John Napier

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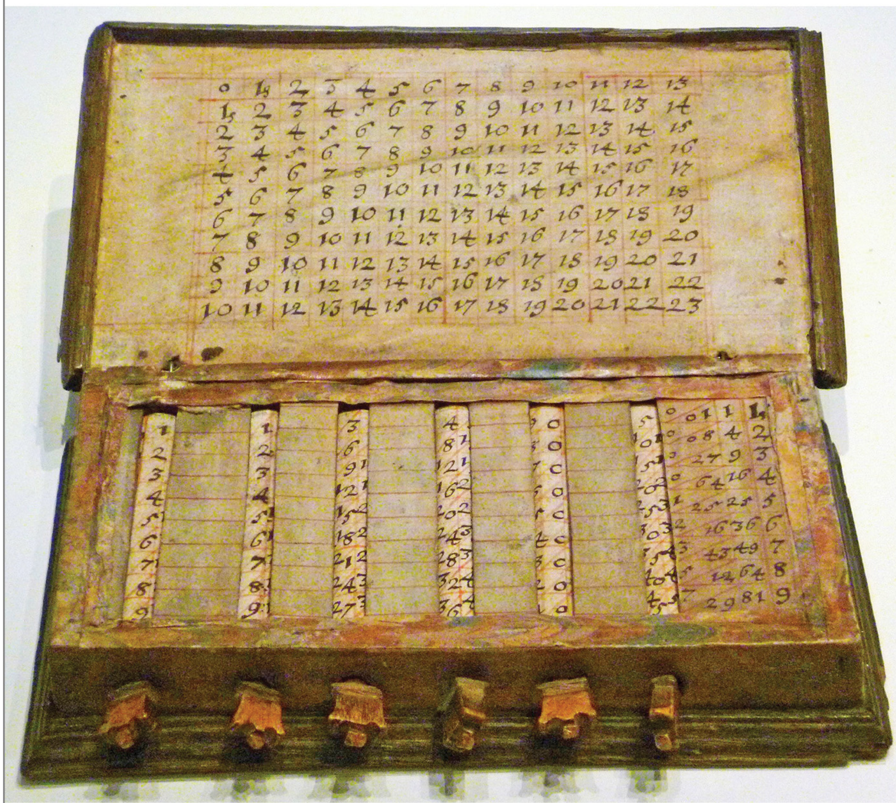

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The Open University, Walton Hall, Milton Keynes, MK7 6AA

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Contents

Introduction

This course provides an overview of Scotsman John Napier (1550-1617) and his new method for reducing the work needed to do complicated calculations. The course introduces how logarithms can turn multiplications into additions, and explores how Napier's work was adopted and improved by other mathematicians.

This OpenLearn course provides a sample of Level 2 study in Mathematics

Learning outcomes

After studying this course, you should be able to:

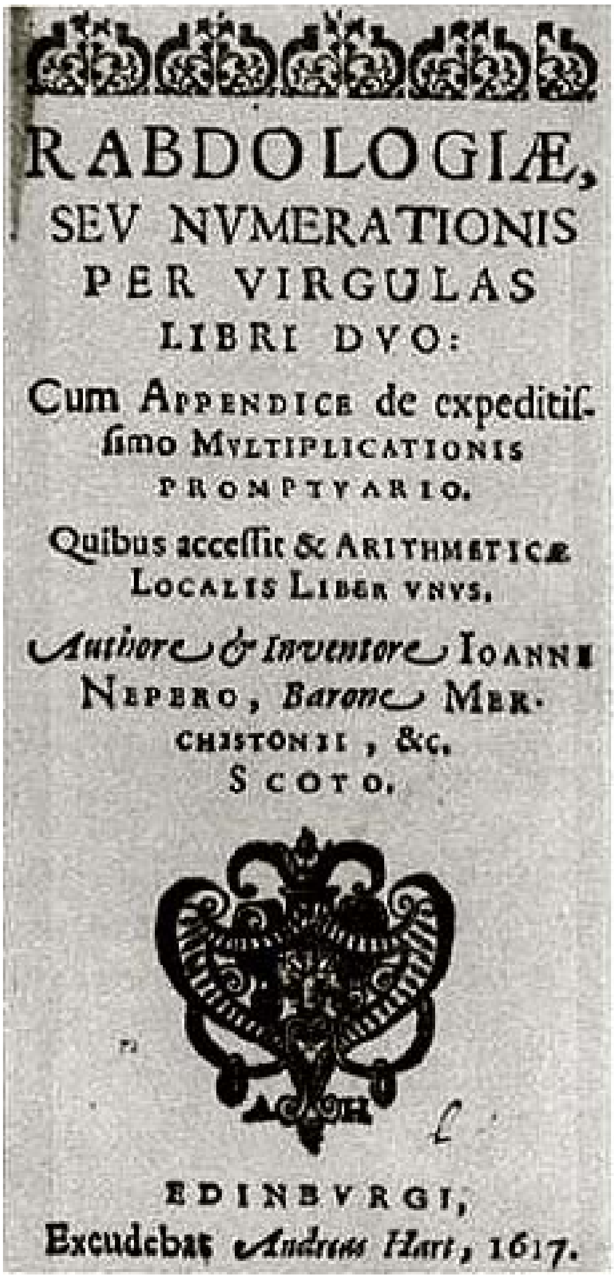
- give examples of factors that motivated Napier to invent calculation aids
- understand how logarithms can turn multiplications into additions
- give examples of how Napier's work became known to mathematicians, astronomers and navigators
- appreciate that presenting mathematical ideas in different ways can affect how easily (or not!) they are understood.

1 Background to Napier and his work

For many years, John Napier (1550–1617) spent his leisure time devising means for making arithmetical calculations easier. Just why a Scots laird at the turn of the seventeenth century should have thus devoted the energies left over from the management of his estates remains a puzzle. Up to the publication of his description of logarithms in 1614, three years before his death, Napier was best known to the world for his Protestant religious treatise *A plaine discovery of the whole Revelation of Saint John* (1594), a well-received work which was translated into several foreign languages. Napier's interest in mathematics, and in computational methods in particular, seems to have started in his early twenties (in the 1570s), and continued mostly unknown to the wider world until the flurry of publishing activity forty years later which revealed first his table of logarithms (*Mirifici logarithmorum canonis descriptio*, 1614), then three further computation aids (*Rabdologiae*, 1617), and after his death an account of how logarithms themselves were calculated (*Mirifici logarithmorum canonis constructio*, 1619). The titles of his works form a revealing contrast.


MIRIFICI
Logarithmorum
Canonis descriptio,
Ejusque usus, in utraque
Trigonometria; ut etiam in
omni Logistica Mathematica,
Amplissimi, Facillimi, &
Authore ac Inventore,
IOANNE NEPERO,
Barone Merchistonii,
&c. Scoto.
EDINBURGI,
Ex officina ANDREÆ HART

Figure 1



RABDOLOGIÆ,
SEV NVMERATIONIS
PER VIRGULAS
LIBRI DVO:
Cum APPENDICE de expeditis-
simo MVLTIPLICATIONIS
PROMPTVARIO.
Quibus accessit & ARITHMETICÆ
LOCALIS LIBER VNVS.
Authore & Inventore IOANNE
NEPERO, Barone MER-
CHISTONII, &c.
SCOTO.

EDINBVRGI,
Excudebat Andreas Hart, 1617.

Figure 2

Question 1

Allow about 5 minutes

Compare the titles of Napier’s theological and mathematical writings. Do you notice any difference that may give a clue as to Napier’s intended readership?

Provide your answer...

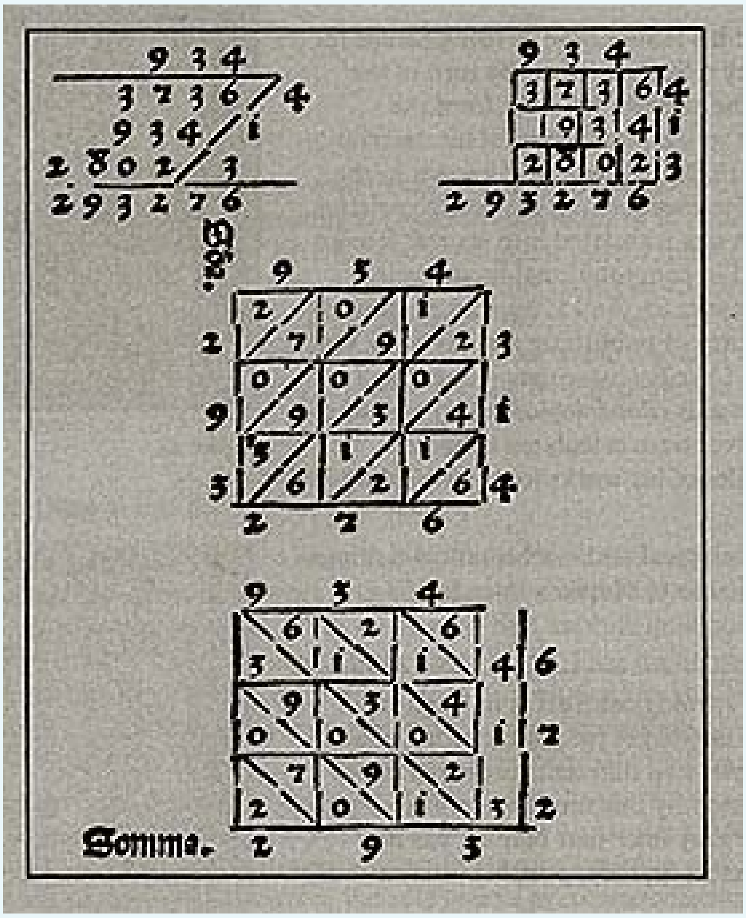
The mathematical titles are in Latin, that of the theological tract is in English. Assuming the contents are likewise in these two languages (which they are, in fact), we might begin to conjecture from this that Napier had different readers in mind. Given the substantial vernacular textbook tradition by this time (seventy or so years after the first appearance of Record’s works), we may infer that Napier was not aiming at the homegrown practitioner but at a more learned, possibly international, audience.

2 Napier’s bones

Before pursuing who logarithms were for (and what they are), you’ll first look briefly at another of Napier’s computational aids. For in the years following his death, it was in fact his numerating rods, the so-called *Napier’s bones*, that were more widely known and used. These consisted of the columns of a multiplication table inscribed on rods, which could make the multiplying of two numbers easier by setting down the partial products more swiftly. This simple contrivance was derived from an ancient multiplication method (called ‘lattice’ multiplication – see Box 1), which involved a special layout on paper of the two numbers to be multiplied so as to display the partial products of the multiplication, to facilitate adding them in the right way.

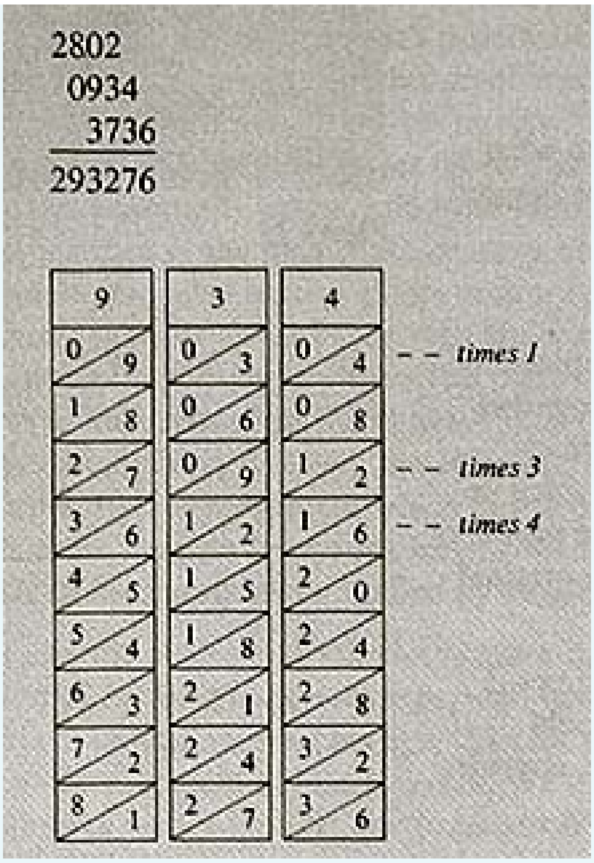
Box 1: Multiplication on Napier’s bones

Consider first various ways of laying out the multiplication of 934 and 314, as given in the *Treviso arithmetic* of 1478 (the first printed arithmetic book).



Note that the result, 293 276, is reached in the middle configuration by adding the figures in the matrix along the diagonal bands, starting from the bottom right (and carrying a digit over to the next band where appropriate).

The same multiplication would be done on Napier’s bones by aligning the rods for 9, 3 and 4, and looking horizontally along the *times 3, times 1* and *times 4* rows to form the partial products to be added, as shown here.


2802
0934
3736
293276
9
3
4
times 1
times 3
times 4

3 Napier's approach to logarithms

Napier's major and more lasting invention, that of logarithms, forms a very interesting case study in mathematical development. Within a century or so what started life as merely an aid to calculation, a set of 'excellent briefe rules', as Napier called them, came to occupy a central role within the body of theoretical mathematics.

The basic idea of what logarithms were to achieve is straightforward: to replace the wearisome task of *multiplying* two numbers by the simpler task of *adding* together two other numbers. To each number there was to be associated another, which Napier called at first an 'artificial number' and later a 'logarithm' (a term which he coined from Greek words meaning something like 'ratio-number'), with the property that from the sum of two such logarithms the result of multiplying the two original numbers could be recovered.

In a sense this idea had been around for a long time. Since at least Greek times it had been known that multiplication of terms in a geometric progression could correspond to addition of terms in an arithmetic progression. For instance, consider

2	4	8	16	32	64
1	2	3	4	5	6

and notice that the *product* of 4 and 8 in the top line, viz 32, lies above the *sum* of 2 and 3 in the bottom line (5). (Here the top line is a *geometric* progression, because each term is twice its predecessor; there is a constant *ratio* between successive terms. The lower line is an *arithmetic* progression, because each term is one more than its predecessor; there is a constant *difference* between successive terms.) Precisely these two lines appear as parallel columns of numbers on an Old Babylonian tablet, though we do not know the scribe's intention in writing them down.

A continuation of these progressions is the subject of a passage in Chuquet's *Triparty* (1484). See the passage in Reading 1 below.

Reading 1: Nicolas Chuquet on exponents

A Boethius says in his first book and in the first chapter, the science of numbers is very great, and among the sciences of the quadrivium it is one in the pursuit of which every man ought to be diligent. And elsewhere he says that the science of numbers ought to be preferred as an acquisition before all others, because of its necessity and because great secrets and other mysteries which there are in the properties of numbers. All sciences partake of it, and it has need of none. […]

To understand the reason why denomination of number is added to denomination, and to have knowledge of the order of numbers which was mentioned in the first chapter, it is necessary in a continuous sequence, like 1, 2, 4, 8, 16, 32, etc, or 3, 9, 27 etc.

Numbers	Denomination
1	0
2	1
4	2
8	3

16	4
32	5
64	6
128	7
256	8
512	9
1 024	10
2 048	11
4 096	12
8 192	13
16 384	14
32 768	15
65 536	16
131 072	17
262 144	18
524 288	19
1 048 576	20

Now it is necessary to know that 1 represents and is in the place of numbers, whose denomination is 0.2 represents […] the first terms, whose denominations is 1.4 holds the place of the second terms, whose denomination is 2. And 8 is the place of the third terms, 16 holds the place of the fourth terms, 32 represents the fifth terms, and so for the others. Now whoever multiplies 1 by 1, it comes to 1, and because 1 multiplied by 1 does not change at all, neither does any other number when it is multiplied by 1 increase or diminish, and for this consideration, whoever multiplies a number by a number, it comes to a number, whose denomination is 0. And whoever adds 0 to 0 makes 0. Afterwards, whoever multiplies 2, which is the first number, by 1, which is a number, the multiplication comes to 2: then afterwards, whoever adds their denominations, which are 0 and 1, it makes 1; thus the multiplication comes to 2^1. And from this it comes that when multiplies numbers by first terms or vice versa, it comes to first terms. Also whoever multiplies 2^1 by 2^1, it comes to 4 which is a second number. Thus the multiplication amounts to 4^2. For 2 multiplied by 2 makes 4 and adding the denominations, that is 1 with 1, makes 2. And from this it comes that whoever multiplies first terms by first terms, it comes to second terms. Likewise whoever multiplies 2^1 by 4^2, it comes to 8^3. For 2 multiplied by 4 and 1 added with 2 makes 8^3. And thus whoever multiplies first terms by second terms, it comes to third terms. Also, whoever multiplies 4^2 by 4^2, it comes to 16 which is a fourth number, and for this reason whoever multiplies second terms by second terms, it comes to fourth terms. Likewise whoever multiplies 4 which is a second number by 8 which is a third number makes 32 which is a fifth number. And thus whoever multiplies second terms by third terms or vice versa, it comes to fifth terms. And third terms by fourth terms comes to 7th terms, and fourth terms by fourth terms, it comes to 8th terms, and so for the others. In this discussion there is manifest a secret which is in the proportional numbers. It is that whoever multiplies a proportional number by itself, it comes to the number of the double of its denomination, as, whoever multiplies 8 which is a third number by itself, it comes to 64 which is a sixth. And 16 which is a fourth number multiplied by itself should come to 256, which is an eighth. And whoever multiplies 128 which is the 7th proportional by 512 which is the 9th, it should come to 65 536 which is the 16th.

Chuquet made the same observation as above, that the product of 4 and 8 (‘whoever multiplies 4 which is a second number by 8 which is a third number’) gives 32, which is above the sum of 2 and 3 (‘makes 32 which is a fifth number’). Chuquet seems virtually to have said that a neat way of multiplying 4 and 8 is to add their associated numbers (‘denominations’) in the arithmetic series and see what the result corresponds to in the geometric series. (Of course, had he wanted to multiply 5 by 9, say, Chuquet would have been stuck.) And in *The sand-reckoner*, long before, Archimedes proved a similar result for any geometric progression.

So the idea that addition in an arithmetic series parallels multiplication in a geometric one was not completely unfamiliar. Nor, indeed, was the notion of reaching the result of a multiplication by means of an addition. For this was quite explicit in trigonometric formulae discovered early in the sixteenth century, such as:

$$\begin{aligned} 2\sin A \sin B &= \cos(A-B) - \cos(A+B) \\ 2\cos A \cos B &= \cos(A-B) + \cos(A+B). \end{aligned}$$

Thus if you wanted to multiply two sines, or two cosines, together – a very nasty calculation on endlessly fiddly numbers – you could reach the answer through the vastly simpler operation of subtracting or adding two other numbers. This method was much used by astronomers towards the end of the sixteenth century, particularly by the great Danish astronomer Tycho Brahe, who was visited by a young friend of Napier, John Craig, in 1590. So Napier was probably aware of these techniques at about the time he started serious work on his own idea, although conceptually it was entirely different.

Napier’s definition of logarithm is rather interesting. This course won’t pursue all its details here, but just enough to see its approach and character. Imagine two points, *P* and *L*, each moving along its own line.

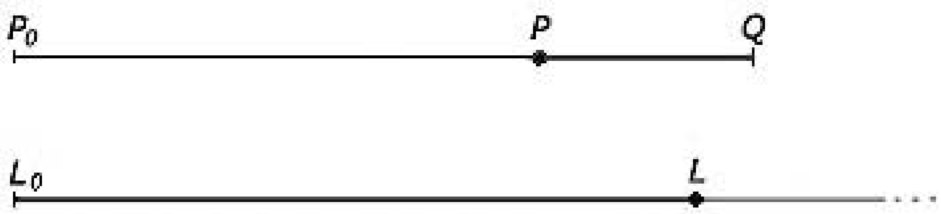
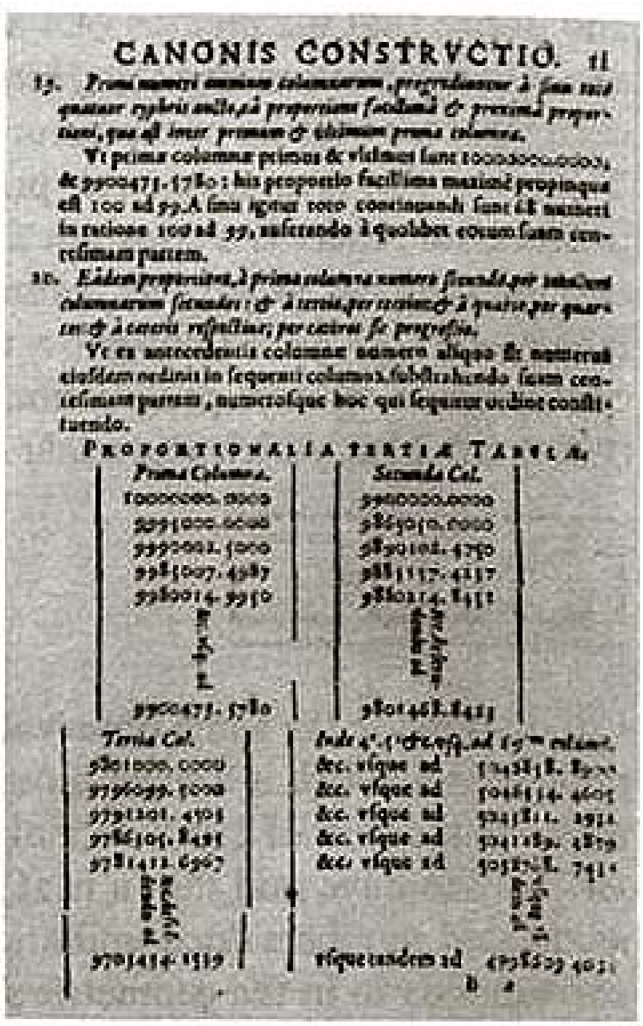


Figure 3

The line $P_0\,Q$ is of fixed, finite length, but *L*’s line is endless. *L* travels along its line at constant speed, but *P* is slowing down. *P* and *L* start (from P_0 and L_0) with the same speed, but thereafter *P*’s speed drops proportionally to the distance it has still to go: at the half-way point between P_0 and *Q*, *P* is travelling at half the speed they both started with; at the three-quarter point, it is travelling with a quarter of the speed; and so on. So *P* is never actually going to get to *Q*, any more than *L* will arrive at the end of *its* line, and at any instant the positions of *P* and *L* uniquely correspond. Then at any instant the distance L_0L is, in Napier's definition, the *logarithm* of the distance *PQ*. (That is, the numbers measuring those distances.) Thus the distance *L* has travelled at any instant is the logarithm of the distance *P* has yet to go.

How does this cohere with the ideas spoken of earlier? The point *L* moves in an arithmetic progression: there is a *constant difference* between the distance it moves in equal time intervals – that is what 'constant speed' means. The point *P*, however, is slowing down in a geometric progression: its motion was defined so that it was the *ratio* of successive distances that remained constant in equal time intervals.


CANONIS CONSTRVCTIO. 11

PROPORTIONALIA TERTIÆ TABULÆ.

Prima Columna. Secunda Col. Tertia Col.

Figure 4 From Napier's *Constructio* (1619). Notice that the seven significant figures afforded by the whole sine of 10^7 are **not** sufficient for the accuracy he desired, and he used four more, beyond the 'decimal point' – one of the earliest occurrences of our decimal point symbol in print, which helped to stabilise this notation in its now-familiar form

Question 2

Allow about 5 minutes

Compare what you have gleaned of Napier's concept of logarithm with earlier ideas about progressions and trigonometrical formulae. What seems to you the most striking difference in overall approach? What problem do you see Napier's work solving?

Provide your answer...

The major difference is surely in his use of the concept of motion, of points moving along lines with speeds defined in various ways. Both exponents of Chuquet's kind and trigonometrical formulae are quite 'static' objects by comparison – there is evidently a deep difference of mathematical style here.

What seems so clever about Napier's approach is that he can cope with any number, in effect, not just the ones that happen to form part of some particular discrete geometrical progression. This is effected by his intuition springing from the *continuous* nature of the straight line and of motion.

4 Napier and motion

So where did the idea of *motion* which is found in Napier's work come from? It was again a concept used by Archimedes, in his study of spirals, so there was a classical precedent for propositions about points moving along lines (see, for example, Proposition 1 of *On Spirals*, in Reading 2 below). Further, although much of the Western mathematical tradition had been rather nervous of the concept of motion hitherto, there had been exceptions to this three centuries or so earlier: both the Merton School in fourteenth-century Oxford and Nicole Oresme at the University of Paris had made prolonged study of issues involving this concept. The details of Napier's education are obscure – we know he spent a year at the University of St Andrews in his early teens, but not what he did or learned thereafter – but it is not implausible that he became aware of mediaeval studies of motion at some stage.

Reading 2: Proposition 1 of *On Spirals*

(b) Proposition 1

If a point move at a uniform rate along any line, and two lengths be taken on it, they will be proportional to the times of describing them.

Two unequal lengths are taken on a straight line, an two lengths on another straight line representing the times; and they are proved to be proportional by taking equimultiples of each length and the corresponding time after the manner of Euclid's *Elements*, V, Def.5.

5 Napier and Briggs

The fact remains that few mathematical inventions have burst on the world so unexpectedly as Napier's logarithms. Although various disparate strands – the idea of doing multiplication via addition, the idea of comparing arithmetic and geometric progressions, the use of the concept of motion – had all been floated at some stage, the enthusiasm with which Napier's work was received makes it clear both that this was perceived as a novel invention and that it fulfilled a pressing need. Foremost among those who welcomed the invention was Henry Briggs, Professor of Geometry at Gresham College, who wrote to the biblical scholar James Ussher in 1615:

> Naper, lord of Markinston, hath set my Head and Hands a Work with his new and admirable Logarithms I hope to see him this Summer if it please God, for I never saw Book which pleased me better or made me more wonder.
>
> (Quoted in D. M. Hallowes (1961–2) 'Henry Briggs, Mathematician', Transactions of the Halifax Antiquarian Society, pp. 80–81.)

Note: Ussher was the scholar whose studies of biblical chronology revealed that the Creation took place at 9 a.m. on October 23rd, 4004 BC.

Briggs did indeed 'see him this Summer', in a visit recorded some time later by the astrologer William Lilly. See Reading 3 for this famous story.

Reading 3: William Lilly on the meeting of Napier and Briggs

I will acquaint you with one memorable Story related unto me by Mr John Marr, an excellent Mathematician and Geometrician, whom I conceive you remember; he was Servant to King James and Charles the First.

At first, when the Lord Napier, or Marchiston, made public his Logarithms, Mr Briggs, then Reader of the Astronomy lecture at Gresham College in London, was so surprised with Admiration of them, that he could have no Quietness in himself, until he had seen that noble Person the Lord Marchiston, whose only Invention they were; he acquaints John Marr herewith, who went into Scotland before Mr Briggs, purposely to be here when these Two so learned Persons should meet: Mr Briggs appoints a certain Day when to meet at Edinborough, but failing thereof; the Lord Napier were speaking of Mr Briggs: 'Ah, John (saith Marchiston), Mr Briggs will not now come'; at the very one knocks at the gate; John Marr hasted down, and it proved Mr Briggs, to his great Contentment; he brings Mr Briggs up into My Lord's chamber, where almost one quarter of an hour was sent, each beholding other almost with Admiration before one word was spoke, at last Mr Briggs began.

'My Lord, I have undertaken this long Journey purposely to see your Person, and to know by what Engine of Wit or Ingenuity you came first to think of this most excellent Help unto Astronomy, *viz.*, the *Logarithms*; but, my Lord, being by you found out, I wonder nobody else found it out before, when now known it is so easy,' He was nobly entertained by the Lord Napier, and every Summer after that, during the Lord's being alive, this venerable Man, Mr Briggs, went purposely into Scotland to visit him.

During this visit, and a further one the next year (1616), Briggs and Napier discussed some simplifications to the idea and presentation of logarithms. It is fortunate they profited from each other's company in this way, for their world-views were very different. Napier not only considered the Pope to be the Antichrist and expected the Day of Judgement quite shortly (probably between 1688 and 1700), but also was 'a great lover of astrology' (according to Lilly), and may even have practised witchcraft, as contemporary rumour had it. Briggs, on the other hand, represented what one might call the Yorkshire common-sense school of thought on occult matters, and like his friend Sir Henry Savile (also a son of Halifax) had no time for astrological practices.

It was these two very different people who worked together on simplifying logarithms. They agreed it would generally be more useful if the logarithm of 1 were to be 0, and the logarithm of 10 were to be 1 (see Box 2). Briggs spent several subsequent years recalculating the tables on this basis. Thus the early history of logarithms exemplifies well a remark which the historian Clifford Truesdell has made in another context: 'the simple ideas are the hardest to achieve; simplicity does not come of itself but must be created' (his full comment is linked below). Napier and Briggs had to work hard to create even the 'simple' major logarithm property that

$$\log(A \times B) = \log A + \log B.$$

Clifford Truesdell's full comment was as follows:

Reading 4: Clifford Truesdell on Euler

We may justly wonder that it took more than sixty years for so simple an extension of Newton's ideas, but the literature of mechanics does not permit us to doubt that it did. As often happens in the history of science, the simple ideas are the hardest to achieve; simplicity does not come of itself but must be created.

Box 2: Napier's, and later, logarithms

Napier's original presentation of logarithms differed markedly from that generally adopted later, and it is worth spelling out the differences.

It follows from our earlier description that it was the length P_0Q whose logarithm is 0, and further that the shorter the length (in the geometric series) the larger its logarithm. Napier chose P_0Q to be 10 000 000, a conventional figure for the 'whole sine', because he intended the table of logarithms to be used trigonometrically – it was the logarithms of sines and tangents he calculated and tabulated in the 1614 *Descriptio*, not the logarithms of numbers in general. A further difference from later practice was that what we think of as the major logarithm property did not hold in our simple form, but was

$$\log(A \times B) = \log A + \log B - \log 1.$$

It was as a result of Briggs' discussions with Napier that log 1 was redefined to be 0, thus simplifying this formula, and making the use of logarithms easier. Briggs, too, developed the calculation of logarithms of ordinary numbers, using the correlation log 10 = 1, log 100 = 2, log 1000 = 3, and so on.

ARITHMETICA

LOGARITHMICA

SIVE

LOGARITHMORVM
CHILIADES TRIGINTA, PRO
numeris naturali serie crescentibus ab vnitate ad
[illegible]

HOS NVMEROS PRIMVS
INVENIT CLARISSIMVS VIR, IOHANNES
NEPERVS Baro Merchistonij: [illegible]

DEVS NOBIS VSVRAM VITÆ DEDIT
ET INGENII, TANQVAM PECVNIÆ,
NVLLA PRÆSTITVTA DIE.

I R

LONDINI,
Excudebat GVLIELMVS
IONES. 1624.

Figure 5 Briggs' recalculation of logarithms, on the new basis which he and Napier had agreed, was published in 1624. The logarithms were calculated to fourteen places of decimals; there was a gap in the tables, between 20 000 and 90 000, which Briggs hoped others would fill

TABLES
des
Sinus, Tangentes & Secantes,
&
Celles des Sinus & Tangentes
par le nombre
LOGARITHME,
supputées depuis un jusqu'à 10000.
Comme aussi les Tables de la Latitude
aggrandissante, & celles des lignes du Compas.
AVEC
Une claire & parfaite instruction, pour par le moyen des
A. VLACQ.
AMSTERDAM.

Figure 6 The gap in Briggs' tables (Figure 5) was filled by Adrian Vlacq, albeit 'only' to ten places of decimals, and published in Holland in 1627, as *Met tweede deel van de nieuwe telkonst*. The calculations of Briggs and Vlacq were the essential basis of all subsequent tables of 'common logarithms'. The work illustrated is of a later French edition of Vlacq's shortened (seven-place) trigonometrical tables.

6 Spreading the word about logarithms

Another person besides Briggs to recognise immediately the importance of Napier's concept was the navigational practitioner Edward Wright, who translated Napier's *Descriptio* into English, as *A description of the admirable table of logarithmes*. Reading 5 below comprises the Preface to that work (the translation of Napier's original Preface, with further sentences added by Napier himself).

Reading 5: John Napier's Preface to *A Description of the Admirable Table of Logarithms*

Seeing there is nothing (right well beloved students in the Mathematics) that is so troublesome to Mathematicall practise, nor that doth more molest and hinder Calculators, than the Multiplications, Division, square and cubical Extractions of great numbers, which besides the tedious expence of time, are for the most part subject to many slippery errors. I began therefore to consider in my minde, by what certaine and ready Art I might remove those hindrances. And having thought upon many things to this purpose, I found at length some excellent briefe rules to be treated of (perhaps) hereafter. But amongst all, none more profitable than this, which together with the hard and tedious Multiplications, Division, and Extractions of rootes, doth also cast away from the worke it selfe, even the very numbers themselves that are to be multiplied, divided and resolved into rootes and putteth other numbers in their place, which performe as much as they can do, onely by Addition and Subtraction, Division by two or Division by three; which secret invention, being (as all other good things are) so much better as it shall be the more common; I thought good heretofore to set forth in Latine for the publique use of Mathematicians. But now some of our good Countrymen in this Island well affected to these studies, and the more publique good, procured a most learned Mathematician to translate the same into our vulgar English tongue, who after he had finished I sent the copy of it to me, to be seene and considered on by myself. I having most willingly and gladly done the same, finde it to be most exact and precisely conformable to my minde and the originall. Therefore it may please you who are inclined to these studies, to receive it from me and the Translator, with as much good will as we recommend it unto you. Fare yee well.

Question 3

Allow about 5 minutes

What explanation does Napier give of why his work appeared first in Latin, then in English? What light does this throw on your answer to Question 1?

Provide your answer...

Napier says his book was written in Latin ‘Tor the publique use of Mathematicians, to ensure it shall be the more common’, that is, presumably, more widespread. The English translation is for the benefit of ‘our Countrymen in this Island’ and ‘the more publique good’. This confirms our hypothesis, from Question 1, that Napier had in mind as his primary audience the international mathematical community. He sounds pleased, all the same, that non-Latin speakers in the United Kingdom might find logarithms useful too.

Knowledge of logarithms spread rapidly in various ways. Wright’s English translation was one source of knowledge; it was dedicated (by Wright’s son, for Wright died before publication) to the East India Company, another of the great trading and exploration companies of the time. This dedication suggests an audience for whom the knowledge was thought to be useful – Wright was navigational consultant to the Company for the last year or two of his life. Knowledge of logarithms spread by word of mouth too; Briggs lectured on them at Gresham College, as did Edmund Gunter, who was appointed Professor of Astronomy there in 1619. Within a decade, editions or similar tables had been published in France, Germany and the Netherlands. As an example of the impact of logarithms abroad, let us consider the response of the astronomer Johannes Kepler.

7 Kepler and logarithms

Kepler was precisely the kind of practitioner for whom logarithms were of greatest benefit: a professional astronomer (and, of course, competent Latin scholar) driven at times to distraction – he tells us – by the magnitude and complexity of the calculations he needed to do. So when he was able to study a copy of Napier's *Descriptio* in about 1619 he welcomed it warmly, dedicating his next book to Napier (not realising he had been dead for two years); and he went further. Napier's book contained the definition of logarithms, and a set of logarithm tables, but not how to get from one to the other. (It is worth bearing in mind that although logarithms *once calculated* are a considerable saving of time, the calculation of the tables themselves was extraordinarily laborious – especially as one did not have the assistance of logarithm tables to help with the necessary computations.) So Kepler set to work recalculating their construction from first principles, basing his argument upon the classical theory of proportion of Euclid's *Elements*, Book V. This approach was consciously quite different from Napier's geometrical and kinematic considerations; Kepler wrote that logarithms were not associated, for him,

> inherently with categories of trajectories, or lines of flow, or any other perceptible qualities, but (if one may say so) with categories of relationship and qualities of thought.
>
> (Quoted in Yu. A. Belyi (1975) 'Johannes Kepler and the development of mathematics', in A. Beer and P. Beer (eds), Kepler: Four Hundred Years, Pergamon Press, p. 656.)

It is interesting to notice that Kepler felt that the appeal to kinematical intuition in Napier's work lacked rigour. He emphatically asserted that what he was supplying was *rigorous proof*, which was, for him, something cast in Euclidean mould. In the event, he had to approximate (just as Napier had done) to express potentially endless numbers to a suitable finite approximation, but Kepler's overall approach was influential nonetheless. You can see something of his style of argument in Reading 6 below.

Reading 6: Charles Hutton on Johannes Kepler's construction of logarithms

Kepler here, first of any, treats of logarithms in the true and genuine way of the measures of ratios, or proportions, as he calls them, and in a very full and scientific manner: and this method of was afterwards followed and abridged by Nercator, Halley, Cotes, and others, as we shall see in the proper places. Kepler forst erects a regular and purely mathematical system of proportions, and the measures of proportions, treated at considerable length in a number of propositions, which are fully and chastely demonstrated by genuine mathematical reasoning, and illustrated by examples in numbers. This part contains and demonstrates both the nature and the principles of the structure of logarithms. And in the second part he applies those principles in the actual construction of his table, which contains only 1000 numbers and their logarithms, in the form as we before described: and in this part he indicates the various contrivances employed in deducing the logarithms of proportions one from another, after a few of the leading ones has been first formed, by the general and more remote principles. He uses the name *logarithms*, given them by the inventor, being the most proper, as expressing the very nature and essence of those artificial numbers and containing as it were a definition in the very name of them; but without taking any notice of the inventor, or the origin of those useful numbers.

As this tract [of 1625] is very curious and important in itself, and is besides very rare and little known, instead of a particular description only, I shall here give a brief translation of both the parts, omitting only the demonstrations of the propositions, and some rather long illustration of them. The book is dedicated to Philip, landgrave of Hesse, but is without either preface or introduction, and commences immediately with the subject of the first part, which is entitled *The Demonstration of the Structure of Logarithms*; and the contents of it are as follow:

Postulate 1 That all proportions equal among themselves by whatever variety of couplets of terms they may be denoted, are measured or expressed by the same quantity.

Axiom 1 If there be any number of quantities of the same kind, the proportion of the extremes is understood to be composed of all the proportions of every adjacent couplet of terms, from the first to the last.

Proposition 1 The mean proportional between two terms, divides the proportion of those terms into two equal proportions.

Axiom 2 Of any number of quantities regularly increasing, the means divide the proportion f the extremes into one proportion more than the number of the means.

Postulate 2 That the proportion between any two terms is divisible into any number of parts, until those parts become less than any proposed quantity.

An example of this section is then inserted in a small table, in dividing the proportion which is between 10 and 7 into 1073741824 equal parts, by as many mean proportionals wanting one, namely, by taking the mean proportional between 10 and 7, than the mean between 10 and this mean, and the mean between 10 and the alst, and so on for 30 mean, or 30 extractions of the square root, the last or 30th of which roots is 99999999966782056900; and the 30th power of 2, which is 1073741824, shows into how many parts the proportion between 60 and 7 o, or between 1000&c, and 700&c, is divided by 1073741824 means, each of which parts is equal to the proportion between 1000&c, and the 30th mean 999&c, that is, the proportion between 1000&c, and 999&c, is the 1073741824th part of the proportion between 10 and 7. Then by assuming the small difference 00000000033217943100, for the measure of the very small element of the proportion of 10 to 7, or for the measure of the proportion of 1000&c, to 999&c, or for the logarithm of this last term, and multiplying it by 1073741824, the number of parts, the product gives 35667.49481.37222.14400, for the logarithm of the less term 7 or 700&c.

Postulate 3 The extremely small quantity or element of the proportion may be measured or denoted by any quantity whatever; as, for instance, by the difference of the terms of that element.

Proposition 2 Of three continued proportionals, the difference of the two first has to the difference of the latter two, the same proportion which the first term has to the 2d, or the 2d to the 3d.

[…]

Proposition 20 When four numbers are proportional, the first to the second as the third to the fourth, and the proportions of 1000 to each of the three former are known, there will also be known the proportion of 1000 to the fourth number.

Corollary 1 By this means other chiliads are added to the former.

Corollary 2 Hence arises the method of performing the Rule-of-Three, when 1000 is not one of the terms. Namely, from the sum of the measures of the proportions of 1000 to the second and third, take that of 1000 to the first, and the remainder is the measure of the proportion of 1000 to the fourth term.

Definition The measure of the proportion between 1000 and any less number as before described, and expressed by a number, is set opposite to that less number in the chiliad, and is called its *logarithm*, that is, the number (*arithmos*) indicating the proportion (*logos*) which 1000 bears to that number, to which the logarithm is annexed.

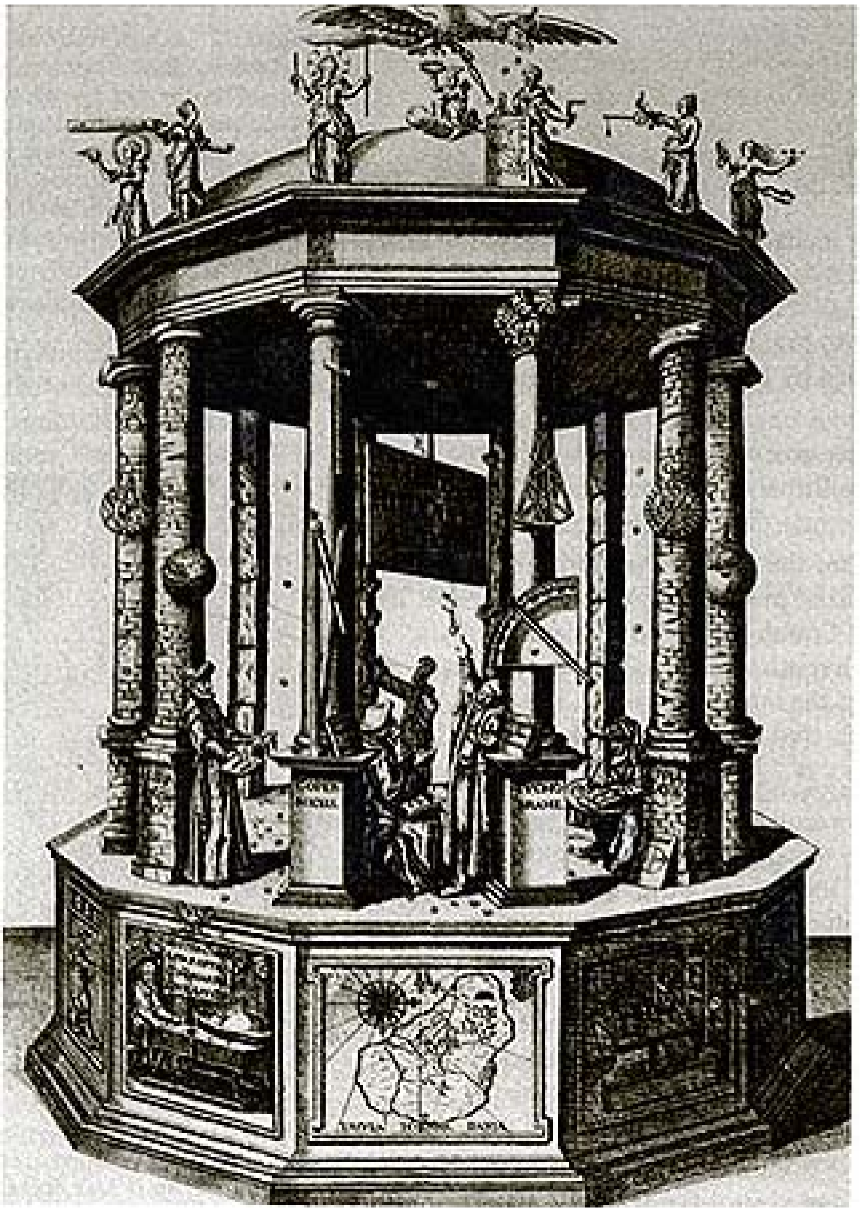


Figure 7 Frontispiece of Kepler’s *Rudolphine Tables* (1627), the set of planetary tables on which he had been working for 26 years, latterly with the aid of logarithms. The design, by Kepler himself, includes amongst a host of symbolic detail the figure of the muse of arithmetic, on the roof, her halo consisting of the numbers 6931472 – the logarithm of 2 (or, which is numerically the same, log sin 30°)

Conclusion

This free course provided an introduction to studying Mathematics at Level 2, especially the idea of a logarithm and how it turns a multiplication into an addition. You have read and analysed primary source material from the history of mathematics to find out more about how new mathematical ideas are produced and shared. These exercises were designed to develop your approach to study and learning at a distance and to help improve your confidence as an independent learner.

Acknowledgements

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Figure 1 Bodleian Library

Figure 2 Keele University, Turner Collection

Figure 7 Deutsches Museum, Munich

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Reading 1 – Nicolas Chuquet on exponents: tr. and ed. H.G. Flegg, C.M. Hay, B. Moss, *Nicolas Chuquet Renaissance Mathematician*, Reidel, 1985, p. 144, 151-153, published in Chapter 7 'Mathematics in Mediaeval Europe';

Reading 2 – Proposition 1 of *On Spirals*: Chapter 'Archimedes and Apollonius';

Reading 3 – William Lilly on the meeting of Napier and Briggs: *Mr Lilly's History of his Life and Times*, London, 1715, pp. 105-106, published in 'William Lilly on the meeting of Napier and Briggs', Chapter 8 'Mathematical Sciences in Tudor and Stuart England';

Reading 4 – Clifford Truesdell on Euler: Chapter 'Euler and his Contemporaries';

Reading 5 – John Napier's Preface: *A Description of the Admirable Table of Logarithms*, preface R.E. Wright, London, published in 'John Napier's Preface to A Description of the Admirable Table of Logarithms', Chapter 9 'Mathematical Sciences in Tudor and Stuart England';

Reading 6 – Charles Hutton on Johannes Kepler's construction of logarithms: C. Hutton, *Mathematical Tables: … to which is prefixed, a large and original history of the discoveries and writings relating to those subjects; …*, London 1785, 1822 edition, pp. 49-54, published in 'Charles Hutton on Johannes Kepler's construction logarithms', Chapter 9 'Mathematical Sciences in Tudor and Stuart England'.