub one plus normal cap gamma sub two plus ellipsis plus normal cap gamma sub n be a contour, and let f be a function that is continuous on normal cap gamma. Then the (**contour**) **integral of** bold-italic f**along** bold cap gamma, denoted by integral over normal cap gamma f of z d z, is

Start of $1

integral over normal cap gamma f of z d z equals sum with variable number of summands integral over normal cap gamma sub one f of z d z plus integral over normal cap gamma sub two f of z d z plus ellipsis plus integral over normal cap gamma sub n f of z d z full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative140)

End of $1

End of Box

### Remarks

1. It is clear that a contour can be split into smooth paths in many different ways. Fortunately, all such splittings lead to the same value for the contour integral. We omit the proof of this result, as it is straightforward but tedious.
2. When evaluating an integral along a contour normal cap gamma equals sum with variable number of summands normal cap gamma sub one plus normal cap gamma sub two plus ellipsis plus normal cap gamma sub n, we often consider each smooth path normal cap gamma sub one comma normal cap gamma sub two comma ellipsis comma normal cap gamma sub n separately, using a convenient parametrisation in each case. For example, consider the contour normal cap gamma equals sum with 3 summands normal cap gamma sub one plus normal cap gamma sub two plus normal cap gamma sub three of Figure 17. To evaluate a contour integral of the form

Start of $1

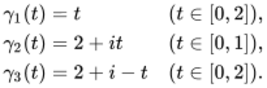
integral over normal cap gamma f of z d z equals sum with 3 summands integral over normal cap gamma sub one f of z d z plus integral over normal cap gamma sub two f of z d z plus integral over normal cap gamma sub three f of z d z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative144)

End of $1

we can use the smooth parametrisations (above) of normal cap gamma sub one, normal cap gamma sub two and normal cap gamma sub three, or we could use another convenient choice of parametrisations, such as

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative148)

End of $1

1. The alternative notation integral over normal cap gamma f is sometimes used for contour integrals when the omission of the integration variable z will cause no confusion.

Start of Example

**Example 5**

Evaluate

Start of $1

integral over normal cap gamma z squared d z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative151)

End of $1

where normal cap gamma is the contour shown in Figure 19.

Start of Figure

Displayed image

Figure 19 A contour normal cap gamma from zero to one plus i

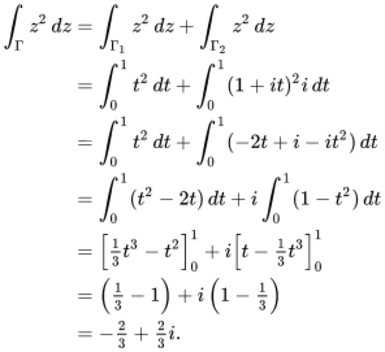
[View description - Figure 19 A contour normal cap gamma from zero to one plus i](" \l "Unit2_Session3_Description3)

End of Figure

**Solution**

We split normal cap gamma into two smooth paths normal cap gamma equals normal cap gamma sub one plus normal cap gamma sub two, where normal cap gamma sub one is the line segment from 0 to 1 with parametrisation gamma sub one of t equals t left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis, and normal cap gamma sub two is the line segment from 1 to one plus i, with parametrisation gamma sub two of t equals one plus i times t left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis. Then

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative163)

End of $1

End of Example

Notice that this answer is the same as that obtained in Example 2 for

Start of $1

integral over normal cap gamma z squared d z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative164)

End of $1

where normal cap gamma is the line segment from zero to one plus i. The reason for this will become clear when we get to Theorem 8, the Contour Independence Theorem.

Start of Exercise

**Exercise 5**

Start of Question

Evaluate

Start of $1

integral over normal cap gamma z macron d z

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative168)

End of $1

for each of the following contours normal cap gamma.

Start of Figure

Displayed image

[View description - Uncaptioned Figure](" \l "Unit2_Session3_Description4)

End of Figure

In part (b) the contour consists of a line segment and a semicircle, traversed once anticlockwise. Take negative one to be the initial (and final) point of this contour.

End of Question

[View answer - Exercise 5](" \l "Unit2_Session3_Answer3)

End of Exercise

This section will conclude by stating some rules for combining contour integrals. To prove them, we split the contour normal cap gamma into constituent smooth paths, and use the Sum Rule and Multiple Rule for real integration given in [Theorem 3](#b1-thm-1-3-new) to prove the results for each path. We omit the details.

Start of Box

**Theorem 5 Combination Rules for Contour Integrals**

Let normal cap gamma be a contour, and let f and g be functions that are continuous on normal cap gamma.

1. **Sum Rule** integral over normal cap gamma left parenthesis f of z plus g of z right parenthesis d z equals integral over normal cap gamma f of z d z plus integral over normal cap gamma g of z d z full stop
2. **Multiple Rule** integral over normal cap gamma lamda times f of z d z equals lamda times integral over normal cap gamma f of z d z comma where lamda element of double-struck cap c full stop

End of Box

## 2.3 Reverse paths and contours

We now introduce the concept of the reverse path (some texts use the name opposite path) of a smooth path normal cap gamma. This is simply the path we obtain by traversing the original path in the opposite direction, starting from the final point of the original path and finishing at the initial point of the original path. In order to define the reverse path formally, we use the fact that as t increases from a to b, so a plus b minus t decreases from b to a.

Start of Box

**Definition**

Let normal cap gamma colon gamma of t left parenthesis t element of left square bracket a comma b right square bracket right parenthesis be a smooth path. Then the **reverse path** of normal cap gamma, denoted by cap gamma tilde, is the path with parametrisation gamma tilde, where

Start of $1

gamma tilde of t equals gamma times left parenthesis a plus b minus t right parenthesis times left parenthesis t element of left square bracket a comma b right square bracket right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative210)

End of $1

End of Box

Note that the initial point gamma tilde of a of cap gamma tilde is the final point gamma of b of normal cap gamma, and the final point gamma tilde of b of cap gamma tilde is the initial point gamma of a of normal cap gamma (see Figure 20). The path cap gamma tilde is smooth because normal cap gamma is smooth. Also note that, as sets, normal cap gamma and cap gamma tilde are the same.

Start of Figure

Displayed image

Figure 20 (a) A smooth path normal cap gamma and (b) its reverse path cap gamma tilde

[View description - Figure 20 (a) A smooth path normal cap gamma and (b) its reverse path cap gamma ...](" \l "Unit2_Session3_Description5)

End of Figure

Start of Exercise

**Exercise 6**

Start of Question

Write down the reverse path of the path normal cap gamma with parametrisation

Start of $1

gamma of t equals two plus i minus t times left parenthesis t element of left square bracket zero comma two right square bracket right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative226)

End of $1

End of Question

[View answer - Exercise 6](" \l "Unit2_Session3_Answer4)

End of Exercise

We can also define a reverse contour. This is done in the natural way – namely by reversing each of the constituent smooth paths of a contour and reversing the order in which they are traversed.

Start of Box

**Definition**

Let normal cap gamma equals sum with variable number of summands normal cap gamma sub one plus normal cap gamma sub two plus ellipsis plus normal cap gamma sub n be a contour. The **reverse contour** cap gamma tilde of normal cap gamma is

Start of $1

cap gamma tilde equals sum with variable number of summands cap gamma tilde sub n plus cap gamma tilde sub n minus one plus ellipsis plus cap gamma tilde sub one full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative235)

End of $1

End of Box

A contour and its reverse contour are illustrated in Figure 21.

Start of Figure

Displayed image

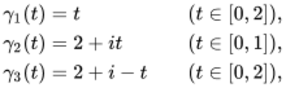
Figure 21 A contour sum with 3 summands normal cap gamma sub one plus normal cap gamma sub two plus normal cap gamma sub three and its reverse contour sum with 3 summands cap gamma tilde sub three plus cap gamma tilde sub two plus cap gamma tilde sub one

[View description - Figure 21 A contour sum with 3 summands normal cap gamma sub one plus normal cap ...](" \l "Unit2_Session3_Description6)

End of Figure

As an example, if normal cap gamma equals sum with 3 summands normal cap gamma sub one plus normal cap gamma sub two plus normal cap gamma sub three is the contour from 0 to i in Figure 22(a), with smooth parametrisations

Start of $1

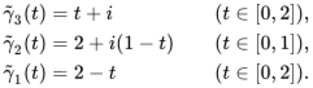


[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative240)

End of $1

then cap gamma tilde equals sum with 3 summands cap gamma tilde sub three plus cap gamma tilde sub two plus cap gamma tilde sub one is the contour from i to 0 in Figure 22(b), with smooth parametrisations

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative243)

End of $1

Start of Figure

Displayed image

Figure 22 (a) The contour normal cap gamma equals sum with 3 summands normal cap gamma sub one plus normal cap gamma sub two plus normal cap gamma sub three (b) The reverse contour cap gamma tilde equals sum with 3 summands cap gamma tilde sub three plus cap gamma tilde sub two plus cap gamma tilde sub one

[View description - Figure 22 (a) The contour normal cap gamma equals sum with 3 summands normal cap ...](" \l "Unit2_Session3_Description7)

End of Figure

Start of Example

**Example 6**

Evaluate

Start of $1

integral over cap gamma tilde z macron d z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative246)

End of $1

where cap gamma tilde is the reverse path of the line segment normal cap gamma from zero to one plus i.

**Solution**

We use the standard parametrisation

Start of $1

gamma of t equals left parenthesis one plus i right parenthesis times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative251)

End of $1

of normal cap gamma. For the reverse path cap gamma tilde, the corresponding parametrisation is

Start of $1

equation sequence part 1 gamma tilde of t equals part 2 gamma times left parenthesis one minus t right parenthesis equals part 3 left parenthesis one plus i right parenthesis times left parenthesis one minus t right parenthesis times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative254)

End of $1

Then gamma tilde times super prime times left parenthesis t right parenthesis equals negative left parenthesis one plus i right parenthesis, so we substitute

Start of $1

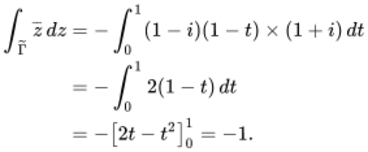
z equals left parenthesis one plus i right parenthesis times left parenthesis one minus t right parenthesis comma z macron equals left parenthesis one minus i right parenthesis times left parenthesis one minus t right parenthesis and d times z equals negative left parenthesis one plus i right parenthesis times d times t

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative256)

End of $1

to give

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative257)

End of $1

End of Example

In [Example 3](#b1-exa2-1) we saw that

Start of $1

integral over normal cap gamma z macron d z equals one comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative258)

End of $1

which is the negative of the value negative one that we obtained in Example 6. This illustrates the general result that if we integrate a function along a reverse contour cap gamma tilde, then the answer is the negative of the integral of the function along normal cap gamma.

Start of Box

**Theorem 6 Reverse Contour Theorem**

Let normal cap gamma be a contour, and let f be a function that is continuous on normal cap gamma. Then the integral of f along the reverse contour cap gamma tilde of normal cap gamma satisfies

Start of $1

integral over cap gamma tilde f of z d z equals negative integral over normal cap gamma f of z d z full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative268)

End of $1

End of Box

Start of Proof

Proof The proof is in two parts. We first prove the result in the case when normal cap gamma is a smooth path, and then extend the proof to contours.

 Let normal cap gamma colon gamma of t left parenthesis t element of left square bracket a comma b right square bracket right parenthesis be a smooth path. Then the parametrisation of cap gamma tilde is

Start of $1

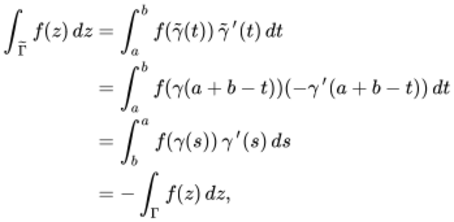
gamma tilde of t equals gamma times left parenthesis a plus b minus t right parenthesis times left parenthesis t element of left square bracket a comma b right square bracket right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative272)

End of $1

It follows that gamma tilde times super prime times left parenthesis t right parenthesis equals negative gamma times super prime times left parenthesis a plus b minus t right parenthesis, by the Chain Rule, so

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative274)

End of $1

where, in the second-to-last line, we have made the real substitution

Start of $1

s equals a plus b minus t comma d times s equals negative d times t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative275)

End of $1

 To extend the proof to a general contour normal cap gamma, we argue as follows.

Let normal cap gamma equals sum with variable number of summands normal cap gamma sub one plus normal cap gamma sub two plus ellipsis plus normal cap gamma sub n, for smooth paths normal cap gamma sub one comma normal cap gamma sub two comma ellipsis comma normal cap gamma sub n. Then

Start of $1

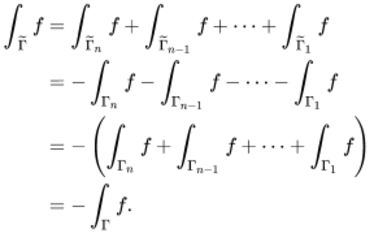
cap gamma tilde equals sum with variable number of summands cap gamma tilde sub n plus cap gamma tilde sub n minus one plus ellipsis plus cap gamma tilde sub one comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative279)

End of $1

and we can apply part (a) to see that

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative280)

End of $1

âˆŽ

End of Proof

In [Example 4](#b1-exa2-2) we saw that

Start of $1

integral over normal cap gamma one divided by z d z equals two times pi times i comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative281)

End of $1

where normal cap gamma is the unit circle z colon absolute value of z equals one. The next exercise asks you to check Theorem 6 for this contour integral.

Start of Exercise

**Exercise 7**

Start of Question

Verify that

Start of $1

integral over cap gamma tilde one divided by z d z equals negative two times pi times i comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative284)

End of $1

where normal cap gamma is the unit circle.

End of Question

[View answer - Exercise 7](" \l "Unit2_Session3_Answer5)

End of Exercise

## 2.4 Further exercises

Here are some further exercises to end this section.

Start of Exercise

**Exercise 8**

Start of Question

Evaluate the following integrals (using the standard parametrisation of the path normal cap gamma in each case).

* 1. integral over normal cap gamma z d z,
  2. integral over normal cap gamma Im of z times d times z,
  3. integral over normal cap gamma z macron d z,

where normal cap gamma is the line segment from 1 to i.

* 1. integral over normal cap gamma z macron d z,
  2. integral over normal cap gamma z squared d z,

where normal cap gamma is the unit circle z colon absolute value of z equals one.

* 1. integral over normal cap gamma one divided by z d z,
  2. integral over normal cap gamma absolute value of z d z,

where normal cap gamma is the upper half of the circle with centre 0 and radius 2 traversed from 2 to negative two.

End of Question

[View answer - Exercise 8](" \l "Unit2_Session3_Answer6)

End of Exercise

Start of Exercise

**Exercise 9**

Start of Question

Evaluate

Start of $1

integral over normal cap gamma Re of z times d times z

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative336)

End of $1

for each of the following contours normal cap gamma from 0 to one plus i.

Start of Figure

Displayed image

[View description - Uncaptioned Figure](" \l "Unit2_Session3_Description8)

End of Figure

End of Question

[View answer - Exercise 9](" \l "Unit2_Session3_Answer7)

End of Exercise

## 3 Evaluating contour integrals

After working through this section, you should be able to:

* state and use the Fundamental Theorem of Calculus for contour integrals
* state and use the Contour Independence Theorem
* use the technique of Integration by Parts
* state and use the Closed Contour Theorem, the Grid Path Theorem, the Zero Derivative Theorem and the Paving Theorem.

## 3.1 The Fundamental Theorem of Calculus

In [Example 5](#b1-exa2-3) we saw that

Start of $1

integral over normal cap gamma z squared d z equals negative two divided by three plus two divided by three times i comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative1)

End of $1

where normal cap gamma is the contour shown in Figure 23. Our method was to write down a smooth parametrisation for each of the two line segments, replace z in the integral by these parametrisations, and then integrate.

Start of Figure

Displayed image

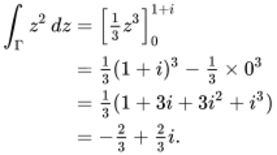
Figure 23 A contour normal cap gamma from zero to one plus i

[View description - Figure 23 A contour normal cap gamma from zero to one plus i](" \l "Unit2_Session4_Description1)

End of Figure

It is, however, tempting to approach this integral as you would a corresponding real integral and write

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative7)

End of $1

The Fundamental Theorem of Calculus for contour integrals tells us that this method of evaluation is permissible under certain conditions. Before stating it, we need the idea of a primitive of a complex function, which is defined in a similar way to the primitive of a real function ([Section 1.3](#b1-s1-ss2)).

Start of Box

**Definition**

Let f and cap f be functions defined on a region script cap r. Then cap f is a **primitive of** bold-italic f**on** bold-script cap r if cap f is analytic on script cap r and

Start of $1

cap f super prime of z equals f of z comma for all z element of script cap r full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative16)

End of $1

End of Box

The function cap f is also called an antiderivative or indefinite integral of f on script cap r.

For example, cap f of z equals one divided by three times z cubed is a primitive of f of z equals z squared on double-struck cap c, since cap f is analytic on double-struck cap c and cap f super prime of z equals z squared, for all z element of double-struck cap c. Another primitive is cap f of z equals one divided by three times z cubed plus two times i; indeed, any function of the form cap f of z equals one divided by three times z cubed plus c, where c element of double-struck cap c, is a primitive of f on double-struck cap c.

Start of Exercise

**Exercise 10**

Start of Question

Write down a primitive cap f of each of the following functions f on the given region script cap r.

1. f of z equals e super three times i times z comma script cap r equals double-struck cap c
2. f of z equals left parenthesis one plus i times z right parenthesis super negative two comma script cap r equals double-struck cap c minus i
3. f of z equals z super negative one comma script cap r equals z colon Re of z greater than zero

End of Question

[View answer - Exercise 10](" \l "Unit2_Session4_Answer1)

End of Exercise

We now state the Fundamental Theorem of Calculus for contour integrals, which gives us a quick way of evaluating a contour integral of a function with a primitive that we can determine. The theorem will be proved later in this section.

Start of Box

**Theorem 7 Fundamental Theorem of Calculus**

Let f be a function that is continuous and has a primitive cap f on a region script cap r, and let normal cap gamma be a contour in script cap r with initial point alpha and final point beta. Then

Start of $1

integral over normal cap gamma f of z d z equals cap f of beta minus cap f of alpha full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative48)

End of $1

End of Box

We often use the notation

Start of $1

left square bracket cap f of z right square bracket sub alpha super beta equals cap f of beta minus cap f of alpha full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative49)

End of $1

Some texts write cap f of z vertical line sub alpha super beta instead of left square bracket cap f of z right square bracket sub alpha super beta.

For an example of the use of the Fundamental Theorem of Calculus, observe that if f of z equals z squared, then f is continuous on double-struck cap c and has a primitive cap f of z equals one divided by three times z cubed there. Hence, for the contour normal cap gamma in Figure 23, we can write

Start of $1

equation sequence part 1 integral over normal cap gamma z squared d z equals part 2 left square bracket one divided by three times z cubed right square bracket sub zero super one plus i equals part 3 one divided by three times left parenthesis one plus i right parenthesis cubed minus one divided by three multiplication zero cubed equals part 4 negative two divided by three plus two divided by three times i full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative57)

End of $1

Start of Exercise

**Exercise 11**

Start of Question

Use the Fundamental Theorem of Calculus to evaluate

Start of $1

integral over normal cap gamma e super three times i times z d z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative58)

End of $1

where normal cap gamma is the semicircular path shown in Figure 24.

Start of Figure

Displayed image

Figure 24 A semicircular path normal cap gamma from two to negative two

[View description - Figure 24 A semicircular path normal cap gamma from two to negative two](" \l "Unit2_Session4_Description2)

End of Figure

End of Question

[View answer - Exercise 11](" \l "Unit2_Session4_Answer2)

End of Exercise

You have seen that

Start of $1

integral over normal cap gamma z squared d z equals negative two divided by three plus two divided by three times i

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative74)

End of $1

both when normal cap gamma is the contour in Figure 23 and also when normal cap gamma is the line segment from zero to one plus i (see [Example 2](#b1-exa2-0)). This is not a coincidence: in fact, it is a particular case of the following important consequence of the Fundamental Theorem of Calculus.

Start of Box

**Theorem 8 Contour Independence Theorem**

Let f be a function that is continuous and has a primitive cap f on a region script cap r, and let normal cap gamma sub one and normal cap gamma sub two be contours in script cap r with the same initial point alpha and the same final point beta. Then

Start of $1

integral over normal cap gamma sub one f of z d z equals integral over normal cap gamma sub two f of z d z full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative87)

End of $1

End of Box

Start of Proof

Proof By the Fundamental Theorem of Calculus for contour integrals, the value of each of these integrals is cap f of beta minus cap f of alpha.

End of Proof

The idea that a contour integral may, under suitable hypotheses, depend only on the endpoints of the contour (and not on the contour itself) has great significance.

Start of Exercise

**Exercise 12**

Start of Question

Use the Fundamental Theorem of Calculus to evaluate the following integrals.

1. integral over normal cap gamma e super negative pi times z d z, where normal cap gamma is any contour from negative i to i.
2. integral over normal cap gamma left parenthesis three times z minus one right parenthesis squared d z, where normal cap gamma is any contour from 2 to two times i plus one divided by three.
3. integral over normal cap gamma hyperbolic sine of z times d times z, where normal cap gamma is any contour from i to 1.
4. integral over normal cap gamma e super sine of z times cosine of z times d times z, where normal cap gamma is any contour from 0 to pi solidus two.
5. integral over normal cap gamma sine of z divided by cosine squared of z d z, where normal cap gamma is any contour from 0 to pi lying in double-struck cap c minus left parenthesis n plus one divided by two right parenthesis times pi colon n element of double-struck cap z.

End of Question

[View answer - Exercise 12](" \l "Unit2_Session4_Answer3)

End of Exercise

Next we give a version of Integration by Parts for contour integrals.

Start of Box

**Theorem 9 Integration by Parts**

Let f and g be functions that are analytic on a region script cap r, and suppose that f super prime and g super prime are continuous on script cap r. Let normal cap gamma be a contour in script cap r with initial point alpha and final point beta. Then

Start of $1

integral over normal cap gamma f of z times g super prime of z d z equals left square bracket f of z times g of z right square bracket sub alpha super beta minus integral over normal cap gamma f super prime of z times g of z d z full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative173)

End of $1

End of Box

Start of Proof

Proof Let cap h of z equals f of z times g of z and h of z equals f super prime of z times g of z plus f of z times g super prime of z. Then h is continuous on script cap r, by hypothesis. Also, h has primitive cap h, since cap h is analytic on script cap r and

cap h super prime of z equals h of z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative182)

by the Product Rule for differentiation. It follows from the Fundamental Theorem of Calculus that integral over normal cap gamma h of z d z equals left square bracket cap h of z right square bracket sub alpha super beta semicolon

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative183)

that is, integral over normal cap gamma left parenthesis f super prime of z times g of z plus f of z times g super prime of z right parenthesis d z equals left square bracket f of z times g of z right square bracket sub alpha super beta full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative184)

Using the Sum Rule ([Theorem 5(a)](#b1-thm2-2)) and rearranging the resulting equation, we obtain integral over normal cap gamma f of z times g super prime of z d z equals left square bracket f of z times g of z right square bracket sub alpha super beta minus integral over normal cap gamma f super prime of z times g of z d z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative185)

as required. âˆŽ

End of Proof

Start of Example

**Example 7**

Use Integration by Parts to evaluate

Start of $1

integral over normal cap gamma z times e super two times z d z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative186)

End of $1

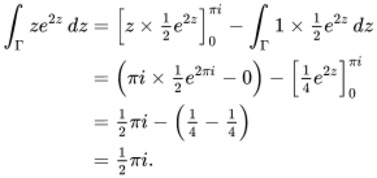
where normal cap gamma is any contour from zero to pi times i.

**Solution**

We take f of z equals z, g of z equals one divided by two times e super two times z and script cap r equals double-struck cap c. Then f and g are analytic on script cap r, and f super prime of z equals one and g super prime of z equals e super two times z are continuous on script cap r.

Integrating by parts, we obtain

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative199)

End of $1

End of Example

Start of Exercise

**Exercise 13**

Start of Question

Use Integration by Parts to evaluate the following integrals.

1. integral over normal cap gamma z times hyperbolic cosine of z times d times z comma where normal cap gamma is any contour from 0 to pi times i.
2. integral over normal cap gamma Log of z times d times z, where normal cap gamma is any contour from 1 to i lying in the cut plane double-struck cap c minus x element of double-struck cap r colon x less than or equals zero.

(Hint: For part (b), take f of z equals Log of z and g of z equals z.)

End of Question

[View answer - Exercise 13](" \l "Unit2_Session4_Answer4)

End of Exercise

The Fundamental Theorem of Calculus is a useful tool when the function f being integrated has an easily determined primitive cap f. However, if the function f has no primitive, or if we are unable to find one, then we have to resort to the definition of an integral and use parametrisation. For example, we cannot use the Fundamental Theorem of Calculus to evaluate

Start of $1

integral over normal cap gamma z macron d z

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative232)

End of $1

along any contour, since the function f of z equals z macron has no primitive on any region.

To see why this is so, suppose that f is a function that is defined on a region in the complex plane. We observe that if *f*is not differentiable, then *f*has no primitive *cap f*. This is because any differentiable complex function can be differentiated as many times as we like. Thus, if f has a primitive cap f, then cap f is differentiable with cap f super prime equals f. Hence f is also differentiable.

It follows that we cannot use the Fundamental Theorem of Calculus to evaluate integrals of non-differentiable functions such as

Start of $1

multiline equation row 1 Blank z long right arrow from bar z macron comma z long right arrow from bar Re of z comma z long right arrow from bar Im of z and z long right arrow from bar absolute value of z full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative243)

End of $1

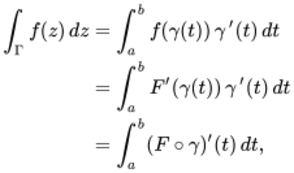
We conclude this section by proving the Fundamental Theorem of Calculus.

Start of Proof

ProofThe proof of the Fundamental Theorem of Calculus is in two parts. We first prove the result in the case when normal cap gamma is a smooth path, and then extend the proof to contours.

 Let normal cap gamma colon gamma of t left parenthesis t element of left square bracket a comma b right square bracket right parenthesis be a smooth path. Then

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative246)

End of $1

by the Chain Rule. Now, if we write left parenthesis cap f ring operator gamma right parenthesis times left parenthesis t right parenthesis as a sum of its real and imaginary parts u of t plus i times v of t, then

Start of $1

integral over a under b left parenthesis cap f ring operator gamma right parenthesis super prime times left parenthesis t right parenthesis d t equals integral over a under b u super prime of t d t plus i times integral over a under b v super prime of t d t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative249)

End of $1

The Fundamental Theorem of Calculus for real integrals ([Theorem 2](#b1-s1-ftc)) tells us that

Start of $1

integral over a under b u super prime of t d t equals u of b minus u of a and integral over a under b v super prime of t d t equals v of b minus v of a full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative250)

End of $1

Hence

Start of $1

equation sequence part 1 integral over normal cap gamma f of z d z equals part 2 left parenthesis u of b minus u of a right parenthesis plus i times left parenthesis v of b minus v of a right parenthesis equals part 3 cap f of beta minus cap f of alpha comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative251)

End of $1

since beta equals gamma of b and alpha equals gamma of a.

 To extend the proof to a general contour normal cap gamma with initial point alpha and final point beta, we argue as follows.

Let normal cap gamma equals sum with variable number of summands normal cap gamma sub one plus normal cap gamma sub two plus ellipsis plus normal cap gamma sub n, for smooth paths normal cap gamma sub one comma normal cap gamma sub two comma ellipsis comma normal cap gamma sub n, and let the initial and final points of normal cap gamma sub k be alpha sub k and beta sub k, for k equals one comma two comma ellipsis comma n. Then

Start of $1

alpha sub one equals alpha comma alpha sub two equals beta sub one comma ellipsis comma alpha sub n equals beta sub n minus one comma beta sub n equals beta full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative263)

End of $1

By part (a),

Start of $1

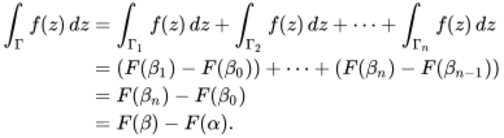
equation sequence part 1 integral over normal cap gamma sub k f of z d z equals part 2 cap f of beta sub k minus cap f of alpha sub k equals part 3 cap f of beta sub k minus cap f of beta sub k minus one comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative264)

End of $1

for k equals one comma two comma ellipsis comma n (where beta sub zero equals alpha). Hence

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative267)

End of $1

âˆŽ

End of Proof

## 3.2 Further exercises

Here are some further exercises to end this section.

Start of Exercise

**Exercise 14**

Start of Question

For each of the following functions f, evaluate

Start of $1

integral over normal cap gamma f of z d z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative269)

End of $1

where normal cap gamma is any contour from negative i to i.

1. f of z equals one
2. f of z equals z
3. f of z equals five times z super four plus three times i times z squared
4. f of z equals left parenthesis one plus two times i times z right parenthesis super nine
5. f of z equals e super negative i times z
6. f of z equals sine of z
7. f of z equals z times e super z squared
8. f of z equals z cubed times hyperbolic cosine of z super four
9. f of z equals z times e super z

End of Question

[View answer - Exercise 14](" \l "Unit2_Session4_Answer5)

End of Exercise

Start of Exercise

**Exercise 15**

Start of Question

Evaluate the following integrals. (In each case pay special attention to the hypotheses of the theorems you use.)

1. integral over normal cap gamma one divided by z d z,

where normal cap gamma is the arc of the circle z colon absolute value of z equals one from negative i to i passing through 1.

1. integral over normal cap gamma Square root of z d z, where normal cap gamma is as in part (a).
2. integral over normal cap gamma sine squared of z times d times z, where normal cap gamma is the unit circle z colon absolute value of z equals one.
3. integral over normal cap gamma one divided by z cubed d z, where normal cap gamma is the circle z colon absolute value of z equals 27.

(Hint: For part (c), use the identity sine squared of z equals one divided by two times left parenthesis one minus cosine of two times z right parenthesis.)

End of Question

[View answer - Exercise 15](" \l "Unit2_Session4_Answer6)

End of Exercise

Start of Exercise

**Exercise 16**

Start of Question

Construct a grid path from alpha to beta in the domain of the function tangent, for each of the following cases.

1. alpha equals one, beta equals six
2. alpha equals pi divided by two plus two times i, beta equals negative three times pi divided by two minus i

End of Question

[View answer - Exercise 16](" \l "Unit2_Session4_Answer7)

End of Exercise

## 4 Summary of Session 2

In this session you have seen how the idea of integration of real functions can be extended to the integration of complex functions along paths in the complex plane. You have seen the surprising result that for a continuous function the integral is independent of the precise path taken.

## Course conclusion

Well done on completing this course, Introduction to complex analysis. As well as being able to understand the terms and definitions, and use the results introduced, you should also find that your skills in understanding complex mathematical texts are improving.

You should now be able to:

* use the definition of derivative to show that a given function is or is not differentiable at a point
* use the Cauchy–Riemann equations to show that a function is or is not differentiable at a point
* interpret the derivative of a complex function at a point as a rotation and a scaling of a small disc
* appreciate how complex integrals can be defined by analogy with real integrals
* define the integral of a complex function along a contour and evaluate such integrals
* state and use several key theorems to evaluate contour integrals.

This OpenLearn course is an extract from the Open University course [M337 Complex analysis](https://www.open.ac.uk/courses/modules/m337).

## Acknowledgements

This free course was written by the Open University School of Mathematics and Statistics.

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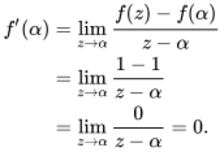
## Solutions

## Exercise 1

#### Answer

1. f of z equals one is defined on the whole of double-struck cap c, so let alpha element of double-struck cap c. Then

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative61)

End of $1

Since alpha is an arbitrary complex number, f is differentiable on the whole of double-struck cap c, and its derivative is the zero function

Start of $1

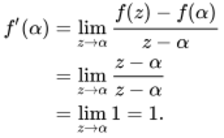
f super prime of z equals zero times left parenthesis z element of double-struck cap c right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative65)

End of $1

1. f of z equals z is defined on the whole of double-struck cap c, so let alpha element of double-struck cap c. Then

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative69)

End of $1

Since alpha is an arbitrary complex number, f is differentiable on the whole of double-struck cap c, and its derivative is the constant function

Start of $1

f super prime of z equals one times left parenthesis z element of double-struck cap c right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative73)

End of $1

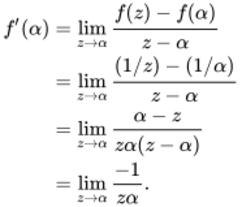
[Back to - Exercise 1](" \l "Unit1_Session2_Exercise1)

## Exercise 2

#### Answer

The domain of f of z equals one solidus z is the region double-struck cap c minus zero. Since f super prime of alpha cannot exist unless f is defined at alpha, we confine our attention to alpha not equals zero. Then

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative88)

End of $1

Now z long right arrow from bar negative one solidus left parenthesis z times alpha right parenthesis is a basic continuous function with domain double-struck cap c minus zero, so we see that

Start of $1

equation sequence part 1 f super prime of alpha equals part 2 lim over z right arrow alpha of negative one divided by z times alpha equals part 3 negative one divided by alpha squared full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative91)

End of $1

Since alpha is an arbitrary non-zero complex number, the derivative of f is

Start of $1

f super prime of z equals negative one divided by z squared times left parenthesis z not equals zero right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative94)

End of $1

The function f is not entire since its domain is not double-struck cap c.

[Back to - Exercise 2](" \l "Unit1_Session2_Exercise2)

## Exercise 3

#### Answer

1. True.
2. False. (The set must be a region.)

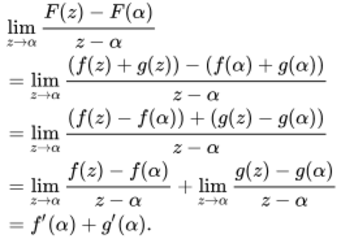
[Back to - Exercise 3](" \l "Unit1_Session2_Exercise3)

## Exercise 4

#### Answer

1. Let cap f equals f plus g. Then

Start of $1

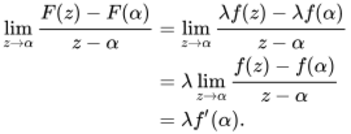


[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative202)

End of $1

1. Let cap f equals lamda times f, for lamda element of double-struck cap c. Then

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative205)

End of $1

[Back to - Exercise 4](" \l "Unit1_Session2_Exercise4)

## Exercise 5

#### Answer

1. By the corollary on differentiating polynomial functions, we have

Start of $1

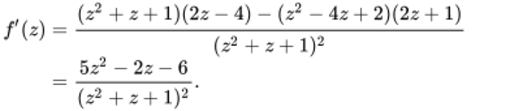
f super prime of z equals four times z cubed plus nine times z squared minus two times z plus four times left parenthesis z element of double-struck cap c right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative231)

End of $1

1. By the Quotient Rule,

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative232)

End of $1

Now, sum with 3 summands z squared plus z plus one equals zero if and only if z equals negative one divided by two times left parenthesis one plus minus Square root of three times i right parenthesis, so the domain of f super prime is

Start of $1

double-struck cap c minus negative one divided by two times left parenthesis one plus Square root of three times i right parenthesis comma negative one divided by two times left parenthesis one minus Square root of three times i right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative236)

End of $1

[Back to - Exercise 5](" \l "Unit1_Session2_Exercise5)

## Exercise 6

#### Answer

The function Arg is discontinuous at each point of the negative real axis. It follows that Log is discontinuous at each point of the negative real axis, and hence that there are no points on it at which Log is differentiable.

[Back to - Exercise 6](" \l "Unit1_Session2_Exercise6)

## Exercise 7

#### Answer

Let z sub n equals alpha times exp of i solidus n, n equals one comma two comma ellipsis. Then left parenthesis z sub n right parenthesis tends to alpha along the circumference of the circle, and

Start of $1

equation sequence part 1 lim over n right arrow normal infinity of absolute value of z sub n minus absolute value of alpha divided by z sub n minus alpha equals part 2 lim over n right arrow normal infinity of absolute value of alpha minus absolute value of alpha divided by z sub n minus alpha equals part 3 zero full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative315)

End of $1

Now let z sub n super prime equals alpha times left parenthesis one plus one solidus n right parenthesis, n equals one comma two comma ellipsis. Then left parenthesis z sub n super prime right parenthesis tends to alpha along the ray from zero through alpha, and

Start of $1

multiline equation row 1 Blank equation sequence part 1 lim over n right arrow normal infinity of absolute value of z sub n super prime minus absolute value of alpha divided by z sub n super prime minus alpha equals part 2 lim over n right arrow normal infinity of absolute value of alpha times left parenthesis one plus one solidus n right parenthesis minus absolute value of alpha divided by alpha times left parenthesis one plus one solidus n right parenthesis minus alpha equals part 3 absolute value of alpha divided by alpha full stop Blank

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative322)

End of $1

Since absolute value of alpha solidus alpha not equals zero for alpha not equals zero, these two limits do not agree. It follows that f of z equals absolute value of z is not differentiable at alpha not equals zero.

[Back to - Exercise 7](" \l "Unit1_Session2_Exercise7)

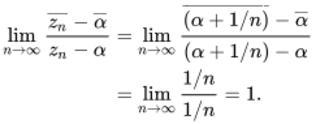
## Exercise 8

#### Answer

Let alpha be an arbitrary complex number. Directions of paths parallel to the imaginary axis through alpha are reversed by f, while directions of paths parallel to the real axis are not. This suggests looking at the sequences z sub n equals alpha plus one solidus n and z sub n super prime equals alpha plus i solidus n, n equals one comma two comma ellipsis.

First let z sub n equals alpha plus one solidus n; then

Start of $1

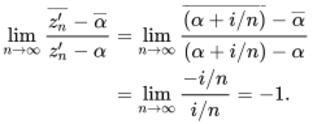


[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative339)

End of $1

Now let z sub n super prime equals alpha plus i solidus n; then

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative341)

End of $1

Since these two limits do not agree, and since alpha is arbitrary, it follows that there are no points of double-struck cap c at which f of z equals z macron is differentiable.

[Back to - Exercise 8](" \l "Unit1_Session2_Exercise8)

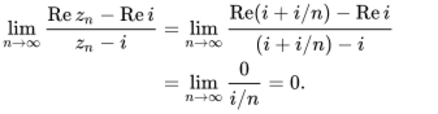
## Exercise 9

#### Answer

1. The fact that Re of z is constant along the imaginary axis, but variable parallel to the real axis, suggests that Re is not differentiable at i (or anywhere else, for that matter). It also suggests looking at the sequences z sub n equals i plus i solidus n and z sub n super prime equals i plus one solidus n, n equals one comma two comma ellipsis.

First let z sub n equals i plus i solidus n; then

Start of $1

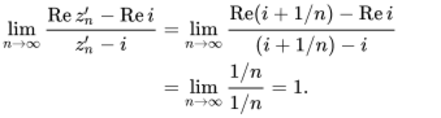


[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative373)

End of $1

Now let z sub n super prime equals i plus one solidus n; then

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative375)

End of $1

Since these two limits do not agree, it follows that Re is not differentiable at i.

1. f is a polynomial function, so f super prime of z equals four times z plus three for all z element of double-struck cap c. Thus f super prime of i equals three plus four times i.
2. f is not differentiable at i, since it is not continuous at i.

[Back to - Exercise 9](" \l "Unit1_Session2_Exercise9)

## Exercise 10

#### Answer

To a close approximation, a small disc centred at i is mapped by f to a small disc centred at

Start of $1

equation sequence part 1 f of i equals part 2 four times i plus three divided by two times i squared plus one equals part 3 negative three minus four times i full stop

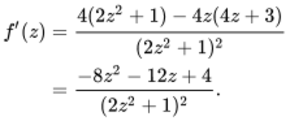
[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative468)

End of $1

In the process the disc is scaled by the factor absolute value of f super prime of i and rotated through the angle Arg of f super prime of i.

By the Quotient Rule,

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative471)

End of $1

So

Start of $1

equation sequence part 1 f super prime of i equals part 2 negative eight times i squared minus 12 times i plus four divided by left parenthesis two times i squared plus one right parenthesis squared equals part 3 12 minus 12 times i full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative472)

End of $1

This has modulus 12 times Square root of two and principal argument negative pi solidus four.

So f scales the disc by the factor 12 times Square root of two and rotates it clockwise through the angle pi solidus four.

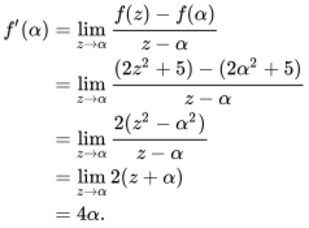
[Back to - Exercise 10](" \l "Unit1_Session2_Exercise10)

## Exercise 11

#### Answer

The function f of z equals two times z squared plus five is defined on the whole of double-struck cap c. Let alpha element of double-struck cap c. Then

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative482)

End of $1

Since alpha is an arbitrary complex number, f is differentiable on the whole of double-struck cap c, and the derivative is the function

Start of $1

f super prime of z equals four times z times left parenthesis z element of double-struck cap c right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative486)

End of $1

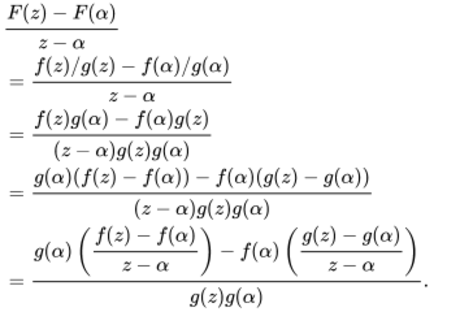
[Back to - Exercise 11](" \l "Unit1_Session2_Exercise11)

## Exercise 12

#### Answer

Let cap f equals f solidus g. Then

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative488)

End of $1

Using the Combination Rules for limits of functions, the continuity of g, and the fact that g of alpha not equals zero, we can take limits to obtain

Start of $1

cap f super prime of alpha equals g of alpha times f super prime of alpha minus f of alpha times g super prime of alpha divided by left parenthesis g of alpha right parenthesis squared full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative491)

End of $1

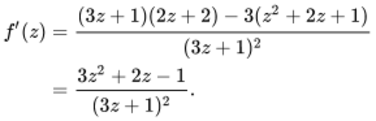
[Back to - Exercise 12](" \l "Unit1_Session2_Exercise12)

## Exercise 13

#### Answer

1. By the Combination Rules,

Start of $1



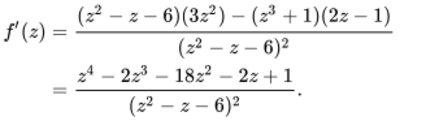
[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative498)

End of $1

The domain of f super prime is double-struck cap c minus left curly bracket negative one solidus three right curly bracket.

1. By the Combination Rules,

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative501)

End of $1

Since z squared minus z minus six equals left parenthesis z plus two right parenthesis times left parenthesis z minus three right parenthesis, the domain of f super prime is double-struck cap c minus negative two comma three.

1. By the Reciprocal Rule,

Start of $1

f super prime of z equals negative left parenthesis two times z plus two right parenthesis divided by left parenthesis sum with 3 summands z squared plus two times z plus two right parenthesis squared full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative505)

End of $1

The roots of sum with 3 summands z squared plus two times z plus two are negative two plus minus Square root of negative four divided by two equals negative one plus minus i. The domain of f super prime is therefore double-struck cap c minus negative one plus i comma negative one minus i.

1. By the Sum Rule and the rule for differentiating integer powers,

Start of $1

f super prime of z equals two times z plus five minus one divided by z squared minus two divided by z cubed full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative510)

End of $1

The domain of f super prime is double-struck cap c minus zero.

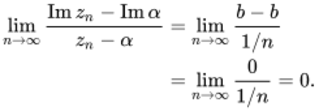
[Back to - Exercise 13](" \l "Unit1_Session2_Exercise13)

## Exercise 14

#### Answer

Consider an arbitrary complex number alpha equals a plus i times b, where a comma b element of double-struck cap r. Let z sub n equals alpha plus one solidus n, n equals one comma two comma ellipsis. Then z sub n right arrow alpha, and

Start of $1

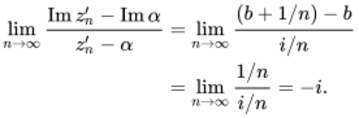


[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative520)

End of $1

Now let z sub n super prime equals alpha plus i solidus n, n equals one comma two comma ellipsis. Then z sub n super prime right arrow alpha, and

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative524)

End of $1

Since the two limits do not agree, it follows that Im fails to be differentiable at each point of double-struck cap c.

[Back to - Exercise 14](" \l "Unit1_Session2_Exercise14)

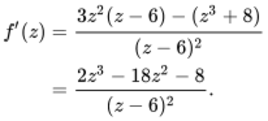
## Exercise 15

#### Answer

To a close approximation, a small disc centred at 2 is mapped by f to small disc centred at f of two equals negative four. In the process, the disc is scaled by the factor absolute value of f super prime of two and rotated through the angle Arg of f super prime of two.

By the Quotient Rule,

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session2_Alternative532)

End of $1

So f scales the disc by the factor 4 and rotates it anticlockwise through the angle pi.

[Back to - Exercise 15](" \l "Unit1_Session2_Exercise15)

## Exercise 16

#### Answer

1. Differentiating v of x comma y equals three times x squared times y minus y cubed with respect to x while keeping y fixed, we obtain

Start of $1

prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals six times x times y full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative110)

End of $1

Differentiating v with respect to y while keeping x fixed, we obtain

Start of $1

prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals three times x squared minus three times y squared full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative114)

End of $1

1. So, at left parenthesis x comma y right parenthesis equals left parenthesis two comma one right parenthesis the partial derivatives have the values

Start of $1

prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis two comma one right parenthesis equals 12 and prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis two comma one right parenthesis equals nine full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative116)

End of $1

[Back to - Exercise 16](" \l "Unit1_Session3_Exercise1)

## Exercise 17

#### Answer

1. Writing f in the form

Start of $1

f times left parenthesis x plus i times y right parenthesis equals u of x comma y plus i times v of x comma y comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative199)

End of $1

we obtain

Start of $1

u of x comma y equals e super x and v of x comma y equals negative e super y full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative200)

End of $1

Hence

Start of $1

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals e super x and prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals negative e super y full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative201)

End of $1

Since e super x is always positive, whereas negative e super y is always negative, the first of the Cauchy–Riemann equations fails to hold for each open x comma y close. It follows that f fails to be differentiable at all points of double-struck cap c.

1. Writing equation sequence part 1 f of z equals part 2 z macron equals part 3 x minus i times y in the form

Start of $1

f times left parenthesis x plus i times y right parenthesis equals u of x comma y plus i times v of x comma y comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative208)

End of $1

we obtain

Start of $1

u of x comma y equals x and v of x comma y equals negative y full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative209)

End of $1

Hence

Start of $1

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals one and prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals negative one full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative210)

End of $1

It follows that the first of the Cauchy–Riemann equations fails to hold for each open x comma y close, so f fails to be differentiable at all points of double-struck cap c.

[Back to - Exercise 17](" \l "Unit1_Session3_Exercise2)

## Exercise 18

#### Answer

1. From the trigonometric identities,

Start of $1

multiline equation row 1 sine of x plus i times y equals sine of x times cosine of i times y plus cosine of x times sine of i times y row 2 Blank equals sine of x times hyperbolic cosine of y plus i times cosine of x times hyperbolic sine of y comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative287)

End of $1

so f times left parenthesis x plus i times y right parenthesis equals u of x comma y plus i times v of x comma y, where

Start of $1

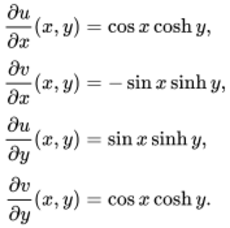
multiline equation row 1 u of x comma y equals sine of x times hyperbolic cosine of y and row 2 v of x comma y equals cosine of x times hyperbolic sine of y full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative289)

End of $1

Hence

Start of $1

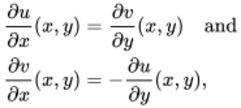


[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative290)

End of $1

These partial derivatives are defined and continuous on the whole of double-struck cap c. Furthermore,

Start of $1



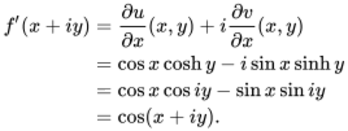
[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative292)

End of $1

so the Cauchy–Riemann equations are satisfied at every point of double-struck cap c.

By the Cauchy–Riemann Converse Theorem, f of z equals sine of z is entire, and

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative295)

End of $1

Hence f super prime has domain double-struck cap c and f super prime of z equals cosine of z.

1. Here equation sequence part 1 f times left parenthesis x plus i times y right parenthesis equals part 2 absolute value of x plus i times y squared equals part 3 x squared plus y squared, so

Start of $1

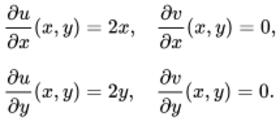
u of x comma y equals x squared plus y squared and v of x comma y equals zero full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative300)

End of $1

Hence

Start of $1



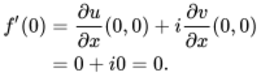
[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative301)

End of $1

The Cauchy–Riemann equations cannot be satisfied unless two times x equals zero and negative two times y equals zero, so f fails to be differentiable at all non-zero points of double-struck cap c.

However, the Cauchy–Riemann equations are satisfied at open zero comma zero close, and the partial derivatives are defined on double-struck cap c and continuous (at open zero comma zero close), so by the Cauchy–Riemann Converse Theorem, f is differentiable at zero, and

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative311)

End of $1

Thus f super prime has domain zero and f super prime of zero equals zero.

(This is the example referred to in [Section 1.1](#a4-hwt) of a function that is differentiable at a point, but not analytic at that point.)

[Back to - Exercise 18](" \l "Unit1_Session3_Exercise3)

## Exercise 19

#### Answer

1. Differentiating u of x comma y equals sum with 3 summands three times x plus x times y plus two times x squared times y squared with respect to x while keeping y fixed, we obtain

Start of $1

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals sum with 3 summands three plus y plus four times x times y squared full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative468)

End of $1

Differentiating with respect to y while keeping x fixed, we obtain

Start of $1

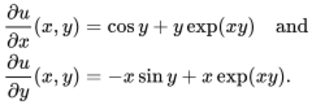
prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals x plus four times x squared times y full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative471)

End of $1

1. Here u of x comma y equals x times cosine of y plus exp of x times y, so

Start of $1

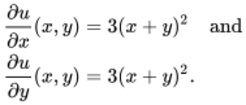


[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative473)

End of $1

1. Here u of x comma y equals left parenthesis x plus y right parenthesis cubed, so

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative475)

End of $1

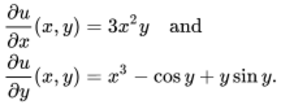
[Back to - Exercise 19](" \l "Unit1_Session3_Exercise4)

## Exercise 20

#### Answer

1. Here u of x comma y equals x cubed times y minus y times cosine of y, so

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative482)

End of $1

So, at left parenthesis x comma y right parenthesis equals left parenthesis one comma zero right parenthesis the partial derivatives have the values

Start of $1

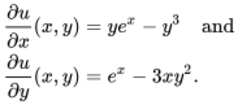
prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis one comma zero right parenthesis equals zero and prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis one comma zero right parenthesis equals zero full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative484)

End of $1

1. Here u of x comma y equals y times e super x minus x times y cubed, so

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative486)

End of $1

So, at left parenthesis x comma y right parenthesis equals left parenthesis one comma zero right parenthesis the partial derivatives have the values

Start of $1

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis one comma zero right parenthesis equals zero and prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis one comma zero right parenthesis equals e full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative488)

End of $1

[Back to - Exercise 20](" \l "Unit1_Session3_Exercise5)

## Exercise 21

#### Answer

Since u of x comma y equals x squared plus two times x times y, it follows that

Start of $1

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals two times x plus two times y and prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals two times x full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative494)

End of $1

The gradient of the graph at left parenthesis x comma y right parenthesis equals left parenthesis one comma two right parenthesis in the x-direction is

Start of $1

equation sequence part 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis one comma two right parenthesis equals part 2 two multiplication one plus two multiplication two equals part 3 six full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative497)

End of $1

The gradient of the graph at left parenthesis x comma y right parenthesis equals left parenthesis one comma two right parenthesis in the y-direction is

Start of $1

equation sequence part 1 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis one comma two right parenthesis equals part 2 two multiplication one equals part 3 two full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative500)

End of $1

[Back to - Exercise 21](" \l "Unit1_Session3_Exercise6)

## Exercise 22

#### Answer

Writing f in the form

Start of $1

f times left parenthesis x plus i times y right parenthesis equals u of x comma y plus i times v of x comma y comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative504)

End of $1

we obtain

Start of $1

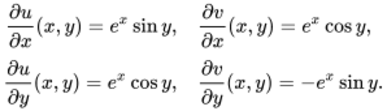
u of x comma y equals e super x times sine of y and v of x comma y equals e super x times cosine of y full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative505)

End of $1

Hence

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative506)

End of $1

If f is differentiable at x plus i times y, then the Cauchy–Riemann equations require that

Start of $1

multiline equation row 1 e super x times sine of y equals negative e super x times sine of y and row 2 e super x times cosine of y equals negative e super x times cosine of y semicolon

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative509)

End of $1

that is,

Start of $1

e super x times sine of y equals zero and e super x times cosine of y equals zero full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative510)

End of $1

But e super x is never zero, so equation sequence part 1 sine of y equals part 2 cosine of y equals part 3 zero, which is impossible. It follows that there is no point of double-struck cap c at which f is differentiable.

[Back to - Exercise 22](" \l "Unit1_Session3_Exercise7)

## Exercise 23

#### Answer

In this case,

Start of $1

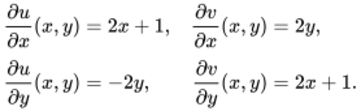
multiline equation row 1 u of x comma y equals x squared plus x minus y squared and row 2 v of x comma y equals two times x times y plus y comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative516)

End of $1

so

Start of $1

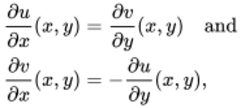


[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative517)

End of $1

These partial derivatives are defined and continuous on the whole of double-struck cap c. Furthermore,

Start of $1



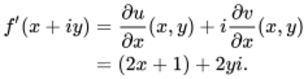
[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative519)

End of $1

so the Cauchy–Riemann equations are satisfied at every point of double-struck cap c.

By the Cauchy–Riemann Converse Theorem, f is entire, and

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative522)

End of $1

(So f super prime of z equals two times z plus one, and in fact f of z equals z squared plus z.)

[Back to - Exercise 23](" \l "Unit1_Session3_Exercise8)

## Exercise 24

#### Answer

1. Here

Start of $1

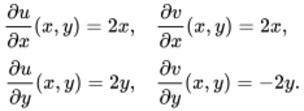
u of x comma y equals x squared plus y squared and v of x comma y equals x squared minus y squared comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative527)

End of $1

so

Start of $1



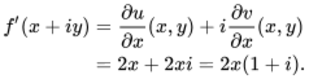
[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative528)

End of $1

The Cauchy–Riemann equations are satisfied only if x equals negative y. So f cannot be differentiable at x plus i times y unless x equals negative y. Since the partial derivatives above exist, and are continuous on double-struck cap c (and in particular when x equals negative y), it follows from the Cauchy–Riemann Converse Theorem that f is differentiable on the set x plus i times y colon x equals negative y.

On this set,

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative537)

End of $1

1. Here

Start of $1

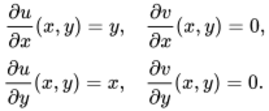
u of x comma y equals x times y and v of x comma y equals zero comma

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative538)

End of $1

so

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative539)

End of $1

The Cauchy–Riemann equations are not satisfied unless y equals zero and negative x equals zero. So f is not differentiable except possibly at 0. Since the partial derivatives above exist, and are continuous at open zero comma zero close, it follows from the Cauchy–Riemann Converse Theorem that f is differentiable at 0. Furthermore,

Start of $1

equation sequence part 1 f super prime of zero equals part 2 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis zero comma zero right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis zero comma zero right parenthesis equals part 3 zero full stop

[View alternative description - Uncaptioned Equation](" \l "Unit1_Session3_Alternative545)

End of $1

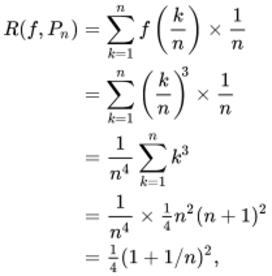
[Back to - Exercise 24](" \l "Unit1_Session3_Exercise9)

## Exercise 1

#### Answer

Each of the n subintervals of cap p sub n has length one solidus n. Therefore

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative94)

End of $1

as required.

Since left parenthesis one solidus n right parenthesis is a basic null sequence, we see that

Start of $1

equation sequence part 1 lim over n right arrow normal infinity of cap r of f comma cap p sub n equals part 2 one divided by four times left parenthesis one plus zero right parenthesis squared equals part 3 one divided by four full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative96)

End of $1

[Back to - Exercise 1](" \l "Unit2_Session2_Exercise1)

## Exercise 2

#### Answer

Since

Start of $1

e super negative x less than or equals e super negative x squared less than or equals one divided by one plus x squared comma for zero less than or equals x less than or equals one comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative211)

End of $1

it follows from the Monotonicity Inequality that

Start of $1

integral over zero under one e super negative x d x less than or equals integral over zero under one e super negative x squared d x less than or equals integral over zero under one one divided by one plus x squared d x full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative212)

End of $1

Hence

Start of $1

left square bracket negative e super negative x right square bracket sub zero super one less than or equals integral over zero under one e super negative x squared d x less than or equals left square bracket tangent super negative one of x right square bracket sub zero super one semicolon

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative213)

End of $1

that is,

Start of $1

one minus e super negative one less than or equals integral over zero under one e super negative x squared d x less than or equals pi divided by four full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative214)

End of $1

Since 0.63 less than one minus e super negative one and pi solidus four less than 0.79, we see that

Start of $1

0.63 less than integral over zero under one e super negative x squared d x less than 0.79 full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session2_Alternative217)

End of $1

(In fact, integral over zero under one e super negative x squared d x equals 0.75 to two decimal places.)

[Back to - Exercise 2](" \l "Unit2_Session2_Exercise2)

## Exercise 3

#### Answer

1. Here gamma of t equals two times left parenthesis one plus i right parenthesis times t left parenthesis t element of left square bracket zero comma one divided by two right square bracket right parenthesis. Let f of z equals z macron. Then

Start of $1

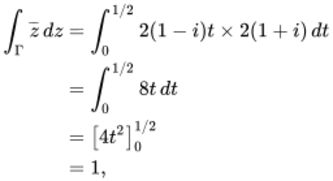
equation sequence part 1 f of gamma of t equals part 2 times times times two left parenthesis right parenthesis plus plus one it macron equals part 3 two times left parenthesis one minus i right parenthesis times t comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative80)

End of $1

and, since gamma times super prime times left parenthesis t right parenthesis equals two times left parenthesis one plus i right parenthesis, we obtain

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative82)

End of $1

in accordance with Example 3.

1. We set out this solution in a similar style to Example 4.

Here gamma of t equals e super three times i times t left parenthesis t element of left square bracket zero comma two times pi solidus three right square bracket right parenthesis. Then

Start of $1

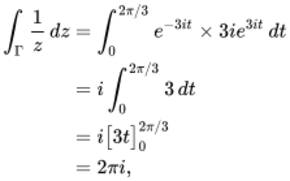
z equals e super three times i times t comma one solidus z equals e super negative three times i times t and d times z equals three times i times e super three times i times t times d times t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative84)

End of $1

Hence

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative85)

End of $1

in accordance with Example 4.

[Back to - Exercise 3](" \l "Unit2_Session3_Exercise1)

## Exercise 4

#### Answer

1. The standard parametrisation of normal cap gamma is

Start of $1

gamma of t equals left parenthesis one plus two times i right parenthesis times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative102)

End of $1

Then

Start of $1

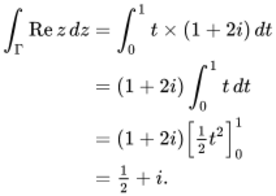
z equals left parenthesis one plus two times i right parenthesis times t comma Re of z equals t comma d times z equals left parenthesis one plus two times i right parenthesis times d times t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative103)

End of $1

Hence

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative104)

End of $1

1. The standard parametrisation of normal cap gamma is

Start of $1

gamma of t equals alpha plus r times e super i times t times left parenthesis t element of left square bracket zero comma two times pi right square bracket right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative106)

End of $1

Then

Start of $1

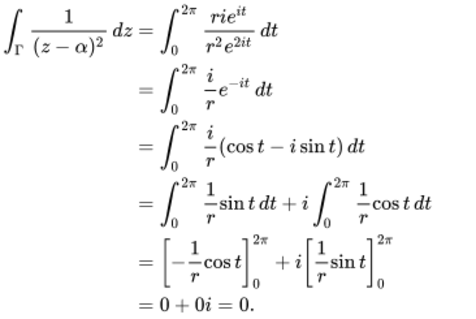
multiline equation row 1 Blank z equals alpha plus r times e super i times t comma one solidus left parenthesis z minus alpha right parenthesis squared equals one solidus left parenthesis r squared times e super two times i times t right parenthesis comma row 2 Blank d times z equals r times i times e super i times t times d times t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative107)

End of $1

Hence

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative108)

End of $1

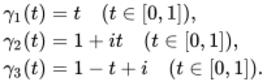
[Back to - Exercise 4](" \l "Unit2_Session3_Exercise2)

## Exercise 5

#### Answer

1. normal cap gamma equals sum with 3 summands normal cap gamma sub one plus normal cap gamma sub two plus normal cap gamma sub three, where normal cap gamma sub one is the line segment from 0 to 1, normal cap gamma sub two is the line segment from 1 to one plus i, and normal cap gamma sub three is the line segment from one plus i to i. We choose to use the associated standard parametrisations

Start of $1

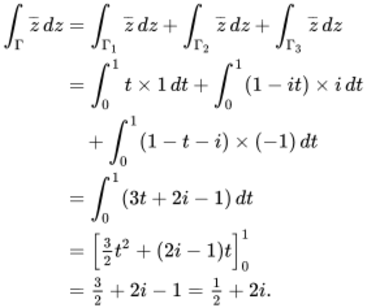


[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative178)

End of $1

Then gamma times sub one super prime times left parenthesis t right parenthesis equals one, gamma times sub two super prime times left parenthesis t right parenthesis equals i, gamma times sub three super prime times left parenthesis t right parenthesis equals negative one. Hence

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative182)

End of $1

1. normal cap gamma equals normal cap gamma sub one plus normal cap gamma sub two, where normal cap gamma sub one is the line segment from negative one to 1, and normal cap gamma sub two is the upper half of the circle with centre 0 from 1 to negative one. We choose to use the parametrisations

Start of $1

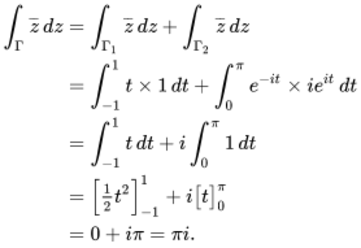
multiline equation row 1 Blank gamma sub one of t equals t times left parenthesis t element of left square bracket negative one comma one right square bracket right parenthesis comma row 2 Blank gamma sub two of t equals e super i times t times left parenthesis t element of left square bracket zero comma pi right square bracket right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative188)

End of $1

Then gamma times sub one super prime times left parenthesis t right parenthesis equals one, gamma times sub two super prime times left parenthesis t right parenthesis equals i times e super i times t. Hence

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative191)

End of $1

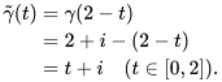
[Back to - Exercise 5](" \l "Unit2_Session3_Exercise3)

## Exercise 6

#### Answer

Since a equals zero and b equals two, the reverse path is cap gamma tilde colon gamma tilde of t (t element of left square bracket zero comma two right square bracket), where

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative231)

End of $1

[Back to - Exercise 6](" \l "Unit2_Session3_Exercise4)

## Exercise 7

#### Answer

In Example 4 we used the parametrisation

Start of $1

gamma of t equals e super i times t times left parenthesis t element of left square bracket zero comma two times pi right square bracket right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative286)

End of $1

For the reverse path cap gamma tilde we use the parametrisation

Start of $1

equation sequence part 1 gamma tilde of t equals part 2 gamma times left parenthesis two times pi minus t right parenthesis equals part 3 e super i times left parenthesis two times pi minus t right parenthesis times left parenthesis t element of left square bracket zero comma two times pi right square bracket right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative288)

End of $1

Since e super two times pi times i equals one, we have

Start of $1

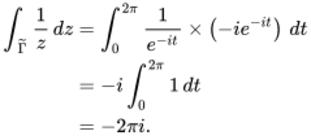
gamma tilde of t equals e super negative i times t times left parenthesis t element of left square bracket zero comma two times pi right square bracket right parenthesis comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative290)

End of $1

and gamma tilde times super prime times left parenthesis t right parenthesis equals negative i times e super negative i times t. Hence

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative292)

End of $1

(Therefore, by Example 4,

Start of $1

integral over cap gamma tilde one divided by z d z equals negative integral over normal cap gamma one divided by z d z full stop right parenthesis

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative293)

End of $1

[Back to - Exercise 7](" \l "Unit2_Session3_Exercise5)

## Exercise 8

#### Answer

1. The standard parametrisation of normal cap gamma, the line segment from 1 to i, is

Start of $1

gamma of t equals one minus t plus i times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis semicolon

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative310)

End of $1

hence

Start of $1

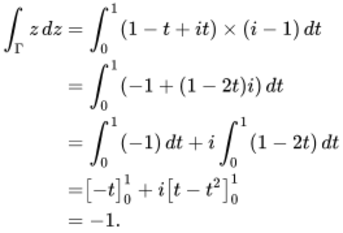
gamma times super prime times left parenthesis t right parenthesis equals i minus one full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative311)

End of $1

* 1. Here f of z equals z, and

Start of $1

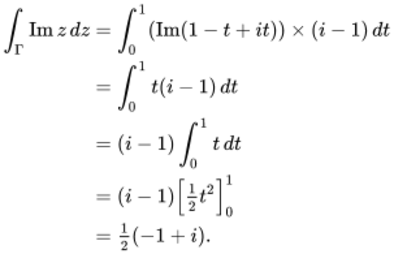


[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative313)

End of $1

* 1. Here f of z equals Im of z, and

Start of $1



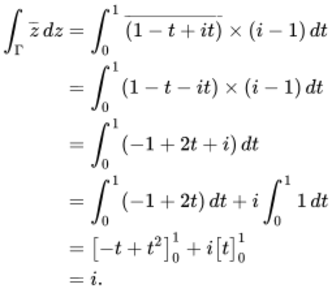
[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative315)

End of $1

(Note that this integral is different from Im of integral over normal cap gamma z d z, which from part (a)(i) is 0.)

* 1. Here f of z equals z macron, and

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative318)

End of $1

(Again, note that this is different from integral integral cap gamma z separator d separator z macron.)

1. We set out this solution in a similar style to [Example 4](#b1-exa2-2).

The standard parametrisation of normal cap gamma, the unit circle z colon absolute value of z equals one, is

Start of $1

gamma of t equals e super i times t times left parenthesis t element of left square bracket zero comma two times pi right square bracket right parenthesis semicolon

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative322)

End of $1

hence

Start of $1

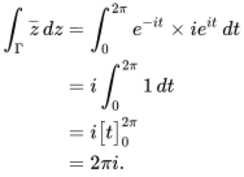
z equals e super i times t comma d times z equals i times e super i times t times d times t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative323)

End of $1

* 1. Here equation sequence part 1 f of z equals part 2 z macron equals part 3 e super negative i times t, and

Start of $1

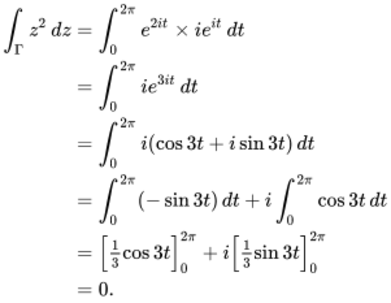


[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative325)

End of $1

* 1. Here equation sequence part 1 f of z equals part 2 z squared equals part 3 e super two times i times t, and

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative327)

End of $1

1. The standard parametrisation of normal cap gamma, the upper half of the circle with centre 0 and radius 2, traversed from 2 to negative two, is

Start of $1

gamma of t equals two times e super i times t times left parenthesis t element of left square bracket zero comma pi right square bracket right parenthesis semicolon

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative330)

End of $1

hence

Start of $1

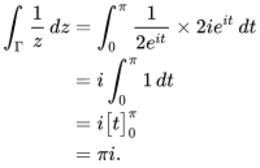
gamma times super prime times left parenthesis t right parenthesis equals two times i times e super i times t full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative331)

End of $1

* 1. Here f of z equals one solidus z, and

Start of $1

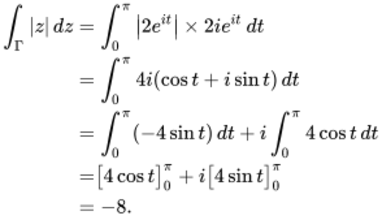


[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative333)

End of $1

* 1. Here f of z equals absolute value of z, and

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative335)

End of $1

[Back to - Exercise 8](" \l "Unit2_Session3_Exercise6)

## Exercise 9

#### Answer

1. normal cap gamma equals normal cap gamma sub one plus normal cap gamma sub two, where normal cap gamma sub one is the line segment from 0 to i and normal cap gamma sub two is the line segment from i to one plus i.

We choose to use the standard parametrisations

Start of $1

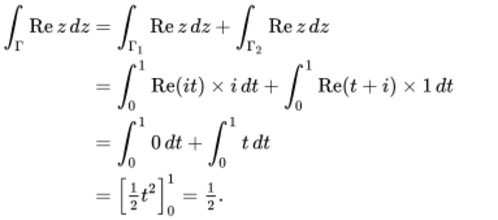
multiline equation row 1 Blank gamma sub one of t equals i times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis comma row 2 Blank gamma sub two of t equals t plus i times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative345)

End of $1

Then gamma times sub one super prime times left parenthesis t right parenthesis equals i, gamma times sub two super prime times left parenthesis t right parenthesis equals one. Hence

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative348)

End of $1

1. normal cap gamma equals normal cap gamma sub one plus normal cap gamma sub two, where normal cap gamma sub one is the line segment from 0 to 1 and normal cap gamma sub two is the line segment from 1 to one plus i.

We choose to use the standard parametrisations

Start of $1

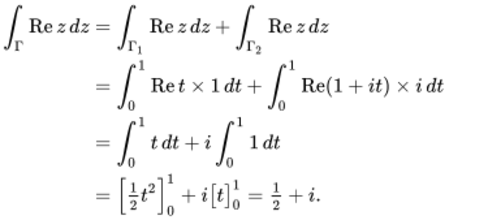
multiline equation row 1 Blank gamma sub one of t equals t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis comma row 2 Blank gamma sub two of t equals one plus i times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative353)

End of $1

Then gamma times sub one super prime times left parenthesis t right parenthesis equals one, gamma times sub two super prime times left parenthesis t right parenthesis equals i. Hence

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session3_Alternative356)

End of $1

(Note that the integrals in parts (a) and (b) have different values.)

[Back to - Exercise 9](" \l "Unit2_Session3_Exercise7)

## Exercise 10

#### Answer

1. cap f of z equals one divided by three times i times e super three times i times z times left parenthesis z element of double-struck cap c right parenthesis
2. equation sequence part 1 cap f of z equals part 2 i times left parenthesis one plus i times z right parenthesis super negative one equals part 3 left parenthesis z minus i right parenthesis super negative one times left parenthesis z element of double-struck cap c minus i right parenthesis
3. cap f of z equals Log of z of Re of z greater than zero

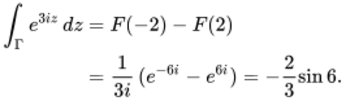
[Back to - Exercise 10](" \l "Unit2_Session4_Exercise1)

## Exercise 11

#### Answer

Let f of z equals e super three times i times z, cap f of z equals e super three times i times z solidus left parenthesis three times i right parenthesis and script cap r script equals double-struck cap c. Then f is continuous on script cap r, and cap f is a primitive of f on script cap r. Thus, by the Fundamental Theorem of Calculus,

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative71)

End of $1

The final simplification follows from the formula

Start of $1

sine of z equals one divided by two times i times left parenthesis e super i times z minus e super negative i times z right parenthesis comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative72)

End of $1

with z equals six.

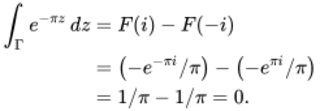
[Back to - Exercise 11](" \l "Unit2_Session4_Exercise2)

## Exercise 12

#### Answer

1. Let f of z equals e super negative pi times z, cap f of z equals negative e super negative pi times z solidus pi and script cap r equals double-struck cap c. Then f is continuous on script cap r, and cap f is a primitive of f on script cap r. Thus, by the Fundamental Theorem of Calculus,

Start of $1

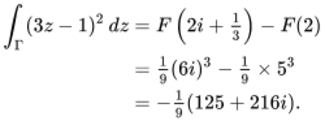


[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative114)

End of $1

1. Let f of z equals left parenthesis three times z minus one right parenthesis squared, cap f of z equals one divided by nine times left parenthesis three times z minus one right parenthesis cubed and script cap r equals double-struck cap c. Then f is continuous on script cap r, and cap f is a primitive of f on script cap r. Thus, by the Fundamental Theorem of Calculus,

Start of $1

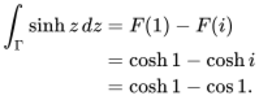


[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative123)

End of $1

1. Let f of z equals hyperbolic sine of z, cap f of z equals hyperbolic cosine of z and script cap r equals double-struck cap c. Then f is continuous on script cap r, and cap f is a primitive of f on script cap r. Thus, by the Fundamental Theorem of Calculus,

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative132)

End of $1

1. The integrand e super sine of z times cosine of z can be written as

Start of $1

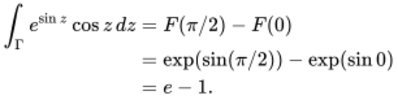
exp of sine of z multiplication sine super prime of z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative134)

End of $1

which equals left parenthesis exp ring operator sine right parenthesis super prime times left parenthesis z right parenthesis comma by the Chain Rule. So let f of z equals exp of sine of z times cosine of z, cap f of z equals exp of sine of z and script cap r equals double-struck cap c. Then f is continuous on script cap r, and cap f is a primitive of f on script cap r. Thus, by the Fundamental Theorem of Calculus,

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative144)

End of $1

Remark: If you have a good deal of experience at differentiating and integrating real and complex functions, then you may have chosen to write down the primitive cap f of z equals e super sine of z of f of z equals e super sine of z times cosine of z straight away.

1. The integrand sine of z solidus cosine squared of z can be written as

Start of $1

negative one divided by cosine squared of z times cosine super prime of z comma

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative148)

End of $1

which equals

Start of $1

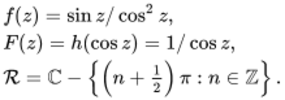
left parenthesis h ring operator cosine right parenthesis super prime times left parenthesis z right parenthesis comma where h of z equals one solidus z full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative149)

End of $1

So let

Start of $1

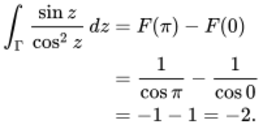


[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative150)

End of $1

Then f is continuous on script cap r, and cap f is a primitive of f on script cap r. Thus, by the Fundamental Theorem of Calculus,

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative156)

End of $1

(In this solution, note that the region script cap r does not contain the point pi solidus two, as cosine of pi solidus two equals zero; thus normal cap gamma cannot be chosen to be a path that contains pi solidus two. In particular, the real integral integral over zero under pi sine of x divided by cosine squared of x d x does not exist.)

[Back to - Exercise 12](" \l "Unit2_Session4_Exercise3)

## Exercise 13

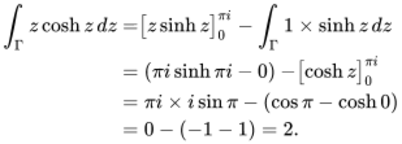
#### Answer

1. We take f of z equals z, g of z equals hyperbolic sine of z and script cap r equals double-struck cap c.

Then f and g are analytic on script cap r, and f super prime of z equals one and g super prime of z equals hyperbolic cosine of z are continuous on script cap r.

Integrating by parts, we obtain

Start of $1



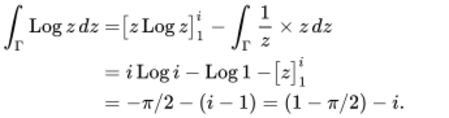
[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative218)

End of $1

1. We take f of z equals Log of z, g of z equals z and script cap r equals double-struck cap c minus x element of double-struck cap r colon x less than or equals zero. Then f and g are analytic on script cap r, and f super prime of z equals one solidus z and g super prime of z equals one are continuous on script cap r.

Integrating by parts, we obtain

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative228)

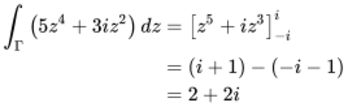
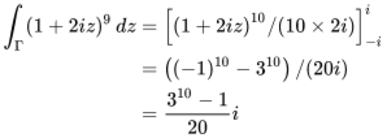
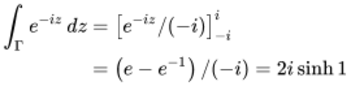
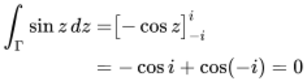
End of $1

[Back to - Exercise 13](" \l "Unit2_Session4_Exercise4)

## Exercise 14

#### Answer

In each case, f is continuous on double-struck cap c and has a primitive on double-struck cap c, so we can apply the Fundamental Theorem of Calculus to evaluate the integral using any contour normal cap gamma from negative i to i.

1. equation sequence part 1 integral over normal cap gamma one d z equals part 2 left square bracket z right square bracket sub negative i super i equals part 3 i minus left parenthesis negative i right parenthesis equals part 4 two times i
2. equation sequence part 1 integral over normal cap gamma z d z equals part 2 left square bracket one divided by two times z squared right square bracket sub negative i super i equals part 3 one divided by two times i squared minus one divided by two times left parenthesis negative i right parenthesis squared equals part 4 zero
3. 
4. 
5. 
6. 
7. A primitive of f of z equals z times e super z squared is

Start of $1

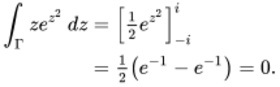
cap f of z equals one divided by two times e super z squared full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative295)

End of $1

Hence

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative296)

End of $1

1. A primitive of f of z equals z cubed times hyperbolic cosine of z super four is

Start of $1

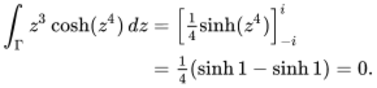
cap f of z equals one divided by four times hyperbolic sine of z super four full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative298)

End of $1

Hence

Start of $1

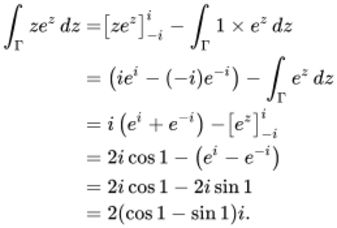


[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative299)

End of $1

1. Let g of z equals z, h of z equals e super z. Then g and h are entire (that is, g and h are differentiable on the whole of double-struck cap c), and g super prime and h super prime are entire and hence continuous. Then, using Integration by Parts ([Theorem 9](#b1-s3-parts)), we have

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative309)

End of $1

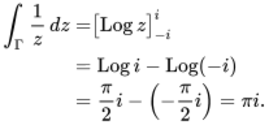
[Back to - Exercise 14](" \l "Unit2_Session4_Exercise5)

## Exercise 15

#### Answer

1. Let f of z equals one solidus z, cap f of z equals Log of z and script cap r equals double-struck cap c minus x element of double-struck cap r colon x less than or equals zero. Then f is continuous on script cap r, cap f is a primitive of f on script cap r, and normal cap gamma is a contour in script cap r. Thus, by the Fundamental Theorem of Calculus,

Start of $1

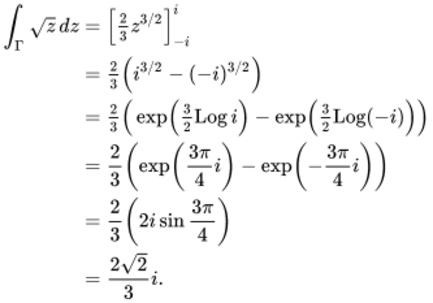


[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative334)

End of $1

1. Let f of z equals Square root of z, cap f of z equals two divided by three times z super three solidus two and script cap r equals double-struck cap c minus x element of double-struck cap r colon x less than or equals zero. Then f is continuous on script cap r, cap f is a primitive of f on script cap r, and normal cap gamma is a contour in script cap r. Thus, by the Fundamental Theorem of Calculus,

Start of $1



[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative345)

End of $1

1. The function

Start of $1

equation sequence part 1 f of z equals part 2 sine squared of z equals part 3 one divided by two times left parenthesis one minus cosine of two times z right parenthesis

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative346)

End of $1

is continuous and has an entire primitive cap f of z equals one divided by two times left parenthesis z minus one divided by two times sine of two times z right parenthesis. Thus, by the Closed Contour Theorem,

Start of $1

integral over normal cap gamma sine squared of z times d times z equals zero full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative348)

End of $1

1. Let f of z equals one solidus z cubed, cap f of z equals negative one solidus left parenthesis two times z squared right parenthesis and script cap r equals double-struck cap c minus zero. Then f is continuous on script cap r, cap f is a primitive of f on script cap r, and normal cap gamma is a contour in script cap r. Thus, by the Closed Contour Theorem,

Start of $1

integral over normal cap gamma one divided by z cubed d z equals zero full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative359)

End of $1

[Back to - Exercise 15](" \l "Unit2_Session4_Exercise6)

## Exercise 16

#### Answer

The domain of tangent is the region

Start of $1

script cap r equals double-struck cap c minus left parenthesis n plus one divided by two right parenthesis times pi colon n element of double-struck cap z full stop

[View alternative description - Uncaptioned Equation](" \l "Unit2_Session4_Alternative368)

End of $1

1. The figure shows one grid path in script cap r from 1 to 6 (there are many others).

Start of Figure

Displayed image

[View description - Uncaptioned Figure](" \l "Unit2_Session4_Description3)

End of Figure

1. The figure shows one grid path in script cap r from pi divided by two plus two times i to negative three times pi divided by two minus i (again, there are many others).

Start of Figure

Displayed image

[View description - Uncaptioned Figure](" \l "Unit2_Session4_Description4)

End of Figure

[Back to - Exercise 16](" \l "Unit2_Session4_Exercise7)

# Figure 1 A chord between two points on a graph

## Description

The figure shows Cartesian axes labelled x and y. It is focused on the upper-right quadrant. A smooth curve that looks like part of the graph of a quadratic function with positive coefficient of x-squared starts in the upper-left quadrant a little above the x-axis and a little to the left of the y-axis. At this point it is nearly horizontal. As x increases the curve slopes upwards with steadily increasing gradient. A line with positive gradient also starts in the upper-left quadrant, below the curve, and slopes upwards into the upper-right quadrant. The curve and the line intersect at two points in the upper-right quadrant, both of which are marked with solid dots. The coordinates of these points are illustrated by vertical and horizontal broken line segments joining each point to the x and y axes. From the lower point of intersection, the vertical broken line segment crosses the x-axis at a point marked c. The horizontal broken line segment crosses the y-axis at a point marked f of c. From the upper point of intersection, the vertical broken line segment crosses the x-axis at a point marked x. The horizontal broken line segment crosses the y-axis at a point marked f of x. From the lower point of intersection a horizontal line segment is drawn to the right. It meets a vertical line segment that extends downwards from the upper point of intersection. They form a right-angled triangle with a segment of the original sloping line as its hypotenuse. The length of the horizontal side of this triangle is marked x minus c. The length of its vertical side is marked f of x minus f of c.

[Back to - Figure 1 A chord between two points on a graph](" \l "Unit1_Session1_Figure1)

# Figure 2 Graph of y equals absolute value of x

## Description

The figure shows Cartesian axes labelled x and y. It is focused on the upper half-plane. The graph of the function y equals the modulus of x is drawn and labelled. In the upper-left quadrant the graph is a line sloping down with gradient negative 1 to the origin. In the upper-right quadrant the graph is a line sloping up with gradient 1 from the origin.

[Back to - Figure 2 Graph of y equals absolute value of x](" \l "Unit1_Session1_Figure2)

# Figure 3 Three regions on which f of z equals one solidus z is analytic

## Description

This figure consists of three copies of the complex plane arranged side by side, from left to right. The axes are not labelled. Each of the three diagrams has a different region, bounded by broken lines or curves, and shaded inside. The left-hand diagram shows a horseshoe-shaped region symmetrical about the vertical axis; the boundary of the shape is drawn as a broken line. The arms of the horseshoe are pointing down and they straddle the horizontal axis. The open end of the horseshoe is at the bottom, so that the origin is not inside the region. The middle diagram shows an open annulus, and both of its boundaries are drawn as broken lines. It consists of the area between two concentric circles, centred at the origin. The right-hand diagram shows a square region centred on the origin, with two vertical and two horizontal sides; its boundaries are drawn as broken lines. It is symmetrical about both axes. The origin itself is marked with a hollow dot.

[Back to - Figure 3 Three regions on which f of z equals one solidus z is analytic](" \l "Unit1_Session2_Figure1)

# Figure 4 Sequences converging to zero from the right and left

## Description

This figure consists of two copies of the complex plane, one above the other. The upper copy shows on the positive real axis a sequence of points that approach zero from the right. The origin is labelled zero and marked with a solid blue dot. The point 1 is labelled, and the first three points in the sequence, starting with 1, are marked with solid black dots. An ellipsis to the left of the third point (one third) indicates that the sequence continues indefinitely getting closer to zero. A horizontal arrow above the real axis, between zero and 1, points from right to left. Above it is the label, open bracket, z sub n, close bracket. The lower copy of the complex plane shows on the negative real axis a sequence of points that approach zero from the left. The origin is labelled zero and marked with a solid blue dot. The point negative 1 is labelled, and the first three points in the sequence, starting with negative 1, are marked with solid black dots. An ellipsis to the right of the third point (negative one third) indicates that the sequence continues indefinitely getting closer to zero. A horizontal arrow above the negative real axis, between negative 1 and zero, points from left to right. Above it is the label, open bracket, z prime sub n, close bracket.

[Back to - Figure 4 Sequences converging to zero from the right and left](" \l "Unit1_Session2_Figure2)

# Uncaptioned Figure

## Description

This figure consists of two copies of the complex plane, arranged side by side. The axes are not labelled. The left-hand copy shows a circle centred at the origin. An arbitrary point on the circle in the upper-right quadrant is labelled alpha and marked with a solid blue dot. A sequence of points on the circumference starts in the upper-left quadrant and approaches the point alpha at decreasing intervals. The first four points in the sequence are marked with solid black dots. An ellipsis between the fourth point and the point alpha indicates that the sequence of points continues indefinitely getting closer to alpha. Above the sequence of points is a curved arrow pointing from left to right. It is labelled: open bracket, z sub n, close bracket. The right-hand copy of the complex plane shows the same circle centred at the origin, and the same arbitrary point alpha on the circle is labelled and marked with a solid blue dot. A sequence of points outside the circle approaches the point alpha at decreasing intervals along the normal to the circle at alpha. The first three points in the sequence are marked with solid black dots. An ellipsis between the third point and the point alpha indicates that the sequence of points continues indefinitely getting closer to alpha. An arrow directed towards alpha is drawn outside the circle parallel to the sequence of points. It is labelled: open bracket, z prime sub n, close bracket.

[Back to - Uncaptioned Figure](" \l "Unit1_Session2_Figure3)

# Figure 5 Images of horizontal and vertical lines under f of z equals z macron

## Description

This figure consists of two copies of the complex plane, arranged side by side. The axes are not labelled. On the left-hand diagram, a bold vertical line, drawn in black, passes through an arbitrary point on the positive horizontal axis. It is marked with a direction arrow pointing upwards, and labelled capital gamma sub 1. A bold horizontal line, drawn in blue, passes through an arbitrary point on the positive vertical axis. It is marked with a direction arrow pointing to the right, and labelled capital gamma sub 2. The two bold lines intersect at right angles in the upper-right quadrant. Between the two copies of the complex plane is a curved horizontal arrow pointing from left to right. It is labelled f of z equals z bar. On the right-hand diagram, a bold vertical line, drawn in black, passes through the same arbitrary point on the positive horizontal axis. It is marked with a direction arrow pointing downwards, and labelled f of capital gamma sub 1. A bold horizontal line, drawn in blue, passes through a point on the negative vertical axis. (This point is the same distance below the origin as the point used on the vertical axis in the left-hand diagram was above it.) The horizontal line is marked direction arrow pointing to the right, and labelled f of capital gamma sub 2. The two bold lines intersect at right angles in the lower-right quadrant.

[Back to - Figure 5 Images of horizontal and vertical lines under f of z equals z macron](" \l "Unit1_Session2_Figure4)

# Figure 6 Scaling and rotating z minus alpha

## Description

This figure consists of a diagram of the complex plane, focused on the right half-plane. The axes are not labelled. A line segment extends from the origin to a point in the lower-right quadrant. The point is marked with a solid dot and labelled z minus alpha. A second line segment extends from the origin to a point in the upper-right quadrant. The second line segment is longer than the first. Above it is the text, scale by the modulus of f prime of alpha. The point at the end of the second line segment is marked with a solid dot and labelled f prime of alpha times open bracket, z minus alpha, close bracket. A curved arrow between the two line segments points in the anticlockwise direction. Next to it is the text, rotate by Arg f prime of alpha.

[Back to - Figure 6 Scaling and rotating z minus alpha](" \l "Unit1_Session2_Figure5)

# Figure 7 Interpreting a derivative as a complex scale factor

## Description

This figure consists of two copies of the complex plane side by side, both focused on the upper-right quadrant. The axes are not labelled. On the left-hand diagram there are two points in the upper-right quadrant, alpha marked with a solid black dot and z marked with a solid blue dot. The point z lies to the right of the point alpha and a little below it. The vector from alpha to z is drawn as a solid blue line with an arrowhead. Between the left-hand and right-hand diagrams is a curved arrow pointing from left to right and labelled f. On the right-hand diagram, three points are marked with solid dots. The first, which is the closest to the origin, is drawn in black labelled f of alpha. The second, also in black, lies above and to the right of the first point, and is labelled f of z. The third point, drawn in blue, lies to the right and a little below the point labelled f of alpha. It is not labelled. Two vectors are also shown, both starting from the point f of alpha. The first vector, drawn in blue, runs from the point f of alpha to the unlabelled blue point. It is an identical copy of the vector from alpha to z that was drawn in the left-hand diagram. The second vector, drawn in black, starts at the point f of alpha and finishes at the point f of z. Above this second vector is the text, scale by the modulus of f prime of alpha. Between the two vectors is a curved arrow, pointing in the anticlockwise direction. Next to it is the text, rotate by Arg f prime of alpha.

[Back to - Figure 7 Interpreting a derivative as a complex scale factor](" \l "Unit1_Session2_Figure6)

# Figure 8 The approximate image of a disc centred at a point alpha, where f super prime of alpha not equals zero

## Description

This figure consists of two copies of the complex plane side by side, both focused on the upper-right quadrant. The axes are not labelled. On the left-hand diagram a point in the upper-right quadrant is marked with a solid black dot and labelled alpha. A small circular disc is drawn with alpha as its centre and a solid black circle as its boundary. Inside the disc a second point, a little to the right and below the point alpha, is marked with a solid green dot. It is labelled z. The disc is shaded blue. Between the left-hand and right-hand diagrams is a curved arrow pointing from left to right. Above it is the label f. On the right-hand diagram the point f of alpha is marked with a solid black dot and labelled. A circular disc is drawn with the point f of alpha as its centre and a solid black circle as its boundary. This disc has radius approximately 50% greater than the radius of the disc surrounding the point alpha in the left-hand diagram. Inside the disc a second point, above and to the right of the point f of alpha, is marked with a solid green dot and labelled f of z. The disc is shaded blue. Outside the disc, above it and slightly to the left, is the text, scale by the modulus of f prime of alpha. Outside the disc, diametrically opposite the first text, is a curved arrow parallel to the circular boundary of the disc, which points in an anticlockwise direction. Next to it is the text, rotate by Arg f prime of alpha.

[Back to - Figure 8 The approximate image of a disc centred at a point alpha, where f super prime of alpha not equals zero](" \l "Unit1_Session2_Figure7)

# Figure 9 Graph of u of x comma y equals x cubed minus three times x times y squared

## Description

The figure consists of a perspective drawing of a three-dimensional set of Cartesian axes with a smooth surface sketched on it. The complex plane is drawn as horizontal, with axes labelled x and y. The x-axis points out of the page and to the left and the y-axis points out of the page and to the right from the origin. The vertical axis is labelled s. A smooth surface shaded darker on top and lighter underneath starts like a sheet with the top edge attached along the y-axis. At this point it is flat. As the x-coordinate increases, the surface rises up in a saddle shape above the x-axis, and dips down on either side. The vertical cross-section of the surface parallel to the s-y plane is shaped like an inverted parabola, symmetrical about a vertical line through the x-axis. On the surface a point is marked with a solid dot and labelled with its coordinates: capital P equals, open bracket, 2 comma 1 comma 2, close bracket.

[Back to - Figure 9 Graph of u of x comma y equals x cubed minus three times x times y squared](" \l "Unit1_Session3_Figure1)

# Figure 10 Intersection of the graph of u of x comma y equals x cubed minus three times x times y squared with the vertical plane y equals one

## Description

This figure is identical to Figure 9, except for the addition of a vertical plane, parallel to the x-axis and intersecting the curved surface. It consists of a perspective drawing of a three-dimensional set of Cartesian axes with a smooth surface sketched on it. The complex plane is drawn as horizontal, with axes labelled x and y. The x-axis points to the left and the y-axis to the right from the origin. The vertical axis is labelled s. A smooth surface shaded darker on top and lighter underneath starts like a sheet with the top edge attached along the y-axis. At this point it is flat. As the x-coordinate increases, the surface rises up in a saddle shape above the x-axis, and dips down on either side. The vertical cross-section of the surface parallel to the s-y plane is shaped like an inverted parabola, symmetrical about a vertical line through the x-axis. On the surface a point is marked with a solid dot and labelled with its coordinates: capital P equals, open bracket, 2 comma 1 comma 2, close bracket. The vertical plane is shaded red, and labelled with its equation, y equals 1. A curve is drawn on the surface to show where the surface intersects the plane. It passes through the point marked P.

[Back to - Figure 10 Intersection of the graph of u of x comma y equals x cubed minus three times x times y squared with the vertical plane y equals one](" \l "Unit1_Session3_Figure2)

# Figure 11 Intersection of the graph of u of x comma y equals x cubed minus three times x times y squared with the vertical plane x equals two

## Description

This figure is identical to Figure 9, except for the addition of a vertical plane, parallel to the y-axis and intersecting the curved surface. It consists of a perspective drawing of a three-dimensional set of Cartesian axes with a smooth surface sketched on it. The complex plane is drawn as horizontal, with axes labelled x and y. The x-axis points to the left and the y-axis to the right from the origin. The vertical axis is labelled s. A smooth surface shaded darker on top and lighter underneath starts like a sheet with the top edge attached along the y-axis. At this point it is flat. As the x-coordinate increases, the surface rises up in a saddle shape above the x-axis, and dips down on either side. The vertical cross-section of the surface parallel to the s-y plane is shaped like an inverted parabola, symmetrical about a vertical line through the x-axis. On the surface a point is marked with a solid dot and labelled with its coordinates: capital P equals, open bracket, 2 comma 1 comma 2, close bracket. The vertical plane is shaded red, and labelled with its equation, x equals 2. A curve is drawn on the surface to show where the surface intersects the plane. It passes through the point marked P.

[Back to - Figure 11 Intersection of the graph of u of x comma y equals x cubed minus three times x times y squared with the vertical plane x equals two](" \l "Unit1_Session3_Figure3)

# Jean le Rond d’Alembert (1717–1783)

## Description

This figure shows the image of Jean le Rond d’Alembert. It is a head and shoulders shot with him looking to the right of the viewer.

[Back to - Jean le Rond d’Alembert (1717–1783)](" \l "Unit1_Session3_Figure4)

# Uncaptioned Figure

## Description

This figure shows the first partial derivatives of u and v with respect to x and y, shown in a two by two grid. Top left: partial derivative of u of x and y with respect to x equals two times x. Bottom left: partial derivative of u of x and y with respect to y equals two times y. Top right: partial derivative of v of x and y with respect to x equals two. Bottom right: partial derivative of v of x and y with respect to y equals four. The partial derivatives of u with respect to x and that of v with respect to y are grouped in a bubble, as are the partial derivatives of u with respect to y and that of v with respect to x.

[Back to - Uncaptioned Figure](" \l "Unit1_Session3_Figure5)

# Figure 12 Graph of u

## Description

This figure consists of a perspective drawing of a three-dimensional set of Cartesian axes. The complex plane is drawn as horizontal, with axes labelled x and y. On the page, the x-axis slopes up and to the right in the positive direction, while the y-axis slopes up and to the left. The third axis is drawn as vertical on the page and is labelled s. It is used to represent the value of the real-valued function u, defined in the text, for each pair of values of x and y. The surface corresponding to the graph of the function u is shaded on the diagram. In the upper-left, lower-left and lower-right quadrants of the x-y plane, the function u takes the value zero, so the x y plane itself is shaded in these quadrants. In the upper-right quadrant, the value of the function at each point on the x-y plane is the minimum of the x and y coordinates of the point. The surface corresponding to the graph in the upper-right quadrant looks a little like a square pyramid, with the origin at one corner of its base, and one of its slant edges sloping up from the origin at an angle of 45 degrees, above the line y equals x on the x-y plane. To emphasise this shape, the slanting face of the pyramid that slopes up from the y-axis is shaded darker than the slanting face that slopes up from the x-axis.

[Back to - Figure 12 Graph of u](" \l "Unit1_Session3_Figure6)

# Figure 13 Graph of the real function f

## Description

This figure consists of a pair of Cartesian axes, labelled x and y. The diagram is focused on the upper-right quadrant. A curve representing part of the graph of an arbitrary real function f of x is drawn. (Actually, to facilitate the comprehensibility of the diagram, the graph is drawn to look like part of a cubic curve. It starts with positive gradient, then as x increases it reaches a local maximum, slopes down to a local minimum, and then slopes up again.) Three points are marked on the positive x-axis, labelled, from left to right, a, c and x. From each of these points a broken vertical line extends upwards to meet the curve. The three points where these three broken vertical lines meet the curve are marked with filled dots. The dot on the curve that lies above the point c on the x-axis is labelled with its coordinates, open bracket, c comma f of c, close bracket. This point is close to the local maximum of the graph. Through the point a line is drawn, tangent to the curve. A line segment is also drawn joining the other two points marked with dots on the curve (the points on the curve above the points a and x on the x-axis). This line segment is parallel to the tangent at c.

[Back to - Figure 13 Graph of the real function f](" \l "Unit1_Session3_Figure7)

# Figure 14 Tangent plane to the graph of u at the point cap p

## Description

The figure consists of a perspective drawing of a three-dimensional set of axes. The two axes in the horizontal plane are labelled x and y. The x-axis points out of the page and to the right; the y-axis points into the page and to the right. The vertical axis is labelled s. The diagram is focused on the section where x, y and s are all positive. The graph of the function u is shown as a convex curved surface shaded blue. The surface is labelled s equals u bracket x comma y close bracket. A point capital P is marked on the surface, and the tangent plane to the surface at the point capital P is shaded in red.

[Back to - Figure 14 Tangent plane to the graph of u at the point cap p](" \l "Unit1_Session3_Figure8)

# Figure 15 Vertical and horizontal line segments from left parenthesis a comma b right parenthesis to open x comma y close

## Description

The figure consists of a circle and two joined line segments. The centre of the circle is marked with a solid dot. This point is labelled with its coordinates, open bracket, a comma b, close bracket. A vertical line segment extends upwards from the centre of the circle, reaching about two-thirds of the way to the circumference. A point half-way up the line segment is marked with a solid dot, and labelled with its coordinates, open bracket, a comma s, close bracket. Another point at the top end of the vertical line segment is also marked with a solid dot, and labelled with its coordinates, open bracket, a comma y, close bracket. From this point at the top of the vertical line segment, a horizontal line segment extends to the right, reaching close to the circumference but remaining inside the circle. Half-way along it, a point is marked with a solid dot, and labelled with coordinates, open bracket, r comma y, close bracket. At the right-hand end of the horizontal line segment, a point is marked with a solid dot, and labelled with coordinates, open bracket, x comma y, close bracket.

[Back to - Figure 15 Vertical and horizontal line segments from left parenthesis a comma b right parenthesis to open x comma y close](" \l "Unit1_Session3_Figure9)

# Pierre-Simon Laplace (1749–1827)

## Description

This figure is a full-length portrait of Laplace as an older man, painted as a tribute after his death by the well-known contemporary artist Jean-Baptiste Paulin Guérin. Laplace is standing richly attired, clean-shaven, and wears a short white wig. At his side is a ceremonial sword, and round his shoulders a sumptuous velvet cloak. Some medals are visible on his chest, and in his right hand he is holding a velvet hat trimmed with fur. In the background are a pedestal with a classical bust, perhaps of a Greek goddess, and a table with a globe, a pair of compasses and some papers, presumably mathematical.

[Back to - Pierre-Simon Laplace (1749–1827)](" \l "Unit1_Session3_Figure10)

# Figure 1 Area under the graph of y equals x squared between a and b

## Description

The figure shows Cartesian axes labelled x and y and is concentrated in the upper-right quadrant. The graph of y equals x squared is drawn and labelled. Points a and b are labelled on the positive x-axis with a being nearer the origin. The line joining these two points on the x-axis is a bold line. The two points have vertical lines drawn to meet the parabola y equals x squared. The area under the graph and above the positive x-axis is shaded.

[Back to - Figure 1 Area under the graph of y equals x squared between a and b](" \l "Unit2_Session1_Figure1)

# Figure 2 Line segment normal cap gamma from alpha to beta

## Description

The figure shows the complex plane with unlabelled axes and concentrated in the upper half-plane. There is a point alpha in the upper-left quadrant marked with a solid dot and a point beta in the upper-right quadrant also marked with a solid dot. The point beta is higher than alpha and further to the right than alpha is to the left of the origin. A bold line joins the two points and is marked with an arrow in the direction from alpha to beta and is labelled capital gamma.

[Back to - Figure 2 Line segment normal cap gamma from alpha to beta](" \l "Unit2_Session1_Figure2)

# Figure 3 Contour with initial point alpha and final point beta coinciding

## Description

The figure shows the complex plane with unlabelled axes. There is a closed contour with irregular shape marked with an arrow in an anticlockwise direction and labelled capital gamma. The contour covers all four quadrants and has the point alpha equals beta marked with a solid dot.

[Back to - Figure 3 Contour with initial point alpha and final point beta coinciding](" \l "Unit2_Session1_Figure3)

# Figure 4 Circular paths normal cap gamma sub one and normal cap gamma sub two

## Description

The figure shows the complex plane with unlabelled axes. There are two circles drawn both with radius 1. The first is in the upper-right quadrant with centre 1 plus i. and is labelled with an arrow in an anticlockwise direction marked as capital gamma sub 1. The second has the origin as its centre and is labelled with an arrow in an anticlockwise direction marked as capital gamma sub 2. The points i and 1 are marked as solid dots and both circles pass through these points.

[Back to - Figure 4 Circular paths normal cap gamma sub one and normal cap gamma sub two](" \l "Unit2_Session1_Figure4)

# Figure 5 Area under the graph of y equals f of x between a and b

## Description

The figure shows the Cartesian axes labelled x and y concentrated in the upper-right quadrant. There is an arbitrary curve labelled y equals f of x contained inside the upper-right quadrant with one local maximum point and one local minimum point and the curve gradually rising bottom left to top right. There are points a and b marked on the positive x-axis with a being closer to the origin. These points are joined by a bold line. Vertical lines join the points a and b to the curve and the area between the curve and above the positive x-axis is shaded.

[Back to - Figure 5 Area under the graph of y equals f of x between a and b](" \l "Unit2_Session2_Figure1)

# Figure 6 Interval open a comma b close split into subintervals

## Description

The figure shows a single horizontal axis labelled x. The points a and b are marked with a being on the left and b on the right. The interval a to b is split into five irregular subintervals. A horizontal brace is drawn beneath the axis and connects a and b; there is a label that says subintervals of open-square-bracket a comma b close-square-bracket.

[Back to - Figure 6 Interval open a comma b close split into subintervals](" \l "Unit2_Session2_Figure2)

# Figure 7 (a) An underestimate (b) An overestimate

## Description

The figure is in two parts: (a) and (b). Each part shows a set of Cartesian axes labelled x and y and focuses on the upper-right quadrant. Both parts have the same curve labelled y equals f of x which is a curve going through the origin from the lower-left quadrant but concentrated mainly in the upper-right quadrant and it has two local maximum points and one local minimum. In both parts the points a and b are marked on the positive x-axis with a being nearer the origin. A bold line joins the points a and b. There are five irregular subintervals shown as rectangles and shaded. Part (a) has the top of each shaded rectangle touching the graph from below. Part (b) has the top of each shaded rectangle touching the graph from above.

[Back to - Figure 7 (a) An underestimate (b) An overestimate](" \l "Unit2_Session2_Figure3)

# Figure 8 Area under the graph of y equals f of x between a and b

## Description

The figure shows a set of Cartesian axes labelled x and y concentrated mainly in the upper-right quadrant. The graph of y equals f of x is shown and it passes through the origin from the lower-left quadrant and has two local maximum points and one local minimum. There are points a and b marked on the positive x-axis and joined by a bold line with the point a nearer the origin. Vertical lines join the points a and b to the curve and the area between the graph and the positive x-axis is shaded.

[Back to - Figure 8 Area under the graph of y equals f of x between a and b](" \l "Unit2_Session2_Figure4)

# Figure 9 Approximating the area under a graph using a sequence of rectangles

## Description

The figure shows a set of Cartesian axes labelled x and y concentrated mainly in the upper-right quadrant. The graph of y equals f of x is shown which passes through the origin from the lower-left quadrant and has two local maximum points and one local minimum. The points x sub zero equals a is marked on the positive x-axis and is close to the origin. The point x sub one is marked on the positive x-axis and is further from the origin. The point x sub n equals b is marked on the positive x-axis and is further away still. There is an ellipses shown on the positive x-axis to demonstrate the sequence x sub zero to x sub n. There are five consecutive touching rectangles under the curve which have base along the horizontal axis. The base of the first rectangle goes from x sub 0 to x sub1 and its height is from x sub 1 to the curve of y equals f of x. The subsequent four rectangles each have arbitrary width and height which is determined by the right-hand side of the rectangle intersecting the curve. The final rectangle’s right-hand side is at x sub n.

[Back to - Figure 9 Approximating the area under a graph using a sequence of rectangles](" \l "Unit2_Session2_Figure5)

# Figure 10 Rectangle of height f of x sub k and width delta times x sub k

## Description

The figure shows a horizontal line with points x sub k minus 1 and point x sub k marked. The distance between the two points is marked with a double-ended arrow as delta x sub k. and creates the base of a shaded rectangle. Part of the curve y equals f of x is shown and a vertical line from the point x sub k meets the curve and this represents the height of the rectangle. The height of the rectangle is labelled with a double-ended arrow as f of x sub k.

[Back to - Figure 10 Rectangle of height f of x sub k and width delta times x sub k](" \l "Unit2_Session2_Figure6)

# Figure 11 Signed area determined by the graph of y equals f of x between a and b

## Description

The figure shows a set of Cartesian axes labelled x and y concentrated in the upper-right and lower-right quadrants. A curve of y equals f of x is labelled. There are points a and b labelled on the positive x-axis with a being nearer the origin. The curve starts in the lower-right quadrant below the positive x-axis directly below the point a, it continues to a local maximum point in the upper-right quadrant and continues to the lower-right quadrant to finish at a point directly below the point b below the positive x-axis. The areas between the curve and the positive x-axis are shaded. The area between the point a and where the curve crosses from the lower-right to the upper-right quadrant is shaded below the positive x-axis and labelled with a minus sign, similarly so too is the area between where the curve crosses back from the upper-right to the lower-right quadrant to the point b. The area above the positive x-axis is shaded and labelled with a plus sign.

[Back to - Figure 11 Signed area determined by the graph of y equals f of x between a and b](" \l "Unit2_Session2_Figure7)

# Figure 12 Monotonicity Inequality

## Description

The figure shows a set of Cartesian axes labelled x and y concentrated in the upper-right quadrant. There are points a and b labelled on the positive x-axis with a being nearer the origin. An arbitrary curve labelled y equals g of x slopes down from left to right and another arbitrary curve labelled y equals f of x is below the first curve but this one has a shallow local minimum and maximum. Both curves start and finish at point directly above the points labelled a and b. The area under g of x to f of x is shaded. The area under f of x to the positive x-axis is shaded in a darker shade.

[Back to - Figure 12 Monotonicity Inequality](" \l "Unit2_Session2_Figure8)

# Figure 13 Modulus Inequality

## Description

The figure shows two sets of Cartesian axes labelled x and y. The first is concentrated in the upper-right and lower-right quadrants. A curve of y equals f of x is labelled. There are points a and b labelled on the positive x-axis with a being nearer the origin. The curve starts in the lower-right quadrant below the positive x-axis directly below the point a, it continues to a local maximum point in the upper-right quadrant and continues to the lower-right quadrant to finish at a point directly below the point b below the positive x-axis. The areas between the curve and the positive x-axis are shaded. The area between the point a and where the curve crosses from the lower-right to the upper-right quadrant is shaded below the positive x-axis and labelled with a minus sign, similarly so too is the area between where the curve crosses back from the upper-right to the lower-right quadrant to the point b. The area above the positive x-axis is shaded and labelled with a plus sign. The second, which is directly beneath the first, is identical apart from a couple of differences. The curve is labelled y equals modulus f of x. The whole curve is now all above the positive x-axis in the upper-right quadrant and all the shaded areas are labelled with plus signs.

[Back to - Figure 13 Modulus Inequality](" \l "Unit2_Session2_Figure9)

# Figure 14 A smooth path from alpha to beta

## Description

This figure shows an unlabelled set of Cartesian axes concentrated in the upper-right and upper-left quadrants. There is an arbitrary point in the upper-left quadrant labelled gamma of a equals alpha and another arbitrary point in the upper-right quadrant labelled gamma of b equals beta. A path joins the two points and is marked with an arrow in a direction from point alpha to beta and labelled capital gamma. The curve starts at alpha then drops to a local minimum in the upper-left quadrant, it continues a local maximum in the upper-right quadrant then drops to point beta.

[Back to - Figure 14 A smooth path from alpha to beta](" \l "Unit2_Session2_Figure10)

# Figure 15 (a) A partition of normal cap gamma (b) A partition of the line segment from zero to one plus i

## Description

This figure has two parts: (a) and (b). Part (a) has a path which starts at the point z sub zero equals alpha and continues in an arbitrary manner from left to right to the final point z sub n equals beta. Along the path are four arrows in a left to right direction labelled capital gamma sub 1, capital gamma sub 2, capital gamma with an ellipsis underneath and finally capital gamma sub n. There are points marked on the path and labelled as follows. The point z sub 1 which is between the arrows capital gamma sub 1 and 2. The point z sub 2 which is between the arrows capital gamma sub 2 and capital gamma with an ellipsis underneath. The point z sub n minus 1 which is between the arrows capital gamma with an ellipsis underneath and capital gamma sub n. Part (b) shows an unlabelled set of Cartesian axes. There is a straight-line segment shown in the upper-right quadrant starting at the origin and this point is labelled z sub zero equals zero. The final point of the line segment is labelled z sub n equals 1 plus i. On the line segment are four arrows marked in a direction from the origin to the point 1 plus i labelled capital gamma sub 1, capital gamma sub 2, capital gamma with an ellipsis underneath and finally capital gamma sub n. There are points marked on the line segment and labelled as follows. The point z sub 1 which is between the arrows capital gamma sub 1 and 2. The point z sub 2 which is between the arrows capital gamma sub 2 and capital gamma with an ellipsis underneath. The point z sub n minus 1 which is between the arrows capital gamma with an ellipsis underneath and capital gamma sub n.

[Back to - Figure 15 (a) A partition of normal cap gamma (b) A partition of the line segment from zero to one plus i](" \l "Unit2_Session2_Figure11)

# Figure 16 A partition of normal cap gamma induced by the parameter values t sub zero comma t sub one comma ellipsis comma t sub n

## Description

This figure has two parts. On the left-hand side is a single unlabelled horizontal axis and on the right is an unlabelled set of Cartesian axes. There is a curved arrow linking the two labelled gamma. The left-hand axis is labelled with four points starting at the left with t sub zero equals a, then t sub 1, t sub k and finally t sub n equals b. A bold line segment joins the points a to b type. Underneath the line segment between the points t sub 1 and t sub k and t sub k and t sub n is an ellipsis. The right-hand set of axes is concentrated in the upper-right quadrant and has an arbitrary path marked with an arrow going from left to right and labelled capital gamma. The path starts at the point labelled z sub zero equals gamma of a and finishes at the point labelled z sub n equals gamma of b. The path is labelled as z sub k equals gamma of t sub k. The point z sub 1 equals gamma of t sub 1 is on the paths and labelled. There are two instances of an ellipsis shown under and along the path.

[Back to - Figure 16 A partition of normal cap gamma induced by the parameter values t sub zero comma t sub one comma ellipsis comma t sub n](" \l "Unit2_Session2_Figure12)

# Figure 17 A path normal cap gamma from zero to i

## Description

This figure shows a diagram of the complex plane with the axes unlabelled concentrated in the upper-right quadrant. The following points are labelled; 2, 2 plus i and i. A path labelled capital gamma is labelled and consists of three joining line segments. The first from zero to 2 with an arrow marked in that direction, from 2 to 2 plus i again with an arrow and from 2 plus i to i with another arrow.

[Back to - Figure 17 A path normal cap gamma from zero to i](" \l "Unit2_Session3_Figure1)

# Figure 18 The contour normal cap gamma equals sum with 4 summands normal cap gamma sub one plus normal cap gamma sub two plus normal cap gamma sub three plus normal cap gamma sub four

## Description

The figure shows a contour capital gamma made up of four paths added together. The first path starts from an arbitrary point, but each of the subsequent paths start from the end of that which was previous. The first path labelled capital gamma sub 1 is a slight curve downwards from left to right. The second labelled capital gamma sub 2 is a vertical line pointing down. The third capital gamma sub 3 is a slight curve downwards from right to left but not as steep as capital gamma sub 1. The final path capital gamma sub 4 is a slight horizontal curve going from left to right. There are arbitrary points marked and labelled initial point and final point to show the start and end of the contour. All four paths are marked with arrows in a general direction from initial point to final point.

[Back to - Figure 18 The contour normal cap gamma equals sum with 4 summands normal cap gamma sub one plus normal cap gamma sub two plus normal cap gamma sub three plus normal cap gamma sub four](" \l "Unit2_Session3_Figure2)

# Figure 19 A contour normal cap gamma from zero to one plus i

## Description

This figure shows a diagram of the complex plane with the axes unlabelled and concentrated in the upper-right quadrant. The following points are labelled: zero, 1 and 1 plus i. A contour is made up of two line segments and labelled capital gamma. The first from zero to point 1 is marked with an arrow from left to right and the second from point 1 to 1 plus i is marked with an arrow in an upwards direction.

[Back to - Figure 19 A contour normal cap gamma from zero to one plus i](" \l "Unit2_Session3_Figure3)

# Uncaptioned Figure

## Description

This figure is in two parts, (a) and (b), and shows two copies of the unlabelled complex plane. Part (a) is concentrated in the upper-right quadrant and shows a contour capital gamma labelled and marked with an arrow in an anticlockwise direction. The contour is made up of three line segments. The first from zero to the labelled point 1, the second from the point 1 to the labelled point 1 plus i and the last from 1 plus i to the labelled point i. Part (b) is concentrated in the upper-right and upper-left quadrants. A closed contour is shown and labelled capital gamma with a marked arrow in an anticlockwise direction. The contour is a line segment and a semicircle with the line segment from the labelled points negative 1 to point 1 and the semicircle from the point 1 to the point negative 1 passing through the labelled point i.

[Back to - Uncaptioned Figure](" \l "Unit2_Session3_Figure4)

# Figure 20 (a) A smooth path normal cap gamma and (b) its reverse path cap gamma tilde

## Description

This figure is in two parts: (a) and (b). Part (a) starts with a horizontal line with two points labelled a and b with a being furthest left and a bold line joining the two points. A curved arrow labelled gamma links this to an unlabelled copy of the complex plane concentrated in the upper-right quadrant. A contour labelled capital gamma is labelled and has labelled initial point gamma of a and final point gamma of b. A direction arrow is shown. The contour travels from the point gamma of a near the origin in a left to right and upwards direction to gamma of b. Part (b) again starts with a horizontal line with two points labelled a and b with a being furthest left and a bold line joining the two points. A curved arrow labelled gamma tilde links this to an unlabelled copy of the complex plane concentrated in the upper-right quadrant. A contour labelled capital gamma tilde is labelled and has labelled initial point gamma tilde of a and final point gamma tilde of b and gamma tilde of b is closest to the origin. A direction arrow is shown. The contour travels from point gamma tilde of a to gamma tilde of b right to left and downwards direction.

[Back to - Figure 20 (a) A smooth path normal cap gamma and (b) its reverse path cap gamma tilde](" \l "Unit2_Session3_Figure5)

# Figure 21 A contour sum with 3 summands normal cap gamma sub one plus normal cap gamma sub two plus normal cap gamma sub three and its reverse contour sum with 3 summands cap gamma tilde sub three plus cap gamma tilde sub two plus cap gamma tilde sub one

## Description

This figure shows two contours side by side, the one on the left is made up of three paths labelled capital gamma sub 1, capital gamma sub 2 and capital gamma sub 3. All have arrows marked in direction of travel. Capital gamma sub 1 is an arc travelling left to right top to bottom and looks like the lower left quarter of a circle. Capital gamma sub 2 is a vertical line travelling top to bottom and capital gamma sub 3 is an arc travelling right to left and top to bottom and looks like the top left quarter of a circle. The contour on the left is the reverse of this contour from capital gamma tilde sub 3 to capital gamma tilde sub 1.

[Back to - Figure 21 A contour sum with 3 summands normal cap gamma sub one plus normal cap gamma sub two plus normal cap gamma sub three and its reverse contour sum with 3 summands cap gamma tilde sub three plus cap gamma tilde sub two plus cap gamma tilde sub one](" \l "Unit2_Session3_Figure6)

# Figure 22 (a) The contour normal cap gamma equals sum with 3 summands normal cap gamma sub one plus normal cap gamma sub two plus normal cap gamma sub three (b) The reverse contour cap gamma tilde equals sum with 3 summands cap gamma tilde sub three plus cap gamma tilde sub two plus cap gamma tilde sub one

## Description

This figure is in two parts: (a) and (b). Each part is a copy of the unlabelled complex plane. Part (a) is concentrated in the upper-right quadrant and shows a contour capital gamma labelled and marked with an arrow in an anticlockwise direction. The contour is made up of three line segments. The first labelled capital gamma sub 1 from zero to the labelled point 2, the second labelled capital gamma sub 2 from the point 2 to the labelled point 2 plus i and the last labelled capital gamma sub 3 from 2 plus i to the labelled point i. Part (b) is identical to the left but it is the reverse contour that is indicated. The contours are labelled with capital gamma tilde.

[Back to - Figure 22 (a) The contour normal cap gamma equals sum with 3 summands normal cap gamma sub one plus normal cap gamma sub two plus normal cap gamma sub three (b) The reverse contour cap gamma tilde equals sum with 3 summands cap gamma tilde sub three plus cap gamma tilde sub two plus cap gamma tilde sub one](" \l "Unit2_Session3_Figure7)

# Uncaptioned Figure

## Description

This figure is in two parts: (a) and (b). Each part is a copy of the unlabelled complex plane concentrated in the upper-right quadrant and showing contours labelled capital gamma made up of two line segments. Part (a) has the first line segment from zero to the unlabelled point i with a direction arrow marked and the second from the unlabelled point i to the labelled point 1 plus i. Part (b) has the first line segment from zero to the unlabelled point 1 and marked with a direction arrow and the second from the unlabelled point 1 to the labelled point 1 plus i.

[Back to - Uncaptioned Figure](" \l "Unit2_Session3_Figure8)

# Figure 23 A contour normal cap gamma from zero to one plus i

## Description

This figure shows an unlabelled copy of the complex plane. The contour capital gamma is shown as two line segments, the first starting at the origin and finishing at the labelled point 1, the second starting at 1 to the labelled point 1 plus i. There is a direction arrow on both segments to indicate the direction of travel.

[Back to - Figure 23 A contour normal cap gamma from zero to one plus i](" \l "Unit2_Session4_Figure1)

# Figure 24 A semicircular path normal cap gamma from two to negative two

## Description

This figure shows an unlabelled copy of the complex plane focused on the upper half-plane. The contour capital gamma is shown as a semicircular path from the point labelled 2 to the point labelled negative 2 and centre origin. A directional arrow shows the anticlockwise direction of the path.

[Back to - Figure 24 A semicircular path normal cap gamma from two to negative two](" \l "Unit2_Session4_Figure2)

# Uncaptioned Figure

## Description

This figure shows a copy of the complex plane with unlabelled axes focused on the upper-right and lower-right quadrants. The points 1, 1 plus i, 6 plus i and 6 are labelled and marked as solid dots. 1 and 6 are on the positive real axis. 1 plus i and 6 plus i are in the upper-right quadrant. The point 1 plus i is directly above the point 1 and the point 6 plus i is directly above the point 6. The points pi over 2 and 3 pi over 2 are labelled and marked as hollow dots on the positive real axis with pi over 2 being to the right but close to the point 1 and the point 3 pi over 2 being to the left but close to the point 6. There is an unlabelled path made up of 3 joining line segments with direction arrows going from points 1 to 1 plus i then 1 plus i to 6 plus i and finally 6 plus i to 6. The entire complex plane except for the hollow dots is shaded.

[Back to - Uncaptioned Figure](" \l "Unit2_Session4_Figure3)

# Uncaptioned Figure

## Description

This figure shows an unlabelled copy of the complex plane. The points pi over 2 plus 2 i, 2 i, negative i and negative 3 pi over 2 minus i are all labelled as solid dots. The points 2 i and negative i are on the imaginary axis. The point pi over 2 plus 2 i is in the upper-right quadrant directly to the right of the point 2 i. The point negative 3 pi over 2 minus i is in the lower-right quadrant directly to the left of the point negative i. The points pi over 2, negative pi over 2 and negative 3 pi over 2 are labelled as hollow dots. An unlabelled path is shown as two horizontal line segments and one vertical line segment with marked direction arrow. The first line segment goes from point pi over 2 plus 2 i to point 2 i. The second from 2 i to minus i along the imaginary axis and the third from negative i to negative 3 pi over 2 minus i.

[Back to - Uncaptioned Figure](" \l "Unit2_Session4_Figure4)

# Uncaptioned Equation

## Alternative description

f of x minus f of c divided by x minus c full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session1_Equation1)

# Uncaptioned Equation

## Alternative description

f super prime of c equals lim over x right arrow c of f of x minus f of c divided by x minus c full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session1_Equation2)

# Uncaptioned Equation

## Alternative description

f of z minus f of alpha divided by z minus alpha comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation1)

# Uncaptioned Equation

## Alternative description

f times left parenthesis alpha plus h right parenthesis minus f of alpha divided by h comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation2)

# Uncaptioned Equation

## Alternative description

lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha times left parenthesis or lim over h right arrow zero of f times left parenthesis alpha plus h right parenthesis minus f of alpha divided by h right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation3)

# Uncaptioned Equation

## Alternative description

f super prime colon z long right arrow from bar f super prime of z

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation4)

# Uncaptioned Equation

## Alternative description

lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation5)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime of alpha equals lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha row 2 Blank equals lim over z right arrow alpha of z squared minus alpha squared divided by z minus alpha row 3 Blank equals lim over z right arrow alpha of z plus alpha full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation6)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime of alpha equals lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha row 2 Blank equals lim over z right arrow alpha of one minus one divided by z minus alpha row 3 Blank equation sequence part 1 equals part 2 lim over z right arrow alpha of zero divided by z minus alpha equals part 3 zero full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation7)

# Uncaptioned Equation

## Alternative description

f super prime of z equals zero times left parenthesis z element of double-struck cap c right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation8)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime of alpha equals lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha row 2 Blank equals lim over z right arrow alpha of z minus alpha divided by z minus alpha row 3 Blank equation sequence part 1 equals part 2 lim over z right arrow alpha of one equals part 3 one full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation9)

# Uncaptioned Equation

## Alternative description

f super prime of z equals one times left parenthesis z element of double-struck cap c right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation10)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime of alpha equals lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha row 2 Blank equals lim over z right arrow alpha of left parenthesis one solidus z right parenthesis minus left parenthesis one solidus alpha right parenthesis divided by z minus alpha row 3 Blank equals lim over z right arrow alpha of alpha minus z divided by z times alpha times left parenthesis z minus alpha right parenthesis row 4 Blank equals lim over z right arrow alpha of negative one divided by z times alpha full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation11)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f super prime of alpha equals part 2 lim over z right arrow alpha of negative one divided by z times alpha equals part 3 negative one divided by alpha squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation12)

# Uncaptioned Equation

## Alternative description

f super prime of z equals negative one divided by z squared times left parenthesis z not equals zero right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation13)

# Uncaptioned Equation

## Alternative description

lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha equals f super prime of alpha full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation14)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 lim over z right arrow alpha of f of z minus f of alpha equals lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha multiplication lim over z right arrow alpha of z minus alpha row 2 Blank equation sequence part 1 equals part 2 f super prime of alpha multiplication zero equals part 3 zero full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation15)

# Uncaptioned Equation

## Alternative description

f of z equals sum with 3 summands f of alpha plus left parenthesis z minus alpha right parenthesis times f super prime of alpha plus e of z comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation16)

# Uncaptioned Equation

## Alternative description

e of z equals f of z minus f of alpha minus left parenthesis z minus alpha right parenthesis times f super prime of alpha

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation17)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 lim over z right arrow alpha of e of z divided by z minus alpha equals lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha minus f super prime of alpha row 2 Blank equation sequence part 1 equals part 2 f super prime of alpha minus f super prime of alpha equals part 3 zero comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation18)

# Uncaptioned Equation

## Alternative description

left parenthesis f plus g right parenthesis super prime times left parenthesis alpha right parenthesis equals f super prime of alpha plus g super prime of alpha

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation19)

# Uncaptioned Equation

## Alternative description

left parenthesis lamda times f right parenthesis super prime times left parenthesis alpha right parenthesis equals lamda times f super prime of alpha

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation20)

# Uncaptioned Equation

## Alternative description

left parenthesis f times g right parenthesis super prime times left parenthesis alpha right parenthesis equals f super prime of alpha times g of alpha plus f of alpha times g super prime of alpha

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation21)

# Uncaptioned Equation

## Alternative description

left parenthesis f divided by g right parenthesis super prime times left parenthesis alpha right parenthesis equals g of alpha times f super prime of alpha minus f of alpha times g super prime of alpha divided by left parenthesis g of alpha right parenthesis squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation22)

# Uncaptioned Equation

## Alternative description

left parenthesis one divided by f right parenthesis super prime times left parenthesis alpha right parenthesis equals negative f super prime of alpha divided by left parenthesis f of alpha right parenthesis squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation23)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 Blank lim over z right arrow alpha of cap f of z minus cap f of alpha divided by z minus alpha Blank row 2 Blank equals lim over z right arrow alpha of f of z times g of z minus f of alpha times g of alpha divided by z minus alpha Blank row 3 Blank equals lim over z right arrow alpha of left parenthesis f of z minus f of alpha right parenthesis times g of z plus f of alpha times left parenthesis g of z minus g of alpha right parenthesis divided by z minus alpha Blank row 4 Blank equals left parenthesis lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha right parenthesis times left parenthesis lim over z right arrow alpha of g of z right parenthesis plus f of alpha times left parenthesis lim over z right arrow alpha of g of z minus g of alpha divided by z minus alpha right parenthesis Blank row 5 Blank equals f super prime of alpha times g of alpha plus f of alpha times g super prime of alpha full stop Blank

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation24)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 Blank lim over z right arrow alpha of cap f of z minus cap f of alpha divided by z minus alpha row 2 Blank equals lim over z right arrow alpha of left parenthesis f of z plus g of z right parenthesis minus left parenthesis f of alpha plus g of alpha right parenthesis divided by z minus alpha row 3 Blank equals lim over z right arrow alpha of left parenthesis f of z minus f of alpha right parenthesis plus left parenthesis g of z minus g of alpha right parenthesis divided by z minus alpha row 4 Blank equals lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha plus lim over z right arrow alpha of g of z minus g of alpha divided by z minus alpha row 5 Blank equals f super prime of alpha plus g super prime of alpha full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation25)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 lim over z right arrow alpha of cap f of z minus cap f of alpha divided by z minus alpha equals lim over z right arrow alpha of lamda times f of z minus lamda times f of alpha divided by z minus alpha row 2 Blank equals lamda times lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha row 3 Blank equals lamda times f super prime of alpha full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation26)

# Uncaptioned Equation

## Alternative description

f of z equals z super n times left parenthesis z element of double-struck cap c right parenthesis

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation27)

# Uncaptioned Equation

## Alternative description

f super prime of z equals n times z super n minus one times left parenthesis z element of double-struck cap c right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation28)

# Uncaptioned Equation

## Alternative description

if f of z equals sum with 3 summands z super four minus three times z squared plus two times z plus one comma then f super prime of z equals four times z cubed minus six times z plus two full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation29)

# Uncaptioned Equation

## Alternative description

p of z equals sum with variable number of summands a sub n times z super n plus ellipsis plus a sub two times z squared plus a sub one times z plus a sub zero times left parenthesis z element of double-struck cap c right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation30)

# Uncaptioned Equation

## Alternative description

p super prime of z equals sum with variable number of summands n times a sub n times z super n minus one plus ellipsis plus two times a sub two times z plus a sub one times left parenthesis z element of double-struck cap c right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation31)

# Uncaptioned Equation

## Alternative description

f of z equals two times z squared plus z divided by z squared plus one comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation32)

# Uncaptioned Equation

## Alternative description

z long right arrow from bar four times z plus one comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation33)

# Uncaptioned Equation

## Alternative description

z long right arrow from bar two times z full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation34)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f super prime of z equals part 2 left parenthesis z squared plus one right parenthesis times left parenthesis four times z plus one right parenthesis minus left parenthesis two times z squared plus z right parenthesis times left parenthesis two times z right parenthesis divided by left parenthesis z squared plus one right parenthesis squared equals part 3 sum with 3 summands negative z squared plus four times z plus one divided by left parenthesis z squared plus one right parenthesis squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation35)

# Uncaptioned Equation

## Alternative description

f super prime of z equals four times z cubed plus nine times z squared minus two times z plus four times left parenthesis z element of double-struck cap c right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation36)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime of z equals left parenthesis sum with 3 summands z squared plus z plus one right parenthesis times left parenthesis two times z minus four right parenthesis minus left parenthesis z squared minus four times z plus two right parenthesis times left parenthesis two times z plus one right parenthesis divided by left parenthesis sum with 3 summands z squared plus z plus one right parenthesis squared Blank row 2 Blank equals five times z squared minus two times z minus six divided by left parenthesis sum with 3 summands z squared plus z plus one right parenthesis squared full stop Blank

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation37)

# Uncaptioned Equation

## Alternative description

double-struck cap c minus negative one divided by two times left parenthesis one plus Square root of three times i right parenthesis comma negative one divided by two times left parenthesis one minus Square root of three times i right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation38)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f super prime of z equals part 2 negative n times z super n minus one divided by left parenthesis z super n right parenthesis squared equals part 3 negative n times z super negative n minus one full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation39)

# Uncaptioned Equation

## Alternative description

f super prime of z equals k times z super k minus one full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation40)

# Uncaptioned Equation

## Alternative description

Log of z equals log of absolute value of z plus i times Arg of z

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation41)

# Uncaptioned Equation

## Alternative description

lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation42)

# Uncaptioned Equation

## Alternative description

lim over n right arrow normal infinity of f of z sub n minus f of alpha divided by z sub n minus alpha

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation43)

# Uncaptioned Equation

## Alternative description

lim over n right arrow normal infinity of f of z sub n minus f of alpha divided by z sub n minus alpha not equals lim over n right arrow normal infinity of f of z sub n super prime minus f of alpha divided by z sub n super prime minus alpha comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation44)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 lim over n right arrow normal infinity of absolute value of z sub n minus absolute value of zero divided by z sub n minus zero equals part 2 lim over n right arrow normal infinity of one solidus n divided by one solidus n equals part 3 one full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation45)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 lim over n right arrow normal infinity of absolute value of z sub n super prime minus absolute value of zero divided by z sub n super prime minus zero equals part 2 lim over n right arrow normal infinity of one solidus n divided by negative one solidus n equals part 3 negative one full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation46)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 lim over n right arrow normal infinity of absolute value of z sub n minus absolute value of alpha divided by z sub n minus alpha equals part 2 lim over n right arrow normal infinity of absolute value of alpha minus absolute value of alpha divided by z sub n minus alpha equals part 3 zero full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation47)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 Blank equation sequence part 1 lim over n right arrow normal infinity of absolute value of z sub n super prime minus absolute value of alpha divided by z sub n super prime minus alpha equals part 2 lim over n right arrow normal infinity of absolute value of alpha times left parenthesis one plus one solidus n right parenthesis minus absolute value of alpha divided by alpha times left parenthesis one plus one solidus n right parenthesis minus alpha equals part 3 absolute value of alpha divided by alpha full stop Blank

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation48)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 lim over n right arrow normal infinity of zn macron minus alpha macron divided by z sub n minus alpha equals lim over n right arrow normal infinity of left parenthesis right parenthesis solidus solidus plus plus alpha one n macron minus alpha macron divided by left parenthesis alpha plus one solidus n right parenthesis minus alpha row 2 Blank equation sequence part 1 equals part 2 lim over n right arrow normal infinity of one solidus n divided by one solidus n equals part 3 one full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation49)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 lim over n right arrow normal infinity of zn prime macron minus alpha macron divided by z sub n super prime minus alpha equals lim over n right arrow normal infinity of left parenthesis right parenthesis solidus solidus plus plus alpha in macron minus alpha macron divided by left parenthesis alpha plus i solidus n right parenthesis minus alpha row 2 Blank equation sequence part 1 equals part 2 lim over n right arrow normal infinity of negative i solidus n divided by i solidus n equals part 3 negative one full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation50)

# Uncaptioned Equation

## Alternative description

w sub n equals f of z sub n minus f of alpha divided by z sub n minus alpha comma n equals one comma two comma ellipsis comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation51)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f of z sub n minus f of zero divided by z sub n minus zero equals part 2 Square root of one solidus n squared minus Square root of zero divided by one solidus n squared minus zero equals part 3 one solidus n divided by one solidus n squared equals part 4 n full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation52)

# Uncaptioned Equation

## Alternative description

f of z minus f of alpha divided by z minus alpha full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation53)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 lim over n right arrow normal infinity of Re of z sub n minus Re of i divided by z sub n minus i equals lim over n right arrow normal infinity of Re of i plus i solidus n minus Re of i divided by left parenthesis i plus i solidus n right parenthesis minus i Blank row 2 Blank equation sequence part 1 equals part 2 lim over n right arrow normal infinity of zero divided by i solidus n equals part 3 zero full stop Blank

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation54)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 lim over n right arrow normal infinity of Re of z sub n super prime minus Re of i divided by z sub n super prime minus i equals lim over n right arrow normal infinity of Re of i plus one solidus n minus Re of i divided by left parenthesis i plus one solidus n right parenthesis minus i Blank row 2 Blank equation sequence part 1 equals part 2 lim over n right arrow normal infinity of one solidus n divided by one solidus n equals part 3 one full stop Blank

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation55)

# Uncaptioned Equation

## Alternative description

left parenthesis f super prime right parenthesis super prime times left parenthesis z right parenthesis equals two divided by z cubed times left parenthesis z not equals zero right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation56)

# Uncaptioned Equation

## Alternative description

f super prime prime of z equals two divided by z cubed comma f super prime prime prime of z equals negative two multiplication three divided by z super four comma f super left parenthesis four right parenthesis of z equals two multiplication three multiplication four divided by z super five comma ellipsis comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation57)

# Uncaptioned Equation

## Alternative description

f super left parenthesis n right parenthesis of z equals left parenthesis negative one right parenthesis super n times n factorial divided by z super n plus one full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation58)

# Uncaptioned Equation

## Alternative description

f of z equals sum with 3 summands f of alpha plus left parenthesis z minus alpha right parenthesis times f super prime of alpha plus e of z comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation59)

# Uncaptioned Equation

## Alternative description

f of z minus f of alpha almost equals f super prime of alpha times left parenthesis z minus alpha right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation60)

# Uncaptioned Equation

## Alternative description

f of z almost equals f of alpha plus f super prime of alpha times left parenthesis z minus alpha right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation61)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f times left parenthesis one plus i right parenthesis equals part 2 one solidus left parenthesis one plus i right parenthesis equals part 3 one divided by two times left parenthesis one minus i right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation62)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f super prime times left parenthesis one plus i right parenthesis equals part 2 negative one divided by left parenthesis one plus i right parenthesis squared equals part 3 negative one divided by two times i equals part 4 i divided by two comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation63)

# Uncaptioned Equation

## Alternative description

f of z equals four times z plus three divided by two times z squared plus one full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation64)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f of i equals part 2 four times i plus three divided by two times i squared plus one equals part 3 negative three minus four times i full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation65)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime of z equals four times left parenthesis two times z squared plus one right parenthesis minus four times z times left parenthesis four times z plus three right parenthesis divided by left parenthesis two times z squared plus one right parenthesis squared row 2 Blank equals negative eight times z squared minus 12 times z plus four divided by left parenthesis two times z squared plus one right parenthesis squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation66)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f super prime of i equals part 2 negative eight times i squared minus 12 times i plus four divided by left parenthesis two times i squared plus one right parenthesis squared equals part 3 12 minus 12 times i full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation67)

# Uncaptioned Equation

## Alternative description

f of z equals two times z squared plus five full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation68)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime of alpha equals lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha row 2 Blank equals lim over z right arrow alpha of left parenthesis two times z squared plus five right parenthesis minus left parenthesis two times alpha squared plus five right parenthesis divided by z minus alpha row 3 Blank equals lim over z right arrow alpha of two times left parenthesis z squared minus alpha squared right parenthesis divided by z minus alpha row 4 Blank equals lim over z right arrow alpha of two times left parenthesis z plus alpha right parenthesis row 5 Blank equals four times alpha full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation69)

# Uncaptioned Equation

## Alternative description

f super prime of z equals four times z times left parenthesis z element of double-struck cap c right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation70)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 Blank cap f of z minus cap f of alpha divided by z minus alpha Blank row 2 Blank equals f of z solidus g of z minus f of alpha solidus g of alpha divided by z minus alpha row 3 Blank equals f of z times g of alpha minus f of alpha times g of z divided by left parenthesis z minus alpha right parenthesis times g of z times g of alpha Blank row 4 Blank equals g of alpha times left parenthesis f of z minus f of alpha right parenthesis minus f of alpha times left parenthesis g of z minus g of alpha right parenthesis divided by left parenthesis z minus alpha right parenthesis times g of z times g of alpha Blank row 5 Blank equals g of alpha times left parenthesis f of z minus f of alpha divided by z minus alpha right parenthesis minus f of alpha times left parenthesis g of z minus g of alpha divided by z minus alpha right parenthesis divided by g of z times g of alpha full stop Blank

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation71)

# Uncaptioned Equation

## Alternative description

cap f super prime of alpha equals g of alpha times f super prime of alpha minus f of alpha times g super prime of alpha divided by left parenthesis g of alpha right parenthesis squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation72)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime of z equals left parenthesis three times z plus one right parenthesis times left parenthesis two times z plus two right parenthesis minus three times left parenthesis sum with 3 summands z squared plus two times z plus one right parenthesis divided by left parenthesis three times z plus one right parenthesis squared row 2 Blank equals three times z squared plus two times z minus one divided by left parenthesis three times z plus one right parenthesis squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation73)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime of z equals left parenthesis z squared minus z minus six right parenthesis times left parenthesis three times z squared right parenthesis minus left parenthesis z cubed plus one right parenthesis times left parenthesis two times z minus one right parenthesis divided by left parenthesis z squared minus z minus six right parenthesis squared Blank row 2 Blank equals z super four minus two times z cubed minus 18 times z squared minus two times z plus one divided by left parenthesis z squared minus z minus six right parenthesis squared full stop Blank

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation74)

# Uncaptioned Equation

## Alternative description

f super prime of z equals negative left parenthesis two times z plus two right parenthesis divided by left parenthesis sum with 3 summands z squared plus two times z plus two right parenthesis squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation75)

# Uncaptioned Equation

## Alternative description

f super prime of z equals two times z plus five minus one divided by z squared minus two divided by z cubed full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation76)

# Uncaptioned Equation

## Alternative description

f of z equals Im of z

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation77)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 lim over n right arrow normal infinity of Im of z sub n minus Im of alpha divided by z sub n minus alpha equals lim over n right arrow normal infinity of b minus b divided by one solidus n row 2 Blank equation sequence part 1 equals part 2 lim over n right arrow normal infinity of zero divided by one solidus n equals part 3 zero full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation78)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 lim over n right arrow normal infinity of Im of z sub n super prime minus Im of alpha divided by z sub n super prime minus alpha equals lim over n right arrow normal infinity of left parenthesis b plus one solidus n right parenthesis minus b divided by i solidus n row 2 Blank equation sequence part 1 equals part 2 lim over n right arrow normal infinity of one solidus n divided by i solidus n equals part 3 negative i full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation79)

# Uncaptioned Equation

## Alternative description

f of z equals z cubed plus eight divided by z minus six

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation80)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime of z equals three times z squared times left parenthesis z minus six right parenthesis minus left parenthesis z cubed plus eight right parenthesis divided by left parenthesis z minus six right parenthesis squared row 2 Blank equals two times z cubed minus 18 times z squared minus eight divided by left parenthesis z minus six right parenthesis squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session2_Equation81)

# Uncaptioned Equation

## Alternative description

exp of x plus i times y equals e super x times left parenthesis cosine of y plus i times sine of y right parenthesis

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation1)

# Uncaptioned Equation

## Alternative description

u of x comma y equals e super x times cosine of y and v of x comma y equals e super x times sine of y comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation2)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f times left parenthesis x plus i times y right parenthesis equals part 2 left parenthesis x plus i times y right parenthesis cubed equals part 3 left parenthesis x cubed minus three times x times y squared right parenthesis plus i times left parenthesis three times x squared times y minus y cubed right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation3)

# Uncaptioned Equation

## Alternative description

u of x comma y equals x cubed minus three times x times y squared and v of x comma y equals three times x squared times y minus y cubed full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation4)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation5)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals three times x squared minus three times y squared comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation6)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis two comma one right parenthesis equals nine full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation7)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation8)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals negative six times x times y comma so prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis two comma one right parenthesis equals negative 12 semicolon

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation9)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals six times x times y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation10)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals three times x squared minus three times y squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation11)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis two comma one right parenthesis equals 12 and prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis two comma one right parenthesis equals nine full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation12)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis equals three times a squared minus three times b squared comma prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis equals six times a times b comma row 2 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis equals negative six times a times b comma prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis equals three times a squared minus three times b squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation13)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis and prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis equals negative prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation14)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis and prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis equals negative prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation15)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f super prime of alpha equals part 2 lim over z right arrow alpha of f of z minus f of alpha divided by z minus alpha equals part 3 lim over n right arrow normal infinity of f of z sub n minus f of alpha divided by z sub n minus alpha full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation16)

# Uncaptioned Equation

## Alternative description

f of z sub n minus f of alpha divided by z sub n minus alpha equals left parenthesis u of x sub n comma y sub n minus u of a comma b divided by left parenthesis x sub n minus a right parenthesis plus i times left parenthesis y sub n minus b right parenthesis right parenthesis plus i of v of x sub n comma y sub n minus v of a comma b divided by left parenthesis x sub n minus a right parenthesis plus i times left parenthesis y sub n minus b right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation17)

# Uncaptioned Equation

## Alternative description

f of z sub n minus f of alpha divided by z sub n minus alpha equals left parenthesis u of x sub n comma b minus u of a comma b divided by x sub n minus a right parenthesis plus i of v of x sub n comma b minus v of a comma b divided by x sub n minus a full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation18)

# Uncaptioned Equation

## Alternative description

u of x sub n comma b minus u of a comma b divided by x sub n minus a right arrow prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis and v of x sub n comma b minus v of a comma b divided by x sub n minus a right arrow prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation19)

# Uncaptioned Equation

## Alternative description

f super prime of alpha equals prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation20)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f of z sub n minus f of alpha divided by z sub n minus alpha equals left parenthesis u of a comma y sub n minus u of a comma b divided by i times left parenthesis y sub n minus b right parenthesis right parenthesis plus i of v of a comma y sub n minus v of a comma b divided by i times left parenthesis y sub n minus b right parenthesis row 2 Blank equals left parenthesis v of a comma y sub n minus v of a comma b divided by y sub n minus b right parenthesis minus i of u of a comma y sub n minus u of a comma b divided by y sub n minus b full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation21)

# Uncaptioned Equation

## Alternative description

f super prime of alpha equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis minus i times prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation22)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis not equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis or prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis not equals negative prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation23)

# Uncaptioned Equation

## Alternative description

f times left parenthesis x plus i times y right parenthesis equals left parenthesis x squared plus y squared right parenthesis plus i times left parenthesis two times x plus four times y right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation24)

# Uncaptioned Equation

## Alternative description

u of x comma y equals x squared plus y squared and v of x comma y equals two times x plus four times y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation25)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis and prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals negative prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation26)

# Uncaptioned Equation

## Alternative description

f times left parenthesis x plus i times y right parenthesis equals u of x comma y plus i times v of x comma y comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation27)

# Uncaptioned Equation

## Alternative description

u of x comma y equals e super x and v of x comma y equals negative e super y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation28)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals e super x and prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals negative e super y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation29)

# Uncaptioned Equation

## Alternative description

f times left parenthesis x plus i times y right parenthesis equals u of x comma y plus i times v of x comma y comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation30)

# Uncaptioned Equation

## Alternative description

u of x comma y equals x and v of x comma y equals negative y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation31)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals one and prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals negative one full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation32)

# Uncaptioned Equation

## Alternative description

u of x comma y equals case statement case 1column 1 comma of minmin left curly bracket right curly bracket comma x comma y comma comma comma x comma greater than greater than y zero comma case 2column 1 comma zero comma full stop otherwise full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation33)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis zero comma zero right parenthesis equals zero comma prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis zero comma zero right parenthesis equals zero comma row 2 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis zero comma zero right parenthesis equals zero comma prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis zero comma zero right parenthesis equals zero full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation34)

# Uncaptioned Equation

## Alternative description

multirelation f of z sub n minus f of zero divided by z sub n minus zero equals u of one solidus n comma zero minus zero divided by one solidus n minus zero equals zero right arrow zero comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation35)

# Uncaptioned Equation

## Alternative description

multirelation f of z sub n minus f of zero divided by z sub n minus zero equals u of one solidus n comma one solidus n minus zero divided by one solidus n plus i solidus n minus zero equals one solidus n divided by one solidus n plus i solidus n equals one divided by one plus i right arrow one divided by one plus i full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation36)

# Uncaptioned Equation

## Alternative description

f super prime times left parenthesis a plus i times b right parenthesis equals prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation37)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals two times x comma prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals two comma row 2 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals two times y comma prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals four full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation38)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f super prime times left parenthesis two minus i right parenthesis equals part 2 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis two comma negative one right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis two comma negative one right parenthesis equals part 3 two multiplication two plus i multiplication two equals part 4 four plus two times i full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation39)

# Uncaptioned Equation

## Alternative description

u of x comma y equals e super x times cosine of y and v of x comma y equals e super x times sine of y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation40)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals e super x times cosine of y comma prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals e super x times sine of y comma row 2 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals negative e super x times sine of y comma prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals e super x times cosine of y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation41)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f super prime of z equals part 2 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals part 3 e super x times cosine of y plus i times e super x times sine of y equals part 4 e super z full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation42)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 sine of x plus i times y equals sine of x times cosine of i times y plus cosine of x times sine of i times y row 2 Blank equals sine of x times hyperbolic cosine of y plus i times cosine of x times hyperbolic sine of y comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation43)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 u of x comma y equals sine of x times hyperbolic cosine of y and row 2 v of x comma y equals cosine of x times hyperbolic sine of y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation44)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals cosine of x times hyperbolic cosine of y comma row 2 prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals negative sine of x times hyperbolic sine of y comma row 3 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals sine of x times hyperbolic sine of y comma row 4 prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals cosine of x times hyperbolic cosine of y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation45)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis and row 2 prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals negative prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation46)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime times left parenthesis x plus i times y right parenthesis equals prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis row 2 Blank equals cosine of x times hyperbolic cosine of y minus i times sine of x times hyperbolic sine of y row 3 Blank equals cosine of x times cosine of i times y minus sine of x times sine of i times y row 4 Blank equals cosine of x plus i times y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation47)

# Uncaptioned Equation

## Alternative description

u of x comma y equals x squared plus y squared and v of x comma y equals zero full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation48)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals two times x comma prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals zero comma row 2 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals two times y comma prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals zero full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation49)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime of zero equals prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis zero comma zero right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis zero comma zero right parenthesis row 2 Blank equation sequence part 1 equals part 2 zero plus i times zero equals part 3 zero full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation50)

# Uncaptioned Equation

## Alternative description

f of x equals f of a plus left parenthesis x minus a right parenthesis times f super prime of c full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation51)

# Uncaptioned Equation

## Alternative description

f of x minus f of a divided by x minus a equals f super prime of c full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation52)

# Uncaptioned Equation

## Alternative description

t of x comma y equals sum with 3 summands u of a comma b plus left parenthesis x minus a right parenthesis times prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus left parenthesis y minus b right parenthesis times prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation53)

# Uncaptioned Equation

## Alternative description

u of x comma y equals sum with 4 summands u of a comma b plus left parenthesis x minus a right parenthesis times prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus left parenthesis y minus b right parenthesis times prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis plus e of x comma y comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation54)

# Uncaptioned Equation

## Alternative description

e of x comma y equals u of x comma y minus u of a comma b minus left parenthesis x minus a right parenthesis times prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis minus left parenthesis y minus b right parenthesis times prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation55)

# Uncaptioned Equation

## Alternative description

e of x comma y divided by Square root of left parenthesis x minus a right parenthesis squared plus left parenthesis y minus b right parenthesis squared right arrow zero as left parenthesis x comma y right parenthesis right arrow left parenthesis a comma b right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation56)

# Uncaptioned Equation

## Alternative description

u of x comma y equals u of a comma y plus left parenthesis x minus a right parenthesis times prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis r comma y right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation57)

# Uncaptioned Equation

## Alternative description

u of a comma y equals u of a comma b plus left parenthesis y minus b right parenthesis times prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma s right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation58)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 u of x comma y minus u of a comma b equals left parenthesis u of x comma y minus u of a comma y right parenthesis plus left parenthesis u of a comma y minus u of a comma b right parenthesis row 2 Blank equals left parenthesis x minus a right parenthesis times prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis r comma y right parenthesis plus left parenthesis y minus b right parenthesis times prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma s right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation59)

# Uncaptioned Equation

## Alternative description

e of x comma y equals left parenthesis x minus a right parenthesis times left parenthesis prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis r comma y right parenthesis minus prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis right parenthesis plus left parenthesis y minus b right parenthesis times left parenthesis prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma s right parenthesis minus prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation60)

# Uncaptioned Equation

## Alternative description

absolute value of x minus a divided by Square root of left parenthesis x minus a right parenthesis squared plus left parenthesis y minus b right parenthesis squared less than or equals one and absolute value of y minus b divided by Square root of left parenthesis x minus a right parenthesis squared plus left parenthesis y minus b right parenthesis squared less than or equals one

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation61)

# Uncaptioned Equation

## Alternative description

absolute value of e of x comma y divided by Square root of left parenthesis x minus a right parenthesis squared plus left parenthesis y minus b right parenthesis squared less than or equals absolute value of prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis r comma y right parenthesis minus prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus absolute value of prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma s right parenthesis minus prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation62)

# Uncaptioned Equation

## Alternative description

f super prime times left parenthesis a plus i times b right parenthesis equals prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation63)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f of z minus f of alpha equals left parenthesis u of x comma y minus u of a comma b right parenthesis plus i times left parenthesis v of x comma y minus v of a comma b right parenthesis row 2 Blank equals left parenthesis sum with 3 summands left parenthesis x minus a right parenthesis times prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus left parenthesis y minus b right parenthesis times prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis plus e sub u of x comma y right parenthesis row 3 Blank prefix plus of i times left parenthesis sum with 3 summands left parenthesis x minus a right parenthesis times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus left parenthesis y minus b right parenthesis times prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis plus e sub v of x comma y right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation64)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f of z minus f of alpha equals left parenthesis x minus a right parenthesis times left parenthesis prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis right parenthesis row 2 Blank sum with 3 summands prefix plus of i times left parenthesis y minus b right parenthesis times left parenthesis prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis minus i times prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis a comma b right parenthesis right parenthesis plus e sub u of x comma y plus i times e sub v of x comma y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation65)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f of z minus f of alpha equals left parenthesis left parenthesis x minus a right parenthesis plus i times left parenthesis y minus b right parenthesis right parenthesis times left parenthesis prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis right parenthesis row 2 Blank prefix plus of e sub u of x comma y plus i times e sub v of x comma y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation66)

# Uncaptioned Equation

## Alternative description

f of z minus f of alpha divided by z minus alpha equals left parenthesis prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis right parenthesis plus left parenthesis e sub u of x comma y plus i times e sub v of x comma y divided by left parenthesis x minus a right parenthesis plus i times left parenthesis y minus b right parenthesis right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation67)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis a comma b right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation68)

# Uncaptioned Equation

## Alternative description

absolute value of e sub u of x comma y plus i times e sub v of x comma y divided by left parenthesis x minus a right parenthesis plus i times left parenthesis y minus b right parenthesis less than or equals absolute value of e sub u of x comma y divided by Square root of left parenthesis x minus a right parenthesis squared plus left parenthesis y minus b right parenthesis squared plus absolute value of e sub v of x comma y divided by Square root of left parenthesis x minus a right parenthesis squared plus left parenthesis y minus b right parenthesis squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation69)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals sum with 3 summands three plus y plus four times x times y squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation70)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals x plus four times x squared times y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation71)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals cosine of y plus y times exp of x times y and row 2 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals negative x times sine of y plus x times exp of x times y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation72)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals three times left parenthesis x plus y right parenthesis squared and row 2 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals three times left parenthesis x plus y right parenthesis squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation73)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals three times x squared times y and row 2 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals x cubed minus cosine of y plus y times sine of y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation74)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis one comma zero right parenthesis equals zero and prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis one comma zero right parenthesis equals zero full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation75)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals y times e super x minus y cubed and row 2 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals e super x minus three times x times y squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation76)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis one comma zero right parenthesis equals zero and prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis one comma zero right parenthesis equals e full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation77)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals two times x plus two times y and prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals two times x full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation78)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis one comma two right parenthesis equals part 2 two multiplication one plus two multiplication two equals part 3 six full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation79)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis one comma two right parenthesis equals part 2 two multiplication one equals part 3 two full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation80)

# Uncaptioned Equation

## Alternative description

f times left parenthesis x plus i times y right parenthesis equals e super x times left parenthesis sine of y plus i times cosine of y right parenthesis

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation81)

# Uncaptioned Equation

## Alternative description

f times left parenthesis x plus i times y right parenthesis equals u of x comma y plus i times v of x comma y comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation82)

# Uncaptioned Equation

## Alternative description

u of x comma y equals e super x times sine of y and v of x comma y equals e super x times cosine of y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation83)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals e super x times sine of y comma prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals e super x times cosine of y comma row 2 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals e super x times cosine of y comma prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals negative e super x times sine of y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation84)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 e super x times sine of y equals negative e super x times sine of y and row 2 e super x times cosine of y equals negative e super x times cosine of y semicolon

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation85)

# Uncaptioned Equation

## Alternative description

e super x times sine of y equals zero and e super x times cosine of y equals zero full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation86)

# Uncaptioned Equation

## Alternative description

f times left parenthesis x plus i times y right parenthesis equals left parenthesis x squared plus x minus y squared right parenthesis plus i times left parenthesis two times x times y plus y right parenthesis

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation87)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 u of x comma y equals x squared plus x minus y squared and row 2 v of x comma y equals two times x times y plus y comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation88)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 Blank prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals two times x plus one comma prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals two times y comma row 2 Blank prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals negative two times y comma prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals two times x plus one full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation89)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis and row 2 prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals negative prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation90)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime times left parenthesis x plus i times y right parenthesis equals prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis row 2 Blank equals left parenthesis two times x plus one right parenthesis plus two times y times i full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation91)

# Uncaptioned Equation

## Alternative description

u of x comma y equals x squared plus y squared and v of x comma y equals x squared minus y squared comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation92)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals two times x comma prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals two times x comma row 2 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals two times y comma prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals negative two times y full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation93)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 f super prime times left parenthesis x plus i times y right parenthesis equals prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis row 2 Blank equation sequence part 1 equals part 2 two times x plus two times x times i equals part 3 two times x times left parenthesis one plus i right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation94)

# Uncaptioned Equation

## Alternative description

u of x comma y equals x times y and v of x comma y equals zero comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation95)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals y comma prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals zero comma row 2 prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals x comma prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis equals zero full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation96)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f super prime of zero equals part 2 prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis zero comma zero right parenthesis plus i times prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis zero comma zero right parenthesis equals part 3 zero full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation97)

# Uncaptioned Equation

## Alternative description

prefix partial differential of of u divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals prefix partial differential of of v divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis and prefix partial differential of of v divided by prefix partial differential of of x times left parenthesis x comma y right parenthesis equals negative prefix partial differential of of u divided by prefix partial differential of of y times left parenthesis x comma y right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation98)

# Uncaptioned Equation

## Alternative description

prefix partial differential of squared of u divided by prefix partial differential of of x squared equals prefix partial differential of squared of v divided by prefix partial differential of of x times prefix partial differential of of y and prefix partial differential of squared of v divided by prefix partial differential of of y times prefix partial differential of of x equals negative prefix partial differential of squared of u divided by prefix partial differential of of y squared full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation99)

# Uncaptioned Equation

## Alternative description

prefix partial differential of squared of v divided by prefix partial differential of of x times prefix partial differential of of y and prefix partial differential of squared of v divided by prefix partial differential of of y times prefix partial differential of of x

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation100)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 prefix partial differential of squared of u divided by prefix partial differential of of x squared equals part 2 prefix partial differential of squared of v divided by prefix partial differential of of x times prefix partial differential of of y equals part 3 prefix partial differential of squared of v divided by prefix partial differential of of y times prefix partial differential of of x equals part 4 negative prefix partial differential of squared of u divided by prefix partial differential of of y squared comma

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation101)

# Uncaptioned Equation

## Alternative description

prefix partial differential of squared of u divided by prefix partial differential of of x squared plus prefix partial differential of squared of u divided by prefix partial differential of of y squared equals zero full stop

[Back to - Uncaptioned Equation](" \l "Unit1_Session3_Equation102)

# Uncaptioned Equation

## Alternative description

integral over a under b x squared d x equals one divided by three times left parenthesis b cubed minus a cubed right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session1_Equation1)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma z squared d z equals one divided by three times left parenthesis beta times cubed minus alpha cubed right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session1_Equation2)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma f of z d z equals integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session1_Equation3)

# Uncaptioned Equation

## Alternative description

integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t equals integral over a under b u of t d t plus i times integral over a under b v of t d t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session1_Equation4)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 integral over normal cap gamma z squared d z equals part 2 one divided by three times left parenthesis beta times cubed minus alpha cubed right parenthesis equals part 3 zero full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session1_Equation5)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma sub one one divided by z d z equals zero comma but integral over normal cap gamma sub two one divided by z d z equals two times pi times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session1_Equation6)

# Uncaptioned Equation

## Alternative description

cap a equals integral over a under b f of x d x full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation1)

# Uncaptioned Equation

## Alternative description

cap p equals left square bracket x sub zero comma x sub one right square bracket comma left square bracket x sub one comma x sub two right square bracket comma ellipsis comma left square bracket x sub n minus one comma x sub n right square bracket comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation2)

# Uncaptioned Equation

## Alternative description

multirelation a equals x sub zero less than or equals x sub one less than or equals x sub two less than or equals ellipsis less than or equals x sub n equals b full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation3)

# Uncaptioned Equation

## Alternative description

absolute value of cap p equals max of delta times x sub one comma delta times x sub two comma ellipsis comma delta times x sub n full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation4)

# Uncaptioned Equation

## Alternative description

cap p equals left square bracket x sub zero comma x sub one right square bracket comma left square bracket x sub one comma x sub two right square bracket comma ellipsis comma left square bracket x sub n minus one comma x sub n right square bracket

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation5)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 cap r of f comma cap p equals part 2 n ary summation from k equals one to n over f of x sub k times delta times x sub k equals part 3 n ary summation from k equals one to n over f of x sub k times left parenthesis x sub k minus x sub k minus one right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation6)

# Uncaptioned Equation

## Alternative description

cap p sub n equals left square bracket zero comma one solidus n right square bracket comma left square bracket one solidus n comma two solidus n right square bracket comma ellipsis comma left square bracket left parenthesis n minus one right parenthesis solidus n comma one right square bracket comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation7)

# Uncaptioned Equation

## Alternative description

cap r of f comma cap p sub n equals one divided by six times left parenthesis one plus one solidus n right parenthesis times left parenthesis two plus one solidus n right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation8)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 cap r of f comma cap p sub n equals n ary summation from k equals one to n over f of k divided by n multiplication one divided by n row 2 Blank equals n ary summation from k equals one to n over left parenthesis k divided by n right parenthesis squared multiplication one divided by n row 3 Blank equals one divided by n cubed times n ary summation from k equals one to n over k squared full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation9)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 n ary summation from k equals one to n over k squared equals part 2 sum with variable number of summands one squared plus two squared plus ellipsis plus n squared equals part 3 one divided by six times n times left parenthesis n plus one right parenthesis times left parenthesis two times n plus one right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation10)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 cap r of f comma cap p sub n equals part 2 one divided by n cubed multiplication one divided by six times n times left parenthesis n plus one right parenthesis times left parenthesis two times n plus one right parenthesis equals part 3 one divided by six times left parenthesis one plus one solidus n right parenthesis times left parenthesis two plus one solidus n right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation11)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 lim over n right arrow normal infinity of cap r of f comma cap p sub n equals part 2 one divided by six times left parenthesis one plus zero right parenthesis times left parenthesis two plus zero right parenthesis equals part 3 one divided by three full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation12)

# Uncaptioned Equation

## Alternative description

sum with variable number of summands one cubed plus two cubed plus ellipsis plus n cubed equals one divided by four times n squared times left parenthesis n plus one right parenthesis squared full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation13)

# Uncaptioned Equation

## Alternative description

cap p sub n equals left square bracket zero comma one solidus n right square bracket comma left square bracket one solidus n comma two solidus n right square bracket comma ellipsis comma left square bracket left parenthesis n minus one right parenthesis solidus n comma one right square bracket comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation14)

# Uncaptioned Equation

## Alternative description

cap r of f comma cap p sub n equals one divided by four times left parenthesis one plus one solidus n right parenthesis squared comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation15)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 cap r of f comma cap p sub n equals n ary summation from k equals one to n over f of k divided by n multiplication one divided by n row 2 Blank equals n ary summation from k equals one to n over left parenthesis k divided by n right parenthesis cubed multiplication one divided by n row 3 Blank equals one divided by n super four times n ary summation from k equals one to n over k cubed row 4 Blank equals one divided by n super four multiplication one divided by four times n squared times left parenthesis n plus one right parenthesis squared row 5 Blank equals one divided by four times left parenthesis one plus one solidus n right parenthesis squared comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation16)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 lim over n right arrow normal infinity of cap r of f comma cap p sub n equals part 2 one divided by four times left parenthesis one plus zero right parenthesis squared equals part 3 one divided by four full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation17)

# Uncaptioned Equation

## Alternative description

lim over n right arrow normal infinity of cap r of f comma cap p sub n equals cap a comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation18)

# Uncaptioned Equation

## Alternative description

integral over a under b f of x d x full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation19)

# Uncaptioned Equation

## Alternative description

integral over zero under one x squared d x equals one divided by three full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation20)

# Uncaptioned Equation

## Alternative description

integral over a under b f of x d x equals negative integral over b under a f of x d x full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation21)

# Uncaptioned Equation

## Alternative description

integral over a under a f of x d x equals zero full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation22)

# Uncaptioned Equation

## Alternative description

integral over a under b f of x d x

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation23)

# Uncaptioned Equation

## Alternative description

integral over a under b f of x d x equals cap f of b minus cap f of a full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation24)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 integral over zero under one x squared d x equals part 2 left square bracket x cubed divided by three right square bracket sub zero super one equals part 3 one cubed divided by three minus zero cubed divided by three equals part 4 one divided by three comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation25)

# Uncaptioned Equation

## Alternative description

integral over a under b f of x d x equals integral over a under c f of x d x plus integral over c under b f of x d x comma for a less than or equals c less than or equals b full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation26)

# Uncaptioned Equation

## Alternative description

integral over a under b f of g of x times g super prime of x d x equals integral over g of a under g of b f of t d t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation27)

# Uncaptioned Equation

## Alternative description

integral over a under b f super prime of x times g of x d x equals left square bracket f of x times g of x right square bracket sub a super b minus integral over a under b f of x times g super prime of x d x full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation28)

# Uncaptioned Equation

## Alternative description

integral over a under b f of x d x less than or equals integral over a under b g of x d x full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation29)

# Uncaptioned Equation

## Alternative description

e super negative x less than or equals e super negative x squared less than or equals one divided by one plus x squared comma for zero less than or equals x less than or equals one comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation30)

# Uncaptioned Equation

## Alternative description

e super negative x less than or equals e super negative x squared less than or equals one divided by one plus x squared comma for zero less than or equals x less than or equals one comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation31)

# Uncaptioned Equation

## Alternative description

integral over zero under one e super negative x d x less than or equals integral over zero under one e super negative x squared d x less than or equals integral over zero under one one divided by one plus x squared d x full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation32)

# Uncaptioned Equation

## Alternative description

left square bracket negative e super negative x right square bracket sub zero super one less than or equals integral over zero under one e super negative x squared d x less than or equals left square bracket tangent super negative one of x right square bracket sub zero super one semicolon

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation33)

# Uncaptioned Equation

## Alternative description

one minus e super negative one less than or equals integral over zero under one e super negative x squared d x less than or equals pi divided by four full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation34)

# Uncaptioned Equation

## Alternative description

0.63 less than integral over zero under one e super negative x squared d x less than 0.79 full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation35)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma f of z d z full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation36)

# Uncaptioned Equation

## Alternative description

cap p equals normal cap gamma sub one comma normal cap gamma sub two comma ellipsis comma normal cap gamma sub n comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation37)

# Uncaptioned Equation

## Alternative description

cap r of f comma cap p equals n ary summation over k equals one under n of f of z sub k times delta times z sub k comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation38)

# Uncaptioned Equation

## Alternative description

absolute value of cap p equals max of absolute value of delta times z sub one comma absolute value of delta times z sub two comma ellipsis comma absolute value of delta times z sub n full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation39)

# Uncaptioned Equation

## Alternative description

lim over n right arrow normal infinity of cap r of f comma cap p sub n comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation40)

# Uncaptioned Equation

## Alternative description

t sub zero comma t sub one comma ellipsis comma t sub n colon multirelation a equals t sub zero less than t sub one less than ellipsis less than t sub n equals b

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation41)

# Uncaptioned Equation

## Alternative description

cap p equals normal cap gamma sub one comma normal cap gamma sub two comma ellipsis comma normal cap gamma sub n

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation42)

# Uncaptioned Equation

## Alternative description

cap r of f comma cap p equals n ary summation from k equals one to n over f of z sub k times delta times z sub k comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation43)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 delta times z sub k equals part 2 z sub k minus z sub k minus one equals part 3 gamma of t sub k minus gamma of t sub k minus one full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation44)

# Uncaptioned Equation

## Alternative description

multirelation gamma times super prime times left parenthesis t sub k right parenthesis almost equals gamma of t sub k minus gamma of t sub k minus one divided by t sub k minus t sub k minus one equals delta times z sub k divided by delta times t sub k comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation45)

# Uncaptioned Equation

## Alternative description

delta times z sub k almost equals gamma times super prime times left parenthesis t sub k right parenthesis times delta times t sub k full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation46)

# Uncaptioned Equation

## Alternative description

multirelation cap r of f comma cap p equals n ary summation from k equals one to n over f of z sub k times delta times z sub k almost equals n ary summation from k equals one to n over f of gamma of t sub k times gamma times super prime times left parenthesis t sub k right parenthesis times delta times t sub k full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation47)

# Uncaptioned Equation

## Alternative description

integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation48)

# Uncaptioned Equation

## Alternative description

t long right arrow from bar f of gamma of t times gamma times super prime times left parenthesis t right parenthesis times left parenthesis t element of left square bracket a comma b right square bracket right parenthesis

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation49)

# Uncaptioned Equation

## Alternative description

integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t equals integral over a under b u of t d t plus i times integral over a under b v of t d t comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation50)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma f of z d z equals integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation51)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma f of z d z full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation52)

# Uncaptioned Equation

## Alternative description

integral over negative normal infinity under normal infinity sine of x divided by x d x equals pi and integral over negative normal infinity under normal infinity e super negative x squared d x equals Square root of pi full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session2_Equation53)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma f of z d z equals integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation1)

# Uncaptioned Equation

## Alternative description

integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t equals integral over a under b u of t d t plus i times integral over a under b v of t d t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation2)

# Uncaptioned Equation

## Alternative description

integral over a under b u of t d t and integral over a under b v of t d t

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation3)

# Uncaptioned Equation

## Alternative description

integral over a under b f of t d t equals integral over a under b u of t d t plus i times integral over a under b v of t d t comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation4)

# Uncaptioned Equation

## Alternative description

z equals gamma of t comma d times z equals gamma times super prime times left parenthesis t right parenthesis times d times t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation5)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma z squared d z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation6)

# Uncaptioned Equation

## Alternative description

gamma of t equals left parenthesis one plus i right parenthesis times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation7)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z squared d z equals integral over zero under one f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t row 2 Blank equals integral over zero under one left parenthesis left parenthesis one plus i right parenthesis times t right parenthesis squared times left parenthesis one plus i right parenthesis d t row 3 Blank equals integral over zero under one two times i times t squared times left parenthesis one plus i right parenthesis d t row 4 Blank equals integral over zero under one left parenthesis negative two plus two times i right parenthesis times t squared d t row 5 Blank equals negative two times integral over zero under one t squared d t plus two times i times integral over zero under one t squared d t row 6 Blank equals left parenthesis negative two plus two times i right parenthesis times integral over zero under one t squared d t row 7 Blank equals left parenthesis negative two plus two times i right parenthesis times left square bracket one divided by three times t cubed right square bracket sub zero super one row 8 Blank equals negative two divided by three plus two divided by three times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation8)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma z macron d z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation9)

# Uncaptioned Equation

## Alternative description

gamma of t equals left parenthesis one plus i right parenthesis times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation10)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f of gamma of t equals part 2 times times left parenthesis right parenthesis plus plus one it macron equals part 3 left parenthesis one minus i right parenthesis times t comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation11)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z macron d z equals integral over zero under one left parenthesis one minus i right parenthesis times t multiplication left parenthesis one plus i right parenthesis d t row 2 Blank equals integral over zero under one two times t d t row 3 Blank equals left square bracket t squared right square bracket sub zero super one row 4 Blank equals one full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation12)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma one divided by z d z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation13)

# Uncaptioned Equation

## Alternative description

gamma of t equals e super i times t times left parenthesis t element of left square bracket zero comma two times pi right square bracket right parenthesis

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation14)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma one divided by z d z equals integral over zero under two times pi e super negative i times t multiplication i times e super i times t d t row 2 Blank equals i times integral over zero under two times pi one d t row 3 Blank equals two times pi times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation15)

# Uncaptioned Equation

## Alternative description

gamma of t equals two times left parenthesis one plus i right parenthesis times t times left parenthesis t element of left square bracket zero comma one divided by two right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation16)

# Uncaptioned Equation

## Alternative description

gamma of t equals e super three times i times t times left parenthesis t element of left square bracket zero comma two times pi solidus three right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation17)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f of gamma of t equals part 2 times times times two left parenthesis right parenthesis plus plus one it macron equals part 3 two times left parenthesis one minus i right parenthesis times t comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation18)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z macron d z equals integral over zero under one solidus two two times left parenthesis one minus i right parenthesis times t multiplication two times left parenthesis one plus i right parenthesis d t row 2 Blank equals integral over zero under one solidus two eight times t d t row 3 Blank equals left square bracket four times t squared right square bracket sub zero super one solidus two row 4 Blank equals one comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation19)

# Uncaptioned Equation

## Alternative description

z equals e super three times i times t comma one solidus z equals e super negative three times i times t and d times z equals three times i times e super three times i times t times d times t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation20)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma one divided by z d z equals integral over zero under two times pi solidus three e super negative three times i times t multiplication three times i times e super three times i times t d t row 2 Blank equals i times integral over zero under two times pi solidus three three d t row 3 Blank equals i times left square bracket three times t right square bracket sub zero super two times pi solidus three row 4 Blank equals two times pi times i comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation21)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma f of z d z

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation22)

# Uncaptioned Equation

## Alternative description

gamma of t equals left parenthesis one plus two times i right parenthesis times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation23)

# Uncaptioned Equation

## Alternative description

z equals left parenthesis one plus two times i right parenthesis times t comma Re of z equals t comma d times z equals left parenthesis one plus two times i right parenthesis times d times t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation24)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma Re of z times d times z equals integral over zero under one t multiplication left parenthesis one plus two times i right parenthesis d t row 2 Blank equals left parenthesis one plus two times i right parenthesis times integral over zero under one t d t row 3 Blank equals left parenthesis one plus two times i right parenthesis times left square bracket one divided by two times t squared right square bracket sub zero super one row 4 Blank equals one divided by two plus i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation25)

# Uncaptioned Equation

## Alternative description

gamma of t equals alpha plus r times e super i times t times left parenthesis t element of left square bracket zero comma two times pi right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation26)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 Blank z equals alpha plus r times e super i times t comma one solidus left parenthesis z minus alpha right parenthesis squared equals one solidus left parenthesis r squared times e super two times i times t right parenthesis comma row 2 Blank d times z equals r times i times e super i times t times d times t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation27)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma one divided by left parenthesis z minus alpha right parenthesis squared d z equals integral over zero under two times pi r times i times e super i times t divided by r squared times e super two times i times t d t Blank row 2 Blank equals integral over zero under two times pi i divided by r times e super negative i times t d t Blank row 3 Blank equals integral over zero under two times pi i divided by r times left parenthesis cosine of t minus i times sine of t right parenthesis d t Blank row 4 Blank equals integral over zero under two times pi one divided by r times sine of t times d times t plus i times integral over zero under two times pi one divided by r times cosine of t times d times t Blank row 5 Blank equals left square bracket negative one divided by r times cosine of t right square bracket sub zero super two times pi plus i times left square bracket one divided by r times sine of t right square bracket sub zero super two times pi Blank row 6 Blank equation sequence part 1 equals part 2 zero plus zero times i equals part 3 zero full stop Blank

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation28)

# Uncaptioned Equation

## Alternative description

gamma of t equals case statement case 1column 1 comma times times two t comma comma less than or equals less than or equals less than or equals zero t one comma case 2column 1 comma plus plus two times times i left parenthesis right parenthesis minus minus t one comma comma less than or equals less than or equals less than or equals one t two comma case 3column 1 comma minus minus plus plus two i times times two left parenthesis right parenthesis minus minus t two comma less than or equals less than or equals less than or equals two t three full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation29)

# Uncaptioned Equation

## Alternative description

normal cap gamma equals sum with variable number of summands normal cap gamma sub one plus normal cap gamma sub two plus ellipsis plus normal cap gamma sub n full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation30)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 Blank gamma sub one of t equals two times t left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis comma row 2 Blank gamma sub two of t equals two plus i times left parenthesis t minus one right parenthesis left parenthesis t element of left square bracket one comma two right square bracket right parenthesis comma row 3 Blank gamma sub three of t equals two plus i minus two times left parenthesis t minus two right parenthesis left parenthesis t element of left square bracket two comma three right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation31)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma f of z d z equals sum with variable number of summands integral over normal cap gamma sub one f of z d z plus integral over normal cap gamma sub two f of z d z plus ellipsis plus integral over normal cap gamma sub n f of z d z full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation32)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma f of z d z equals sum with 3 summands integral over normal cap gamma sub one f of z d z plus integral over normal cap gamma sub two f of z d z plus integral over normal cap gamma sub three f of z d z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation33)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 gamma sub one of t equals t left parenthesis t element of left square bracket zero comma two right square bracket right parenthesis comma row 2 gamma sub two of t equals two plus i times t left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis comma row 3 gamma sub three of t equals two plus i minus t left parenthesis t element of left square bracket zero comma two right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation34)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma z squared d z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation35)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z squared d z equals integral over normal cap gamma sub one z squared d z plus integral over normal cap gamma sub two z squared d z row 2 Blank equals integral over zero under one t squared d t plus integral over zero under one left parenthesis one plus i times t right parenthesis squared times i d t row 3 Blank equals integral over zero under one t squared d t plus integral over zero under one left parenthesis negative two times t plus i minus i times t squared right parenthesis d t row 4 Blank equals integral over zero under one left parenthesis t squared minus two times t right parenthesis d t plus i times integral over zero under one left parenthesis one minus t squared right parenthesis d t row 5 Blank equals left square bracket one divided by three times t cubed minus t squared right square bracket sub zero super one plus i times left square bracket t minus one divided by three times t cubed right square bracket sub zero super one row 6 Blank equals left parenthesis one divided by three minus one right parenthesis plus i times left parenthesis one minus one divided by three right parenthesis row 7 Blank equals negative two divided by three plus two divided by three times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation36)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma z squared d z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation37)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma z macron d z

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation38)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 Blank gamma sub one of t equals t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis comma row 2 Blank gamma sub two of t equals one plus i times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis comma row 3 Blank gamma sub three of t equals one minus t plus i times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation39)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z macron d z equals sum with 3 summands integral over normal cap gamma sub one z macron d z plus integral over normal cap gamma sub two z macron d z plus integral over normal cap gamma sub three z macron d z row 2 Blank equals integral over zero under one t multiplication one d t plus integral over zero under one left parenthesis one minus i times t right parenthesis multiplication i d t row 3 Blank prefix plus of integral over zero under one left parenthesis one minus t minus i right parenthesis multiplication left parenthesis negative one right parenthesis d t row 4 Blank equals integral over zero under one left parenthesis three times t plus two times i minus one right parenthesis d t row 5 Blank equals left square bracket three divided by two times t squared plus left parenthesis two times i minus one right parenthesis times t right square bracket sub zero super one row 6 Blank equation sequence part 1 equals part 2 three divided by two plus two times i minus one equals part 3 one divided by two plus two times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation40)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 Blank gamma sub one of t equals t times left parenthesis t element of left square bracket negative one comma one right square bracket right parenthesis comma row 2 Blank gamma sub two of t equals e super i times t times left parenthesis t element of left square bracket zero comma pi right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation41)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z macron d z equals integral over normal cap gamma sub one z macron d z plus integral over normal cap gamma sub two z macron d z row 2 Blank equals integral over negative one under one t multiplication one d t plus integral over zero under pi e super negative i times t multiplication i times e super i times t d t row 3 Blank equals integral over negative one under one t d t plus i times integral over zero under pi one d t row 4 Blank equals left square bracket one divided by two times t squared right square bracket sub negative one super one plus i times left square bracket t right square bracket sub zero super pi row 5 Blank equation sequence part 1 equals part 2 zero plus i times pi equals part 3 pi times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation42)

# Uncaptioned Equation

## Alternative description

gamma tilde of t equals gamma times left parenthesis a plus b minus t right parenthesis times left parenthesis t element of left square bracket a comma b right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation43)

# Uncaptioned Equation

## Alternative description

gamma of t equals two plus i minus t times left parenthesis t element of left square bracket zero comma two right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation44)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 gamma tilde of t equals gamma times left parenthesis two minus t right parenthesis row 2 Blank equals two plus i minus left parenthesis two minus t right parenthesis row 3 Blank equals t plus i times left parenthesis t element of left square bracket zero comma two right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation45)

# Uncaptioned Equation

## Alternative description

cap gamma tilde equals sum with variable number of summands cap gamma tilde sub n plus cap gamma tilde sub n minus one plus ellipsis plus cap gamma tilde sub one full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation46)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 gamma sub one of t equals t left parenthesis t element of left square bracket zero comma two right square bracket right parenthesis comma row 2 gamma sub two of t equals two plus i times t left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis comma row 3 gamma sub three of t equals two plus i minus t left parenthesis t element of left square bracket zero comma two right square bracket right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation47)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 gamma tilde sub three of t equals t plus i left parenthesis t element of left square bracket zero comma two right square bracket right parenthesis comma row 2 gamma tilde sub two of t equals two plus i times left parenthesis one minus t right parenthesis left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis comma row 3 gamma tilde sub one of t equals two minus t left parenthesis t element of left square bracket zero comma two right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation48)

# Uncaptioned Equation

## Alternative description

integral over cap gamma tilde z macron d z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation49)

# Uncaptioned Equation

## Alternative description

gamma of t equals left parenthesis one plus i right parenthesis times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation50)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 gamma tilde of t equals part 2 gamma times left parenthesis one minus t right parenthesis equals part 3 left parenthesis one plus i right parenthesis times left parenthesis one minus t right parenthesis times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation51)

# Uncaptioned Equation

## Alternative description

z equals left parenthesis one plus i right parenthesis times left parenthesis one minus t right parenthesis comma z macron equals left parenthesis one minus i right parenthesis times left parenthesis one minus t right parenthesis and d times z equals negative left parenthesis one plus i right parenthesis times d times t

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation52)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over cap gamma tilde z macron d z equals negative integral over zero under one left parenthesis one minus i right parenthesis times left parenthesis one minus t right parenthesis multiplication left parenthesis one plus i right parenthesis d t row 2 Blank equals negative integral over zero under one two times left parenthesis one minus t right parenthesis d t row 3 Blank equation sequence part 1 equals part 2 negative left square bracket two times t minus t squared right square bracket sub zero super one equals part 3 negative one full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation53)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma z macron d z equals one comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation54)

# Uncaptioned Equation

## Alternative description

integral over cap gamma tilde f of z d z equals negative integral over normal cap gamma f of z d z full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation55)

# Uncaptioned Equation

## Alternative description

gamma tilde of t equals gamma times left parenthesis a plus b minus t right parenthesis times left parenthesis t element of left square bracket a comma b right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation56)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over cap gamma tilde f of z d z equals integral over a under b f of gamma tilde of t times gamma tilde times super prime times left parenthesis t right parenthesis d t row 2 Blank equals integral over a under b f of gamma times left parenthesis a plus b minus t right parenthesis times left parenthesis negative gamma times super prime times left parenthesis a plus b minus t right parenthesis right parenthesis d t row 3 Blank equals integral over b under a f of gamma of s times gamma times super prime times left parenthesis s right parenthesis d s row 4 Blank equals negative integral over normal cap gamma f of z d z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation57)

# Uncaptioned Equation

## Alternative description

s equals a plus b minus t comma d times s equals negative d times t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation58)

# Uncaptioned Equation

## Alternative description

cap gamma tilde equals sum with variable number of summands cap gamma tilde sub n plus cap gamma tilde sub n minus one plus ellipsis plus cap gamma tilde sub one comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation59)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over cap gamma tilde f equals integral over cap gamma tilde sub n f plus integral over cap gamma tilde sub n minus one f postfix plus times ellipsis plus integral over cap gamma tilde sub one f row 2 Blank equals negative integral over normal cap gamma sub n f minus integral over normal cap gamma sub n minus one f postfix minus times ellipsis minus integral over normal cap gamma sub one f row 3 Blank equals negative left parenthesis integral over normal cap gamma sub n f plus integral over normal cap gamma sub n minus one f postfix plus times ellipsis plus integral over normal cap gamma sub one f right parenthesis row 4 Blank equals negative integral over normal cap gamma f full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation60)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma one divided by z d z equals two times pi times i comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation61)

# Uncaptioned Equation

## Alternative description

integral over cap gamma tilde one divided by z d z equals negative two times pi times i comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation62)

# Uncaptioned Equation

## Alternative description

gamma of t equals e super i times t times left parenthesis t element of left square bracket zero comma two times pi right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation63)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 gamma tilde of t equals part 2 gamma times left parenthesis two times pi minus t right parenthesis equals part 3 e super i times left parenthesis two times pi minus t right parenthesis times left parenthesis t element of left square bracket zero comma two times pi right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation64)

# Uncaptioned Equation

## Alternative description

gamma tilde of t equals e super negative i times t times left parenthesis t element of left square bracket zero comma two times pi right square bracket right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation65)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over cap gamma tilde one divided by z d z equals integral over zero under two times pi one divided by e super negative i times t multiplication left parenthesis negative i times e super negative i times t right parenthesis d t row 2 Blank equals negative i times integral over zero under two times pi one d t row 3 Blank equals negative two times pi times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation66)

# Uncaptioned Equation

## Alternative description

integral over cap gamma tilde one divided by z d z equals negative integral over normal cap gamma one divided by z d z full stop right parenthesis

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation67)

# Uncaptioned Equation

## Alternative description

gamma of t equals one minus t plus i times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis semicolon

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation68)

# Uncaptioned Equation

## Alternative description

gamma times super prime times left parenthesis t right parenthesis equals i minus one full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation69)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z d z equals integral over zero under one left parenthesis one minus t plus i times t right parenthesis multiplication left parenthesis i minus one right parenthesis d t row 2 Blank equals integral over zero under one left parenthesis negative one plus left parenthesis one minus two times t right parenthesis times i right parenthesis d t row 3 Blank equals integral over zero under one left parenthesis negative one right parenthesis d t plus i times integral over zero under one left parenthesis one minus two times t right parenthesis d t row 4 Blank equals left square bracket negative t right square bracket sub zero super one plus i times left square bracket t minus t squared right square bracket sub zero super one row 5 Blank equals negative one full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation70)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma Im of z times d times z equals integral over zero under one left parenthesis Im of one minus t plus i times t right parenthesis multiplication left parenthesis i minus one right parenthesis d t row 2 Blank equals integral over zero under one t times left parenthesis i minus one right parenthesis d t row 3 Blank equals left parenthesis i minus one right parenthesis times integral over zero under one t d t row 4 Blank equals left parenthesis i minus one right parenthesis times left square bracket one divided by two times t squared right square bracket sub zero super one row 5 Blank equals one divided by two times left parenthesis negative one plus i right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation71)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z macron d z equals integral over zero under one left parenthesis right parenthesis plus plus minus minus one t times times it macron multiplication left parenthesis i minus one right parenthesis d t row 2 Blank equals integral over zero under one left parenthesis one minus t minus i times t right parenthesis multiplication left parenthesis i minus one right parenthesis d t row 3 Blank equals integral over zero under one left parenthesis sum with 3 summands negative one plus two times t plus i right parenthesis d t row 4 Blank equals integral over zero under one left parenthesis negative one plus two times t right parenthesis d t plus i times integral over zero under one one d t row 5 Blank equals left square bracket negative t plus t squared right square bracket sub zero super one plus i times left square bracket t right square bracket sub zero super one row 6 Blank equals i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation72)

# Uncaptioned Equation

## Alternative description

gamma of t equals e super i times t times left parenthesis t element of left square bracket zero comma two times pi right square bracket right parenthesis semicolon

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation73)

# Uncaptioned Equation

## Alternative description

z equals e super i times t comma d times z equals i times e super i times t times d times t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation74)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z macron d z equals integral over zero under two times pi e super negative i times t multiplication i times e super i times t d t row 2 Blank equals i times integral over zero under two times pi one d t row 3 Blank equals i times left square bracket t right square bracket sub zero super two times pi row 4 Blank equals two times pi times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation75)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z squared d z equals integral over zero under two times pi e super two times i times t multiplication i times e super i times t d t row 2 Blank equals integral over zero under two times pi i times e super three times i times t d t row 3 Blank equals integral over zero under two times pi i times left parenthesis cosine of three times t plus i times sine of three times t right parenthesis d t row 4 Blank equals integral over zero under two times pi left parenthesis negative sine of three times t right parenthesis d t plus i times integral over zero under two times pi cosine of three times t times d times t row 5 Blank equals left square bracket one divided by three times cosine of three times t right square bracket sub zero super two times pi plus i times left square bracket one divided by three times sine of three times t right square bracket sub zero super two times pi row 6 Blank equals zero full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation76)

# Uncaptioned Equation

## Alternative description

gamma of t equals two times e super i times t times left parenthesis t element of left square bracket zero comma pi right square bracket right parenthesis semicolon

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation77)

# Uncaptioned Equation

## Alternative description

gamma times super prime times left parenthesis t right parenthesis equals two times i times e super i times t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation78)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma one divided by z d z equals integral over zero under pi one divided by two times e super i times t multiplication two times i times e super i times t d t row 2 Blank equals i times integral over zero under pi one d t row 3 Blank equals i times left square bracket t right square bracket sub zero super pi row 4 Blank equals pi times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation79)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma absolute value of z d z equals integral over zero under pi absolute value of two times e super i times t multiplication two times i times e super i times t d t row 2 Blank equals integral over zero under pi four times i times left parenthesis cosine of t plus i times sine of t right parenthesis d t row 3 Blank equals integral over zero under pi left parenthesis negative four times sine of t right parenthesis d t plus i times integral over zero under pi four times cosine of t times d times t row 4 Blank equals left square bracket four times cosine of t right square bracket sub zero super pi plus i times left square bracket four times sine of t right square bracket sub zero super pi row 5 Blank equals negative eight full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation80)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma Re of z times d times z

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation81)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 Blank gamma sub one of t equals i times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis comma row 2 Blank gamma sub two of t equals t plus i times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation82)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma Re of z times d times z equals integral over normal cap gamma sub one Re of z times d times z plus integral over normal cap gamma sub two Re of z times d times z Blank row 2 Blank equals integral over zero under one Re of i times t multiplication i d t plus integral over zero under one Re of t plus i multiplication one d t Blank row 3 Blank equals integral over zero under one zero d t plus integral over zero under one t d t Blank row 4 Blank equation sequence part 1 equals part 2 left square bracket one divided by two times t squared right square bracket sub zero super one equals part 3 one divided by two full stop Blank

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation83)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 Blank gamma sub one of t equals t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis comma row 2 Blank gamma sub two of t equals one plus i times t times left parenthesis t element of left square bracket zero comma one right square bracket right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation84)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma Re of z times d times z equals integral over normal cap gamma sub one Re of z times d times z plus integral over normal cap gamma sub two Re of z times d times z Blank row 2 Blank equals integral over zero under one Re of t multiplication one d t plus integral over zero under one Re of one plus i times t multiplication i d t Blank row 3 Blank equals integral over zero under one t d t plus i times integral over zero under one one d t Blank row 4 Blank equation sequence part 1 equals part 2 left square bracket one divided by two times t squared right square bracket sub zero super one plus i times left square bracket t right square bracket sub zero super one equals part 3 one divided by two plus i full stop Blank

[Back to - Uncaptioned Equation](" \l "Unit2_Session3_Equation85)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma z squared d z equals negative two divided by three plus two divided by three times i comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation1)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z squared d z equals left square bracket one divided by three times z cubed right square bracket sub zero super one plus i row 2 Blank equals one divided by three times left parenthesis one plus i right parenthesis cubed minus one divided by three multiplication zero cubed row 3 Blank equals one divided by three times left parenthesis sum with 4 summands one plus three times i plus three times i squared plus i cubed right parenthesis row 4 Blank equals negative two divided by three plus two divided by three times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation2)

# Uncaptioned Equation

## Alternative description

cap f super prime of z equals f of z comma for all z element of script cap r full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation3)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma f of z d z equals cap f of beta minus cap f of alpha full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation4)

# Uncaptioned Equation

## Alternative description

left square bracket cap f of z right square bracket sub alpha super beta equals cap f of beta minus cap f of alpha full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation5)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 integral over normal cap gamma z squared d z equals part 2 left square bracket one divided by three times z cubed right square bracket sub zero super one plus i equals part 3 one divided by three times left parenthesis one plus i right parenthesis cubed minus one divided by three multiplication zero cubed equals part 4 negative two divided by three plus two divided by three times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation6)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma e super three times i times z d z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation7)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma e super three times i times z d z equals cap f of negative two minus cap f of two row 2 Blank equation sequence part 1 equals part 2 one divided by three times i times left parenthesis e super negative six times i minus e super six times i right parenthesis equals part 3 negative two divided by three times sine of six full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation8)

# Uncaptioned Equation

## Alternative description

sine of z equals one divided by two times i times left parenthesis e super i times z minus e super negative i times z right parenthesis comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation9)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma z squared d z equals negative two divided by three plus two divided by three times i

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation10)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma sub one f of z d z equals integral over normal cap gamma sub two f of z d z full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation11)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma e super negative pi times z d z equals cap f of i minus cap f of negative i row 2 Blank equals left parenthesis negative e super negative pi times i solidus pi right parenthesis minus left parenthesis negative e super pi times i solidus pi right parenthesis row 3 Blank equation sequence part 1 equals part 2 one solidus pi minus one solidus pi equals part 3 zero full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation12)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma left parenthesis three times z minus one right parenthesis squared d z equals cap f times left parenthesis two times i plus one divided by three right parenthesis minus cap f of two row 2 Blank equals one divided by nine times left parenthesis six times i right parenthesis cubed minus one divided by nine multiplication five cubed row 3 Blank equals negative one divided by nine times left parenthesis 125 plus 216 times i right parenthesis full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation13)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma hyperbolic sine of z times d times z equals cap f of one minus cap f of i row 2 Blank equals hyperbolic cosine of one minus hyperbolic cosine of i row 3 Blank equals hyperbolic cosine of one minus cosine of one full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation14)

# Uncaptioned Equation

## Alternative description

exp of sine of z multiplication sine super prime of z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation15)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma e super sine of z times cosine of z times d times z equals cap f times left parenthesis pi solidus two right parenthesis minus cap f of zero row 2 Blank equals exp of sine of pi solidus two minus exp of sine of zero row 3 Blank equals e minus one full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation16)

# Uncaptioned Equation

## Alternative description

negative one divided by cosine squared of z times cosine super prime of z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation17)

# Uncaptioned Equation

## Alternative description

left parenthesis h ring operator cosine right parenthesis super prime times left parenthesis z right parenthesis comma where h of z equals one solidus z full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation18)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 Blank f of z equals sine of z solidus cosine squared of z comma row 2 Blank equation sequence part 1 cap f of z equals part 2 h of cosine of z equals part 3 one solidus cosine of z comma row 3 Blank script cap r equals double-struck cap c minus left parenthesis n plus one divided by two right parenthesis times pi colon n element of double-struck cap z full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation19)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma sine of z divided by cosine squared of z d z equals cap f of pi minus cap f of zero row 2 Blank equals one divided by cosine of pi minus one divided by cosine of zero row 3 Blank equation sequence part 1 equals part 2 negative one minus one equals part 3 negative two full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation20)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma f of z times g super prime of z d z equals left square bracket f of z times g of z right square bracket sub alpha super beta minus integral over normal cap gamma f super prime of z times g of z d z full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation21)

# Uncaptioned Equation

## Alternative description

cap h super prime of z equals h of z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation22)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma h of z d z equals left square bracket cap h of z right square bracket sub alpha super beta semicolon

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation23)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma left parenthesis f super prime of z times g of z plus f of z times g super prime of z right parenthesis d z equals left square bracket f of z times g of z right square bracket sub alpha super beta full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation24)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma f of z times g super prime of z d z equals left square bracket f of z times g of z right square bracket sub alpha super beta minus integral over normal cap gamma f super prime of z times g of z d z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation25)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma z times e super two times z d z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation26)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z times e super two times z d z equals left square bracket z multiplication one divided by two times e super two times z right square bracket sub zero super pi times i minus integral over normal cap gamma one multiplication one divided by two times e super two times z d z row 2 Blank equals left parenthesis pi times i multiplication one divided by two times e super two times pi times i minus zero right parenthesis minus left square bracket one divided by four times e super two times z right square bracket sub zero super pi times i row 3 Blank equals one divided by two times pi times i minus left parenthesis one divided by four minus one divided by four right parenthesis row 4 Blank equals one divided by two times pi times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation27)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z times hyperbolic cosine of z times d times z equals left square bracket z times hyperbolic sine of z right square bracket sub zero super pi times i minus integral over normal cap gamma one multiplication hyperbolic sine of z times d times z row 2 Blank equals left parenthesis pi times i times hyperbolic sine of pi times i minus zero right parenthesis minus left square bracket hyperbolic cosine of z right square bracket sub zero super pi times i row 3 Blank equals pi times i multiplication i times sine of pi minus left parenthesis cosine of pi minus hyperbolic cosine of zero right parenthesis row 4 Blank equation sequence part 1 equals part 2 zero minus left parenthesis negative one minus one right parenthesis equals part 3 two full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation28)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma Log of z times d times z equals left square bracket z times Log of z right square bracket sub one super i minus integral over normal cap gamma one divided by z multiplication z d z Blank row 2 Blank equals i times Log of i minus Log of one minus left square bracket z right square bracket sub one super i Blank row 3 Blank equation sequence part 1 equals part 2 negative pi solidus two minus left parenthesis i minus one right parenthesis equals part 3 left parenthesis one minus pi solidus two right parenthesis minus i full stop Blank

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation29)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma z macron d z

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation30)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 Blank z long right arrow from bar z macron comma z long right arrow from bar Re of z comma z long right arrow from bar Im of z and z long right arrow from bar absolute value of z full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation31)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma f of z d z equals integral over a under b f of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t row 2 Blank equals integral over a under b cap f super prime of gamma of t times gamma times super prime times left parenthesis t right parenthesis d t row 3 Blank equals integral over a under b left parenthesis cap f ring operator gamma right parenthesis super prime times left parenthesis t right parenthesis d t comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation32)

# Uncaptioned Equation

## Alternative description

integral over a under b left parenthesis cap f ring operator gamma right parenthesis super prime times left parenthesis t right parenthesis d t equals integral over a under b u super prime of t d t plus i times integral over a under b v super prime of t d t full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation33)

# Uncaptioned Equation

## Alternative description

integral over a under b u super prime of t d t equals u of b minus u of a and integral over a under b v super prime of t d t equals v of b minus v of a full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation34)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 integral over normal cap gamma f of z d z equals part 2 left parenthesis u of b minus u of a right parenthesis plus i times left parenthesis v of b minus v of a right parenthesis equals part 3 cap f of beta minus cap f of alpha comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation35)

# Uncaptioned Equation

## Alternative description

alpha sub one equals alpha comma alpha sub two equals beta sub one comma ellipsis comma alpha sub n equals beta sub n minus one comma beta sub n equals beta full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation36)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 integral over normal cap gamma sub k f of z d z equals part 2 cap f of beta sub k minus cap f of alpha sub k equals part 3 cap f of beta sub k minus cap f of beta sub k minus one comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation37)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma f of z d z equals sum with variable number of summands integral over normal cap gamma sub one f of z d z plus integral over normal cap gamma sub two f of z d z plus ellipsis plus integral over normal cap gamma sub n f of z d z row 2 Blank equals sum with variable number of summands left parenthesis cap f of beta sub one minus cap f of beta sub zero right parenthesis plus ellipsis plus left parenthesis cap f of beta sub n minus cap f of beta sub n minus one right parenthesis row 3 Blank equals cap f of beta sub n minus cap f of beta sub zero row 4 Blank equals cap f of beta minus cap f of alpha full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation38)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma f of z d z comma

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation39)

# Uncaptioned Equation

## Alternative description

cap f of z equals one divided by two times e super z squared full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation40)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z times e super z squared d z equals left square bracket one divided by two times e super z squared right square bracket sub negative i super i row 2 Blank equation sequence part 1 equals part 2 one divided by two times left parenthesis e super negative one minus e super negative one right parenthesis equals part 3 zero full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation41)

# Uncaptioned Equation

## Alternative description

cap f of z equals one divided by four times hyperbolic sine of z super four full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation42)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z cubed times hyperbolic cosine of z super four d z equals left square bracket one divided by four times hyperbolic sine of z super four right square bracket sub negative i super i row 2 Blank equation sequence part 1 equals part 2 one divided by four times left parenthesis hyperbolic sine of one minus hyperbolic sine of one right parenthesis equals part 3 zero full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation43)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma z times e super z d z equals left square bracket z times e super z right square bracket sub negative i super i minus integral over normal cap gamma one multiplication e super z d z row 2 Blank equals left parenthesis i times e super i minus left parenthesis negative i right parenthesis times e super negative i right parenthesis minus integral over normal cap gamma e super z d z row 3 Blank equals i times left parenthesis e super i plus e super negative i right parenthesis minus left square bracket e super z right square bracket sub negative i super i row 4 Blank equals two times i times cosine of one minus left parenthesis e super i minus e super negative i right parenthesis row 5 Blank equals two times i times cosine of one minus two times i times sine of one row 6 Blank equals two times left parenthesis cosine of one minus sine of one right parenthesis times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation44)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma one divided by z d z equals left square bracket Log of z right square bracket sub negative i super i row 2 Blank equals Log of i minus Log of negative i row 3 Blank equation sequence part 1 equals part 2 pi divided by two times i minus left parenthesis negative pi divided by two times i right parenthesis equals part 3 pi times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation45)

# Uncaptioned Equation

## Alternative description

multiline equation row 1 integral over normal cap gamma Square root of z d z equals left square bracket two divided by three times z super three solidus two right square bracket sub negative i super i row 2 Blank equals two divided by three times left parenthesis i super three solidus two minus left parenthesis negative i right parenthesis super three solidus two right parenthesis row 3 Blank equals two divided by three times left parenthesis exp of three divided by two times Log of i minus exp of three divided by two times Log of negative i right parenthesis row 4 Blank equals two divided by three times left parenthesis exp of three times pi divided by four times i minus exp of negative three times pi divided by four times i right parenthesis row 5 Blank equals two divided by three times left parenthesis two times i times sine of three times pi divided by four right parenthesis row 6 Blank equals two times Square root of two divided by three times i full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation46)

# Uncaptioned Equation

## Alternative description

equation sequence part 1 f of z equals part 2 sine squared of z equals part 3 one divided by two times left parenthesis one minus cosine of two times z right parenthesis

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation47)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma sine squared of z times d times z equals zero full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation48)

# Uncaptioned Equation

## Alternative description

integral over normal cap gamma one divided by z cubed d z equals zero full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation49)

# Uncaptioned Equation

## Alternative description

script cap r equals double-struck cap c minus left parenthesis n plus one divided by two right parenthesis times pi colon n element of double-struck cap z full stop

[Back to - Uncaptioned Equation](" \l "Unit2_Session4_Equation50)