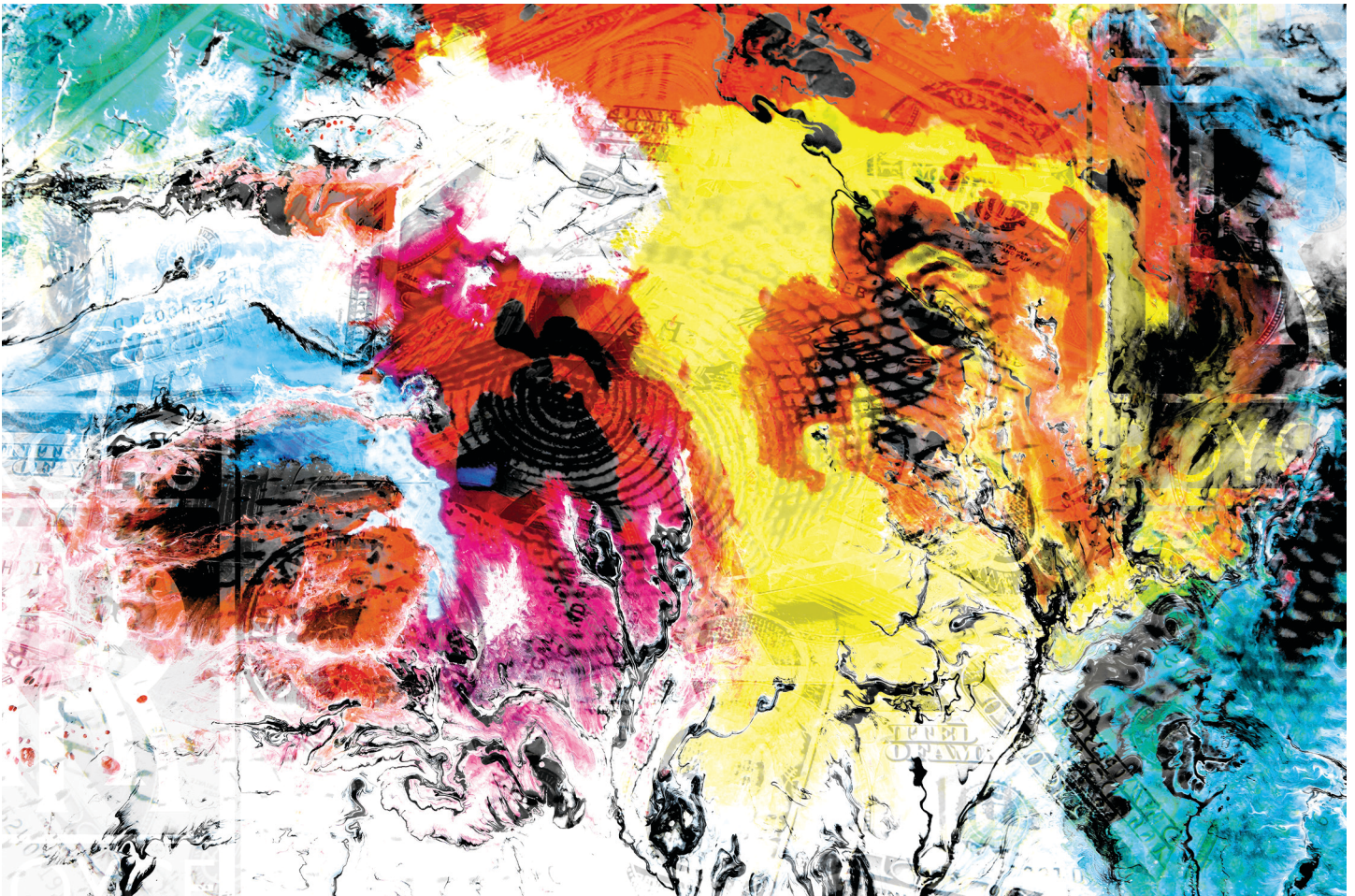


White dwarfs and neutron stars



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Introduction

In this course, you will explore the exotic end-points in the lives of most stars – namely white dwarfs and neutron stars. To do this we consider the properties of so-called degenerate matter and how these properties govern the later stages of stellar evolution. To set the scene, we begin by outlining the life cycle of stars in general, before moving onto consider their end-points. We next look at planetary nebulae and white dwarfs, which are the end points of low- and medium-mass stars, and then see how high-mass stars end their lives as supernovae, and may produce neutron star remnants.

This course shall refer to masses, radii and luminosities of stars in terms of the mass, radius and luminosity of the Sun (represented by $M_{\odot} = 1.99 \times 10^{30}$ kg, $R_{\odot} = 6.96 \times 10^5$ km and $L_{\odot} = 3.83 \times 10^{26}$ W respectively) and will sometimes refer to energies in the alternative units of electronvolts (where $1 \text{ eV} = 1.60 \times 10^{-19}$ J, with $1 \text{ keV} = 10^3 \text{ eV}$ and $1 \text{ MeV} = 10^6 \text{ eV}$ for convenience).

This OpenLearn course is an adapted extract from the Open University course [*S384 Astrophysics of stars and exoplanets*](#).

Learning outcomes

After studying this course, you should be able to:

- appreciate how stars end their lives
- understand how planetary nebulae and supernova remnants form
- calculate the physical properties of white dwarfs and neutron stars
- understand the properties of electron-degenerate and neutron-degenerate matter
- calculate the energetics of supernovae explosions.

1 The life cycle of stars

Stars form from clouds of gas that collapse under the influence of gravity until their central temperatures and pressures become high enough for nuclear fusion to be initiated in their cores. At this point, the clouds stop contracting further and are maintained in **hydrostatic equilibrium** with a balance between the inward force of gravity and the outward force due to their internal pressure. Stars are typically initially composed of about 75% hydrogen and 25% helium, with small traces of other elements.

Stars are observed with a wide range of masses, ranging from around one-tenth that of the Sun to over a hundred times the Sun's mass. Other characteristics of stars include their radii, surface temperature and luminosity. Usually it is only stars' surface temperatures (characterised by their colour) and luminosities (characterised by their brightness) that are directly observable. When these two quantities are plotted against each other for a population of stars, on a so-called **Hertzsprung–Russell diagram** (see Figure 1), certain patterns emerge in the stars' locations.

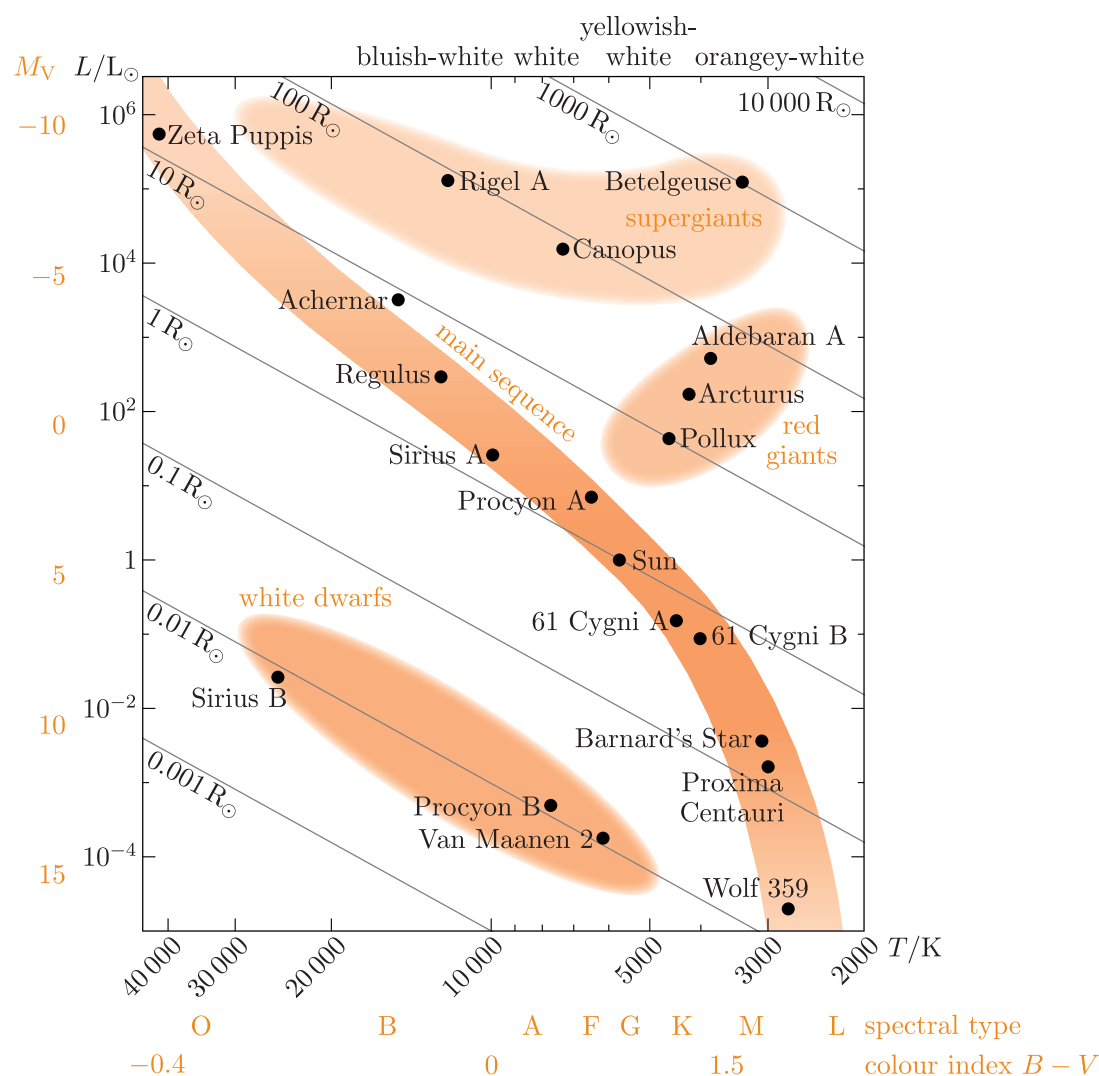


Figure 1 A schematic H–R diagram indicating the locations of four regions where most stars are found. The vertical axis shows stellar luminosity in solar units or absolute visual magnitude; the horizontal axis shows surface temperature in kelvins, spectral type, or colour index defined as the difference between B and V magnitudes. (Notice that

temperature increases to the *left*.) Diagonal lines of constant stellar radius are also shown and some notable named stars are indicated.

Most stars are found in a band running from the top left to bottom right of the H–R diagram, known as the **main sequence**. This is where all stars spend most of their lives as they undergo nuclear fusion in their cores, converting hydrogen into helium, and releasing energy. On the main sequence, the hottest, most luminous stars sit at the top left; these stars are the most massive and the largest. The coolest, least luminous stars sit at the bottom right; these stars are the least massive and the smallest. The most massive stars have relatively short main-sequence lifetimes of only a few tens of millions of years, whilst the least massive ones have main-sequence lifetimes of hundreds of billions of years (longer than the current age of the Universe).

When most of the hydrogen in the core of a star is depleted, its outer layers will expand and the star will move towards the right of the main sequence, such that its surface appears cooler. This is the region known as the **red giant branch**. To begin with, hydrogen fusion occurs in a shell around the inert helium core, but as the central temperature and pressure rise further, helium fusion is initiated in the centre of the star. The star then moves back down to somewhat lower luminosities and hotter temperatures, at the lower end of the red giant branch. This new fusion process converts helium into carbon and oxygen in the star's core, again releasing energy. Low mass stars ($< 3 M_{\odot}$) and intermediate mass stars (in the range $3\text{--}8 M_{\odot}$) ultimately begin a second ascent of the red giant branch (known as the **asymptotic giant branch**, or AGB) when core helium is exhausted and helium fusion only continues in a shell surrounding an inert carbon/oxygen core.

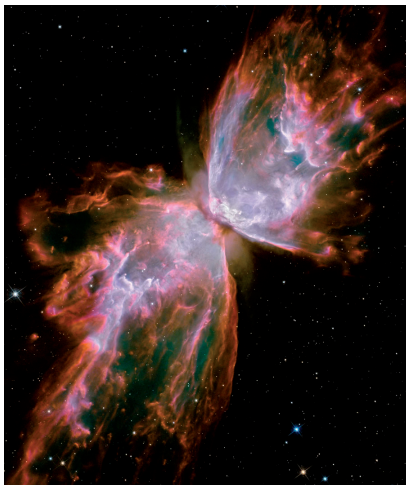
Massive stars ($> 8 M_{\odot}$) leave the upper main sequence when their core hydrogen is exhausted and then track back and forth along the **supergiant branch** as new sources of fusion in turn become active and are then depleted. Unlike lower mass stars whose fusion ends with carbon and oxygen, they can initiate further fusion reactions, converting carbon and oxygen into elements such as neon, magnesium, silicon and ultimately iron. They will end up with an 'onion-like' structure with shells of less massive elements surrounding shells of heavier elements towards the core. Once the core is composed of iron, no further fusion reactions are possible as reactions to produce nuclei heavier than iron require more energy to be put in than is released.

The subsequent evolution of stars once all nuclear reactions cease is the main topic of this course, which will be explored in the following sections.

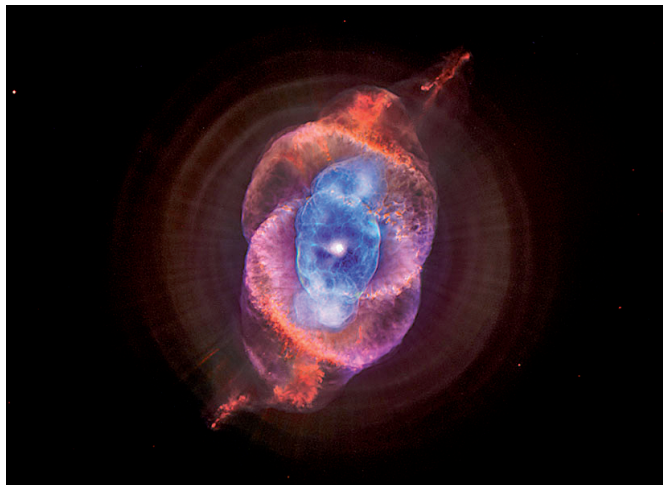
2 Planetary nebulae

As a low-mass or intermediate-mass star ascends the AGB, mass loss from the star becomes more and more significant. A strong stellar wind may develop in which a substantial fraction of the mass is carried away from the star. This usually corresponds to most of the hydrogen envelope! One consequence of extreme mass loss can be the formation of a planetary nebula, described below. Another effect is that even intermediate-mass stars, which had main-sequence masses in the range $3\text{--}8\text{ M}_{\odot}$, end up with a mass $< 1.4\text{ M}_{\odot}$. The significance of this final mass will become clear when white dwarfs are discussed later.

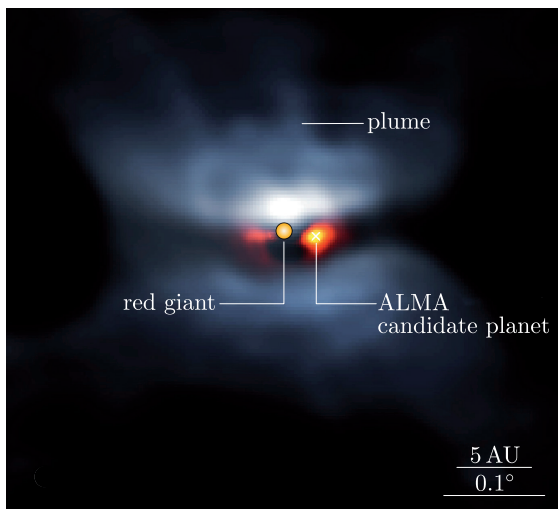
The mass lost from stars at the end of their AGB phase is not immediately visible in optical light, because dust grains in the cool ejecta obscure the star (although it may be seen in the infrared). The underlying helium-rich star – with a carbon and oxygen core – gets smaller and hotter, and a fast, radiation-driven stellar wind develops. Once the central star's surface temperature reaches $\sim 10\,000\text{ K}$ it ionises the ejected envelope, which is then seen as a **planetary nebula**, expanding at $\sim 30\text{--}60\text{ km s}^{-1}$. (Note: the term *planetary nebula* is a misnomer originating from when these objects were first observed as fuzzy planet-like objects through 18th century telescopes.)



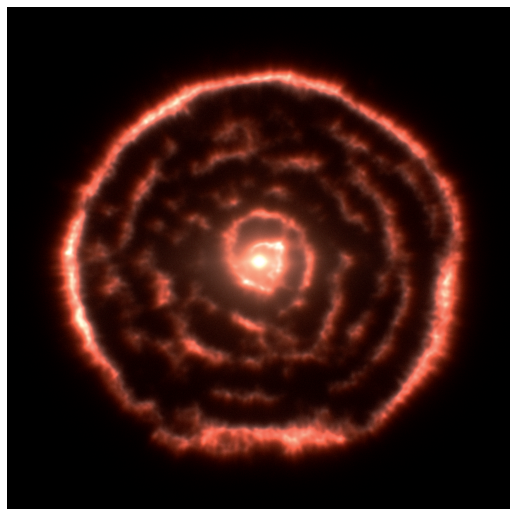
(a)



(b)



(c)



(d)

Figure 2(a) Hubble Space Telescope (HST) image of the bipolar planetary nebula NGC 6302. (b) A composite of data from the Chandra X-ray Observatory and HST

showing the multipolar planetary nebula NGC 6543. (c) A composite view of the AGB star L₂ Puppis in visible light (blue) and radio (orange) revealing a giant planet shaping the outflow from the star. (d) ALMA radio observations reveal a spiral structure in the material around the AGB star R Sculptoris.

The majority of planetary nebulae are asymmetric, with a small fraction being largely spherical or elliptical. Most are bipolar (with lobes either side of a central star) or multipolar (with at least two axes of symmetry), as shown in Figures 2a and 2b. The origin of these asymmetries has been the subject of considerable debate over recent decades. However, recent observations with the ALMA radio telescope and the SPHERE coronagraphic adaptive optics system on the Very Large Telescope (VLT) show that many of the asymmetries in planetary nebulae derive from structures that first formed while the star was on the AGB. They appear to be the result of orbiting companions, either a binary companion star or a massive planet (see Figure 2c). These can either transfer angular momentum to the ejecta and form a constraining torus, leading to a bipolar nebula, or create an orbital motion of the AGB star around the system's centre of mass, giving rise to a spiral pattern in the ejecta (see Figure 2d).

The physical diameter of a planetary nebula can be derived from its angular extent and distance. Because the expansion velocity of the nebula gas can be measured by its Doppler shift, the duration of the expansion – and hence of the post-AGB phase – can be calculated. By extrapolating the observed expansion of the nebula backwards, it can be shown that the whole evolution of a planetary nebula lasts only about 20,000 years.

The star at the centre of the planetary nebula continues to get smaller and hotter. Its hydrogen-burning shell is extinguished when it gets too close to the surface and hence too cool, and the same happens eventually to the helium-burning shell. What remains is a hot carbon-and-oxygen core, surrounded by thin shells of helium and hydrogen, with a surface temperature of ~ 100,000 K. All nuclear burning has been extinguished by this stage, and there is no prospect of thermonuclear reactions being re-ignited. Observations show that, although the nebula continues to expand, the central star now begins a long slide to lower luminosity and temperature at almost-constant radius. The star has become a cooling white dwarf.

3 Degeneracy

Before considering the physics of white dwarfs, let's pause to look at the concept of **degeneracy**. In this context, degeneracy means that there is more than one quantum state with the same energy.

In quantum mechanics, all particles display wave-like behaviour. In particular, any particle has a characteristic wavelength, known as the **de Broglie wavelength**, defined by

$$\lambda_{\text{dB}} = \frac{h}{p} \quad (1)$$

where h is Planck's constant ($= 6.63 \times 10^{-34} \text{ J s}$ or $4.14 \times 10^{-15} \text{ eV Hz}^{-1}$) and p is the particle's momentum.

Now, inside a star, a particle's kinetic energy can be written as $E_k = \frac{3}{2}k_B T$, where T is the temperature and k_B is Boltzmann's constant ($= 1.38 \times 10^{-23} \text{ J K}^{-1}$). Another expression for the kinetic energy of a particle (with mass m and speed v) is simply $E_k = \frac{1}{2}mv^2$. So equating these two we obtain $3k_B T = mv^2$ or $v = (3k_B T/m)^{1/2}$. (Strictly, particles exist with a range of speeds, but $(3k_B T/m)^{1/2}$ is close to the average.) The momentum of a particle is given by $p = mv$, so $p = (3mk_B T)^{1/2}$. Hence the de Broglie wavelength of a particle may be written as

$$\lambda_{\text{dB}} = \frac{h}{(3mk_B T)^{1/2}}. \quad (2)$$

Degeneracy becomes relevant when particles are packed so closely together that their separation is similar to their de Broglie wavelength. It turns out to be important at various stages in the lives of stars. Firstly, it determines the lower mass limit for stars: below a certain mass (about 0.075 times that of the Sun) matter in the star's core becomes degenerate as the star collapses and hydrogen fusion cannot begin. Such a star will instead become a **brown dwarf**. Secondly, degeneracy also determines how helium fusion begins: for stars with a mass below about 2.25 times that of the Sun, after hydrogen fusion finishes in the core, matter in the star's core becomes degenerate and helium fusion begins explosively in an event called a **helium flash**. Both these phenomena are therefore due to degeneracy. This concept will now be explored in detail as we consider the compact remnants left behind at the end of a star's life.

3.1 The de Broglie wavelength of electrons and nucleons

Let's begin by thinking in terms of the quantum mechanical properties of particles. A volume of space containing a gas may be considered to be occupied by particles whose wave properties give rise to standing waves that fit neatly within that volume. They therefore have quantised values of wavelength, and also of energy and momentum. This quantisation means that there is a discrete (rather than continuous) distribution of possible particle energies.

The number of possible quantum states available to particles below some energy is finite; often it is very large, but nevertheless it is finite. In the very dense interiors of some astronomical objects, such a finite number of available quantum states, though large, can be insufficient for the huge number of particles squeezed into the confined volume. Under these circumstances, physics then takes new twists, giving rise to some remarkable properties.

The critical point of having too few available quantum states for the number of particles present is reached when the separations of the gas particles become smaller than their de Broglie wavelength. At that point, the wave properties of the gas dominate its classical properties, and a quantum gas is said to exist. The following activity explores how the de Broglie wavelengths for protons and electrons compare in the core of the Sun.

Activity 1

- a. The de Broglie wavelength of non-relativistic particles is given by Equation (2) as $\lambda_{\text{dB}} = h/(3mk_{\text{B}}T)^{1/2}$ where m is the particle's mass, k_{B} is Boltzmann's constant ($= 1.38 \times 10^{-23} \text{ J K}^{-1}$) and T is the temperature. Calculate the de Broglie wavelength, in the core of the Sun at a temperature of $T_{\text{c},\odot} = 15.6 \times 10^6 \text{ K}$, of:
 - i. a proton (with mass $m_{\text{p}} = 1.67 \times 10^{-27} \text{ kg}$)
 - ii. an electron (with mass $m_{\text{e}} = 9.11 \times 10^{-31} \text{ kg}$)
- b. Which has the larger de Broglie wavelength, and by what factor?
- c. How does this ratio vary from star to star?

Discussion

- a. (i) For a proton in the core of the Sun:

$$\begin{aligned}\lambda_{\text{dB}}(\text{p}) &= h/(3m_{\text{p}}k_{\text{B}}T_{\text{c},\odot})^{1/2} \\ &= \frac{6.63 \times 10^{-34} \text{ J s}}{\sqrt{3 \times 1.67 \times 10^{-27} \text{ kg} \times 1.38 \times 10^{-23} \text{ J K}^{-1} \times 15.6 \times 10^6 \text{ K}}} \\ &= 6.38 \times 10^{-13} \text{ m.}\end{aligned}$$

- (ii) For an electron in the core of the Sun:

$$\begin{aligned}\lambda_{\text{dB}}(\text{e}) &= h/(3m_{\text{e}}k_{\text{B}}T_{\text{c},\odot})^{1/2} \\ &= \frac{6.63 \times 10^{-34} \text{ J s}}{\sqrt{3 \times 9.11 \times 10^{-31} \text{ kg} \times 1.38 \times 10^{-23} \text{ J K}^{-1} \times 15.6 \times 10^6 \text{ K}}} \\ &= 2.73 \times 10^{-11} \text{ m.}\end{aligned}$$

- b. The ratio of their de Broglie wavelengths is therefore:

$$\frac{\lambda_{\text{dB}}(\text{e})}{\lambda_{\text{dB}}(\text{p})} = \frac{h}{(3m_{\text{e}}k_{\text{B}}T_{\text{c},\odot})^{1/2}} \times \frac{(3m_{\text{p}}k_{\text{B}}T_{\text{c},\odot})^{1/2}}{h} = \left(\frac{m_{\text{p}}}{m_{\text{e}}}\right)^{1/2}$$

$$= \left(\frac{1.67 \times 10^{-27} \text{ kg}}{9.11 \times 10^{-31} \text{ kg}}\right)^{1/2} = 42.8$$

i.e. the de Broglie wavelength of the electron is greater than that of the proton, by a factor of about 40.

- c. From part (b), this ratio depends only on the mass of the particles, and hence is independent of the environment and hence of the temperature. The electron's wavelength is about 40 times longer than the proton's wavelength in all stars.

Note that neutrons have a similar de Broglie wavelength to protons because both types of particle have similar mass. Since the de Broglie wavelength of electrons is roughly 40 times longer than that of nucleons (i.e. protons or neutrons), then as the stellar core contracts and its density increases, electrons run out of space and hence reach the quantum limit corresponding to degeneracy sooner than nucleons.

3.2 Conditions for degeneracy

Degeneracy can be described in two equivalent ways:

1. The separation between particles is *less than* their de Broglie wavelength,

$$l < \lambda_{\text{dB}} = h/(3mk_{\text{B}}T)^{1/2} \quad (3)$$

2. The number of particles per unit volume, n , is *greater than* the number of available quantum states n_{Q} known as the **quantum concentration**,

$$n > n_{\text{Q}} = (2\pi mk_{\text{B}}T/h^2)^{3/2} \quad (4)$$

Before going further, complete the following activity to confirm to yourself that these two conditions are equivalent.

Activity 2

Beginning with the second statement of the condition for degeneracy, $n > n_{\text{Q}}$, use

the definition of the quantum concentration to derive an expression for the mean separation of the particles, in terms of the de Broglie wavelength. Show that this leads to the first statement for the condition of degeneracy.

Hint: if the mean separation of particles is l , then the number of particles per unit volume is $n = 1/l^3$.

.....

Discussion

The second condition is $n > n_{\text{Q}}$, i.e. $n > (2\pi mk_{\text{B}}T/h^2)^{3/2}$.

Because $n = 1/l^3$, substituting gives $1/l^3 > (2\pi m k_B T / h^2)^{3/2}$.

Taking the (1/3)-power and rearranging the result gives

$$\begin{aligned} l &< h / (2\pi m k_B T)^{1/2} \\ &< (3/2\pi)^{1/2} \times h / (3m k_B T)^{1/2} \\ &< (3/2\pi)^{1/2} \times \lambda_{dB} \\ &< 0.7 \lambda_{dB}. \end{aligned}$$

That is, from the second degeneracy condition, $n > n_Q$, we obtain the first degeneracy condition, $l < \lambda_{dB}$.

You have now shown that the two conditions for degeneracy are actually equivalent; they are merely alternative expressions of the same physics. In the following activity, you will see how a third such expression is derived, which is also equivalent.

Activity 3

- Rewrite the second description of degeneracy, $n > n_Q$, to find a limit for the thermal energy $k_B T$. State a third, equivalent condition for degeneracy as a limit on the temperature.
- Calculate values of the quantity $h^2 n^{2/3} / (2\pi m k_B)$ in the core of the Sun for (i) protons and (ii) electrons. Assume that the mass fractions of hydrogen and helium are $X_H = X_{He} = 0.5$ and that the core density of the Sun is

$\rho_{c,\odot} = 1.48 \times 10^5 \text{ kg m}^{-3}$. Note that in the solar core, all atoms are fully ionised and that the number density is $n = \rho X / m$ for each type of particle.

The mass of an electron is $m_e = 9.11 \times 10^{-31} \text{ kg}$, the mass of a hydrogen nucleus is $m_p = 1.67 \times 10^{-27} \text{ kg}$ and the mass of a helium nucleus is $m_{He} = 4u$ where u is the atomic mass unit given by $u = 1.66 \times 10^{-27} \text{ kg}$. As usual Planck's constant is $h = 6.63 \times 10^{-34} \text{ J s}$ and Boltzmann's constant is $k_B = 1.38 \times 10^{-23} \text{ J K}^{-1}$.

- Consider the results to part (b), and state whether (i) protons and (ii) electrons are degenerate in the Sun's core. Assume that the core temperature of the Sun is $T_{c,\odot} = 15.6 \times 10^6 \text{ K}$.

.....

Answer

- a. Since $n_Q = (2\pi mk_B T/h^2)^{3/2}$, so the degeneracy condition $n > n_Q$ implies that $n > (2\pi mk_B T/h^2)^{3/2}$.

Taking the $(2/3)$ -power and multiplying both sides by $h^2/(2\pi m)$ gives

$$h^2 n^{2/3}/(2\pi m) > k_B T, \text{ i.e. } k_B T < h^2 n^{2/3}/(2\pi m).$$

The third, equivalent condition for degeneracy, in relation to temperature, is: the gas is degenerate if its temperature $T < h^2 n^{2/3}/(2\pi m k_B)$.

- b. (i) For protons,

$$\begin{aligned} n_p &= \rho_{c,\odot} X_H / m_p = 1.48 \times 10^5 \text{ kg m}^{-3} \times 0.5 / 1.67 \times 10^{-27} \text{ kg} \\ &= 4.43 \times 10^{31} \text{ m}^{-3} \text{ (i.e. hydrogen nuclei per m}^3\text{)}. \end{aligned}$$

Therefore, for protons, the temperature-related degeneracy condition is

$$\begin{aligned} \frac{h^2 n_p^{2/3}}{2\pi m_p k_B} &= \frac{(6.63 \times 10^{-34} \text{ J s})^2 \times (4.43 \times 10^{31} \text{ m}^{-3})^{2/3}}{(2 \times \pi \times 1.67 \times 10^{-27} \text{ kg} \times 1.38 \times 10^{-23} \text{ J K}^{-1})} \\ &= 3.78 \times 10^3 \text{ J s}^2 \text{ m}^{-2} \text{ kg}^{-1} \text{ K} \\ &\approx 3800 \text{ K}. \end{aligned}$$

(ii) In the solar core, all atoms are ionised. The electrons are provided by the hydrogen (1 electron per atom) and helium (2 electrons per atom), which each account for 0.5 of the composition by mass. So

$$\begin{aligned} n_e &= n_p + 2n_{\text{He}} = \frac{\rho_{c,\odot} X_H}{m_p} + \frac{2\rho_{c,\odot} X_{\text{He}}}{m_{\text{He}}} = \rho_{c,\odot} \left(\frac{X_H}{m_p} + \frac{2X_{\text{He}}}{4u} \right) \\ &= 1.48 \times 10^5 \text{ kg m}^{-3} \left(\frac{0.5}{1.67 \times 10^{-27} \text{ kg}} + \frac{2 \times 0.5}{4 \times 1.66 \times 10^{-27} \text{ kg}} \right) \\ &= 6.66 \times 10^{31} \text{ m}^{-3}. \end{aligned}$$

Therefore, for electrons, the temperature-related degeneracy condition is

$$\begin{aligned} \frac{h^2 n_e^{2/3}}{2\pi m_e k_B} &= \frac{(6.63 \times 10^{-34} \text{ J s})^2 \times (6.66 \times 10^{31} \text{ m}^{-3})^{2/3}}{(2 \times \pi \times 9.11 \times 10^{-31} \text{ kg} \times 1.38 \times 10^{-23} \text{ J K}^{-1})} \\ &= 9.12 \times 10^6 \text{ J s}^2 \text{ m}^{-2} \text{ kg}^{-1} \text{ K} \\ &\approx 9.1 \times 10^6 \text{ K}. \end{aligned}$$

- c. The temperature condition for degeneracy is $T < n^{2/3} h^2 / (2\pi m k_B)$. Since

$T_{c,\odot} = 15.6 \times 10^6 \text{ K}$, so the temperature in the core of the Sun is (i) much too high for proton degeneracy to have set in, and (ii) marginally too high for electron degeneracy to have set in.

You have seen in the previous activity that a third equivalent expression for the condition of degeneracy can be written as follows:

$$T < h^2 n^{2/3} / (2\pi m k_B) \quad (5)$$

Sometimes, this third condition is used to describe a degenerate gas as cold, because its temperature falls below some limit. However, this can be misleading because electrons may become degenerate at temperatures of millions of kelvins, which is obviously not cold in the common usage of the word.

3.3 Fermions

Electrons, along with protons and neutrons, are **fermions**, which means they have particle spin $= \pm 1/2$. The occupation of quantum states by fermions is restricted by the

Pauli exclusion principle, which says that no more than one identical fermion can occupy a given quantum state. The *only* way to distinguish one fermion from another of the same particle is its **spin**, and because this has only two possible values, $+1/2$ or $-1/2$, at most two fermions of opposite spin can occupy a given quantum state, so that the number of fermions per quantum state is $g_s = 2$.

At low temperatures, all quantum states with energy less than their **chemical potential** are filled, and all quantum states with energy greater than their chemical potential are empty. In a cold electron gas, the energy of the most energetic degenerate electron is called the **Fermi energy**, E_F , and the momentum of particles with this energy is called the **Fermi momentum**, p_F . The Fermi energy is the sum of a particle's kinetic energy and rest-mass energy. At low speeds, in the so-called non-relativistic limit, the **Fermi kinetic energy** and Fermi momentum for electrons are related by

$$E_F = p_F^2 / 2m_e \quad (6)$$

where m_e is the electron mass. The total number of degenerate electrons in the gas is evaluated as

$$N_e = \frac{4\pi g_s V}{3h^3} p_F^3. \quad (7)$$

Because the number density of degenerate electrons is simply the total number of electrons per unit volume, $n_e = N_e/V$, and there are two spin states for electrons ($g_s = 2$),

this equation can be rearranged to express the magnitude of the Fermi momentum as

$$p_F = \left(\frac{3n_e}{8\pi} \right)^{1/3} h. \quad (8)$$

- Combine Equations (6) and (8) to express the Fermi kinetic energy in terms of the electron number density.
- For electrons we have

$$E_F = \frac{p_F^2}{2m_e} = \left(\frac{3n_e}{8\pi} \right)^{2/3} \frac{h^2}{2m_e}.$$

Fermi energies are conveniently expressed in units of eV, keV or MeV. For comparison, the gap between atomic energy levels in a hydrogen atom is a few eV, the rest mass energy of an electron is 511 keV and that of a proton or neutron is about 940 MeV.

3.4 The pressure of a non-relativistic degenerate gas

In the non-relativistic limit (i.e. when $v \ll c$), the total energy of the gas may be expressed as

$$E_{NR} = N_e \left(\frac{3p_F^2}{10m_e} + m_e c^2 \right) \quad (9)$$

The second term in the brackets in Equation (9) is simply the rest-mass energy per particle, and the first term is the kinetic energy per particle. Furthermore, the kinetic energy per unit volume is the kinetic energy per particle multiplied by the number of particles per unit volume. So the kinetic energy per unit volume is $3p_F^2 n_e / (10m_e)$.

Now, the pressure provided by non-relativistic particles is 2/3 of the kinetic energy per unit volume, so

$$P_{NR} = \frac{2}{3} \times \frac{3p_F^2 n_e}{10m_e} = \frac{n_e p_F^2}{5m_e}.$$

Then, using Equation (8), we have

$$P_{NR} = \frac{n_e}{5m_e} \times \left(\frac{3n_e}{8\pi} \right)^{2/3} h^2 = \frac{h^2}{5m_e} \left(\frac{3}{8\pi} \right)^{2/3} n_e^{5/3}.$$

The **equation of state** for non-relativistic degenerate electrons may therefore be written as follows:

$$P_{NR} = K_{NR} n_e^{5/3} \quad \text{where} \quad K_{NR} = \frac{h^2}{5m_e} \left(\frac{3}{8\pi} \right)^{2/3}. \quad (10)$$

3.5 The pressure of an ultra-relativistic degenerate gas

We can also develop a similar expression for the equation of state for ultra-relativistic particles (i.e. when $v \sim c$). In this case, the total energy of the gas may be expressed as

$$E_{UR} = \frac{3}{4} N_e p_F c. \quad (11)$$

Because, in this case, the rest-mass energy is negligible, the kinetic energy per particle is $3p_F c/4$. Furthermore, the kinetic energy per unit volume is (once again) the kinetic energy

per particle multiplied by the number of particles per unit volume. So the kinetic energy per unit volume is $3p_F n_e c/4$.

Now, it turns out that the pressure provided by ultra-relativistic particles is 1/3 of the kinetic energy per unit volume, so

$$P_{UR} = \frac{1}{3} \times \frac{3p_F n_e c}{4} = \frac{n_e p_F c}{4}.$$

Then using Equation (8) we have

$$P_{UR} = \frac{n_e c}{4} \times \left(\frac{3n_e}{8\pi} \right)^{1/3} h = \frac{hc}{4} \left(\frac{3}{8\pi} \right)^{1/3} n_e^{4/3}.$$

The equation of state for ultra-relativistic degenerate electrons may therefore be written as follows:

$$P_{UR} = K_{UR} n_e^{4/3} \quad \text{where} \quad K_{UR} = \frac{hc}{4} \left(\frac{3}{8\pi} \right)^{1/3}. \quad (12)$$

Notice that the equations of state for degenerate electrons, whether non-relativistic or ultra-relativistic (or, indeed, somewhere in between), show that the pressure is *independent* of temperature and depends in each case only on the number density of electrons. In both the non-relativistic and ultra-relativistic cases, it is the independence of the pressure from the temperature that helps give a degenerate gas its interesting properties.

- What is the key difference between the equation of state for a non-relativistic electron-degenerate gas and that of an ultra-relativistic electron-degenerate gas?
- The pressure of an ultra-relativistic electron-degenerate gas has a weaker dependence on density than that of a non-relativistic gas (i.e. $P_{UR} \propto n_e^{4/3}$ compared to $P_{NR} \propto n_e^{5/3}$).

So as the density increases, the pressure in an ultra-relativistic gas increases less markedly.

Whether the gas of electrons is considered to be non-relativistic or ultra-relativistic, it is clear that white dwarfs are supported against further collapse by **electron degeneracy pressure**. Notice that the equations of state for a degenerate gas (whether non-relativistic or ultra-relativistic) show that the electron degeneracy pressure does *not* depend on temperature. This is quite different from a so-called ideal gas (which is a good approximation for most normal gases) where the equation of state is $P = nk_B T$, where n is the particle number density, k_B is Boltzmann's constant and T is the temperature. This decoupling of pressure and temperature in a degenerate gas is what is responsible for many of the unusual properties of white dwarfs.

4 White dwarfs

We now have all the tools in place to consider the physics of white dwarfs – the end points in the evolution of low- and intermediate-mass stars.

4.1 The electron density

Because we will be considering electron degeneracy, we begin by deriving a new way of expressing the electron density. The general equation for the number density n of some type of particle is $n = \rho X/m$, where ρ is the gas density, X is the mass fraction and m is the particle mass.

We can write this for electrons just as easily as for nucleons: $n_e = \rho X_e/m_e$. However, it is more convenient to write $X_e \equiv Y_e m_e/m_H$ where Y_e is the number of electrons per nucleon. This leads to the expression

$$n_e = \rho Y_e/m_H. \quad (13)$$

- What is the value of Y_e in: (a) pure ionised hydrogen; (b) pure ionised helium?
- (a) Hydrogen has one electron and one nucleon (its proton), so $Y_e = 1$; (b) helium has two electrons and four nucleons (two protons and two neutrons), so $Y_e = 0.5$.

You can readily convince yourself that $Y_e = 0.5$ for the dominant isotopes of carbon-12 and oxygen-16 also (since carbon-12 has 12 nucleons and 6 electrons, while oxygen-16 has 16 nucleons and 8 electrons). For white dwarfs, the amount of hydrogen is negligible – it has all been burnt to helium or on to carbon and/or oxygen – so $Y_e = 0.5$ for white dwarfs also.

For an electron-degenerate gas, you know that the pressure is independent of temperature, and is given by a constant multiplied by some power of the electron density. Using Equation 13, we can therefore write:

$$\text{for non-relativistic gas, } P_{\text{NR}} = K_{\text{NR}}(\rho Y_e/m_H)^{5/3}.$$

$$\text{for ultra-relativistic gas, } P_{\text{UR}} = K_{\text{UR}}(\rho Y_e/m_H)^{4/3}$$

The so-called Clayton model for the internal structure of a star gives the core pressure required to support a star in terms of its core density as:

$$P_c \approx (\pi/36)^{1/3} G M^{2/3} \rho_c^{4/3}. \quad (14)$$

Here G is Newton's gravitational constant ($= 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$). By saying that

degenerate electrons *provide* the required internal pressure to support the star, that is, by equating this to the degenerate-electron pressure, it is possible to write these equations purely in terms of the mass and core density, which therefore allows us to write one variable in terms of the other, as the following activity demonstrates.

Activity 4

- By equating the core pressure in a star to the degenerate pressure of non-relativistic electrons, derive an expression for the core density in terms of its mass M and the number of electrons per nucleon, Y_e . Leave physical constants unevaluated.
- Using the relation between electron number density n_e and gas density ρ_c in the core of the star, express the core electron density as a function of stellar mass.

Answer

- Equating the core pressure $P_c = (\pi/36)^{1/3} GM^{2/3} \rho_c^{4/3}$ to the pressure of non-relativistic degenerate electrons $P_{NR} = K_{NR} (\rho_c Y_e / m_H)^{5/3}$, where

$$K_{NR} = \frac{h^2}{5m_e} \left(\frac{3}{8\pi} \right)^{2/3}, \text{ we have}$$

$$K_{NR} (\rho_c Y_e / m_H)^{5/3} = (\pi/36)^{1/3} GM^{2/3} \rho_c^{4/3}.$$

Collecting terms in ρ on the left-hand side, and all others on the right, we get

$$\rho_c^{1/3} = \left(\frac{\pi}{36} \right)^{1/3} \frac{G}{K_{NR}} \left(\frac{m_H}{Y_e} \right)^{5/3} M^{2/3}.$$

Cubing this gives

$$\rho_c = \left(\frac{\pi}{36} \right) \left(\frac{G}{K_{NR}} \right)^3 \left(\frac{m_H}{Y_e} \right)^5 M^2$$

and substituting for K_{NR} gives

$$\rho_c = \left(\frac{\pi}{36} \right) \left(\frac{5m_e}{h^2} \right)^3 \left(\frac{8\pi}{3} \right)^2 G^3 \left(\frac{m_H}{Y_e} \right)^5 M^2.$$

Consolidating the two numerical factors, we obtain

$$\rho_c = \left(\frac{16\pi^3}{81} \right) \left(\frac{5m_e}{h^2} \right)^3 G^3 \left(\frac{m_H}{Y_e} \right)^5 M^2.$$

- The electron number density in the core of the star is $n_e = \rho_c Y_e / m_H$, so substituting in the core density from part (a) gives

$$n_e = \left(\frac{16\pi^3}{81} \right) \left(\frac{5m_e}{h^2} \right)^3 G^3 \left(\frac{m_H}{Y_e} \right)^4 M^2.$$

Activity 4 shows that in the non-relativistic case, the density in the core of a star may be written as

$$\rho_c = \left(\frac{16\pi^3}{81} \right) \left(\frac{5m_e}{h^2} \right)^3 G^3 \left(\frac{m_H}{Y_e} \right)^5 M^2 \quad (15)$$

or, in terms of the degenerate-electron number density, as

$$n_e = \left(\frac{16\pi^3}{81} \right) \left(\frac{5m_e}{h^2} \right)^3 G^3 \left(\frac{m_H}{Y_e} \right)^4 M^2. \quad (16)$$

4.2 The Chandrasekhar limit

We have now derived an expression for the electron number density inside a star in the non-relativistic case, but is the material inside a white dwarf really non-relativistic? In the non-relativistic limit, the Fermi kinetic energy of degenerate electrons is given by $E_F = p_F^2 / (2m_e)$ (Equation 6), where p_F is the Fermi momentum given by

$p_F = (3n_e/8\pi)^{1/3} h$ (Equation 8). Therefore, the Fermi kinetic energy of the degenerate, non-relativistic electrons in the core of a white dwarf is

$$E_F = \left[\left(\frac{3n_e}{8\pi} \right)^{1/3} h \right]^2 \frac{1}{2m_e}$$

So, using Equation 16, this becomes

$$E_F = \left[\frac{3}{8\pi} \left(\frac{16\pi^3}{81} \right) \left(\frac{5m_e}{h^2} \right)^3 G^3 \left(\frac{m_H}{Y_e} \right)^4 M^2 \right]^{2/3} \frac{h^2}{2m_e}$$

$$E_F = \frac{25}{2} \left(\frac{2\pi^2}{27} \right)^{2/3} \left(\frac{G}{h} \right)^2 \left(\frac{m_H}{Y_e} \right)^{8/3} M^{4/3} m_e. \quad (17)$$

The expression above relates the Fermi kinetic energy to a set of physical constants and the mass of the white dwarf, M . Substituting for these values therefore we have $E_F = 364(M/M_\odot)^{4/3}$ keV. Evaluating this expression for a white dwarf with mass $0.4 M_\odot$

reveals that the Fermi kinetic energy in this case is about 107 keV which is already more than 20% of the electron rest-mass energy ($m_e c^2 = 511$ keV). This makes it doubtful that the non-relativistic treatment is reliable for such a white dwarf. In more massive white dwarfs, the Fermi kinetic energy is even higher (because $E_F \propto M^{4/3}$) and the non-relativistic treatment will be progressively less reliable. Equation 17 implies that $E_F \approx m_e c^2$ when $M = 1.3 M_\odot$, indicating that the ultra-relativistic treatment is certainly required instead when masses become this large.

Activity 5

- By equating the core pressure in a star to the degenerate pressure of ultra-relativistic electrons, derive an expression for the mass of a star supported by ultra-relativistic electrons.
- Evaluate the mass, assuming $Y_e = 0.5$, in SI units and solar masses. Note that $G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$, $h = 6.63 \times 10^{-34} \text{ J s}$, $c = 3.00 \times 10^8 \text{ m s}^{-1}$ and $m_H = 1.67 \times 10^{-27} \text{ kg}$.

Discussion

- For the ultra-relativistic case, equating P_c to P_{UR} gives

$$P_{UR} = K_{UR}(\rho_c Y_e / m_H)^{4/3} = (\pi/36)^{1/3} G M^{2/3} \rho_c^{4/3} \text{ where } K_{UR} = \frac{hc}{4} \left(\frac{3}{8\pi} \right)^{1/3}$$

Note that, in contrast to the non-relativistic case, the density term $\rho_c^{4/3}$ is the same on both sides, so cancels out, leaving

$$K_{UR} \left(\frac{Y_e}{m_H} \right)^{4/3} = \left(\frac{\pi}{36} \right)^{1/3} G M^{2/3}.$$

Collecting M on one side and swapping left and right sides gives

$$M^{2/3} = \left(\frac{36}{\pi} \right)^{1/3} \left(\frac{K_{UR}}{G} \right) \left(\frac{Y_e}{m_H} \right)^{4/3}.$$

Taking the (3/2)-power and substituting for K_{UR} gives

$$M = \left(\frac{36}{\pi} \right)^{1/2} \left(\frac{(hc/4)(3/8\pi)^{1/3}}{G} \right)^{3/2} \left(\frac{Y_e}{m_H} \right)^2$$

$$M = \left(\frac{3 \times 36}{4^3 \times 8\pi^2} \right)^{1/2} \left(\frac{hc}{G} \right)^{3/2} \left(\frac{Y_e}{m_H} \right)^2$$

$$M = \frac{1}{\pi} \left(\frac{27}{128} \right)^{1/2} \left(\frac{hc}{G} \right)^{3/2} \left(\frac{Y_e}{m_H} \right)^2.$$

- Putting in the numbers, we have

$$M = \frac{1}{\pi} \left(\frac{27}{128} \right)^{1/2} \left(\frac{hc}{G} \right)^{3/2} \left(\frac{Y_e}{m_H} \right)^2$$

$$M = \frac{1}{\pi} \left(\frac{27}{128} \right)^{1/2} \left(\frac{6.63 \times 10^{-34} \text{ J s} \times 3.00 \times 10^8 \text{ m s}^{-1}}{6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}} \right)^{3/2} \times \left(\frac{0.5}{1.67 \times 10^{-27} \text{ kg}} \right)^2$$

So $M = 2.12 \times 10^{30} \text{ kg}$. Converting to solar units, this

is $M = 2.12 \times 10^{30} \text{ kg} / (1.99 \times 10^{30} \text{ kg } M_{\odot}^{-1}) = 1.07 M_{\odot}$.

The mass of a star when the ultra-relativistic limit is reached may be viewed as the maximum stellar mass that can be supported by degenerate electron pressure. The approximate calculation in the previous activity shows that this maximum mass is of the order of the mass of the Sun.

The maximum mass for a white dwarf is called the **Chandrasekhar limit**; the most realistic computations estimate it as $M_{\text{Ch}} \approx 1.4 M_{\odot}$.

If the mass is increased further, then the pressure required to support the star also increases. But the ultra-relativistic degenerate electrons will not be able to increase their pressure to support it. Something has to give.... That something is the material supporting the star. There *is* a stable configuration of material at higher mass, but it does not consist of degenerate electrons. Rather, the electrons and protons of higher-mass objects are forced to combine as neutrons, and the stable object above the maximum white-dwarf mass is called a neutron star. Such objects will be studied later in this course.

4.3 The white dwarf mass-radius relationship

The mean density of a star is equal to its mass divided by its volume, $\bar{\rho} = 3M/(4\pi R^3)$. It

can be related to the core density if the density profile of the star is known. So-called **polytropic stellar models** are ones in which the pressure at some radius is proportional to the density at that radius to some power. For a star described by a polytropic model with $P(r) \propto \rho(r)^{5/3}$ (a reasonable approximation), the mean density turns out to be one-sixth

of the core density: $\bar{\rho} = \rho_c/6$. It is therefore possible to write a relation between the radius, mass and core density of a star.

Where the core density is dictated by non-relativistic degenerate electrons, we have already seen how the core density and mass are related (Equation 15). We can eliminate the density from this equation and find an expression for the radius of the object as a function of its mass, as shown by the following activity.

Activity 6

- Rearrange the definition of the mean density $\bar{\rho} = 3M/(4\pi R^3)$ to get an expression for the radius in terms of the mass and mean density.
- Use the result for a polytropic model with $P \propto \rho^{5/3}$ (i.e. $\bar{\rho} = \rho_c/6$) to re-express the radius in terms of mass and *core* density.

- c. Substitute in the expression for the core density of non-relativistic degenerate electrons (Equation 15) to derive an expression for the radius of a white dwarf as a function of mass. Give your final answer in units normalised to the solar mass and solar radius. Note that $G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$,
- $$h = 6.63 \times 10^{-34} \text{ J s}, m_e = 9.11 \times 10^{-31} \text{ kg} \text{ and } m_H = 1.67 \times 10^{-27} \text{ kg}.$$
- d. Re-express the final result in solar masses and Earth radii, where the Earth's radius is $R_\oplus = 6.37 \times 10^6 \text{ m}$.
-

Discussion

- a. The radius as a function of mass and average density is

$$R = \left(\frac{3M}{4\pi\bar{\rho}} \right)^{1/3}.$$

- b. If the average density is 1/6 of the core density, then

$$R = \left(\frac{18M}{4\pi\rho_c} \right)^{1/3}.$$

- c. Substituting in the core density from Equation 15 gives

$$R = \left(\frac{18M}{4\pi} \right)^{1/3} \left(\frac{16\pi^3}{81} \right)^{-1/3} \left(\frac{5m_e}{h^2} \right)^{-1} G^{-1} \left(\frac{m_H}{Y_e} \right)^{-5/3} M^{-2/3}.$$

Rearranging the terms with negative powers:

$$R = \left(\frac{9M}{2\pi} \right)^{1/3} \left(\frac{81}{16\pi^3} \right)^{1/3} \left(\frac{h^2}{5m_e} \right) \frac{Y_e^{5/3}}{Gm_H^{5/3}} M^{-2/3}.$$

Now consolidate the mass terms and the numerical factors:

$$R = \left(\frac{729}{32\pi^4} \right)^{1/3} \left(\frac{h^2}{5m_e} \right) \frac{Y_e^{5/3}}{Gm_H^{5/3}} M^{-1/3}.$$

Now, for convenience, express the mass in solar units:

$$R = \left(\frac{729}{32\pi^4} \right)^{1/3} \left(\frac{h^2}{5m_e} \right) \frac{Y_e^{5/3}}{Gm_H^{5/3}} M^{-1/3} \times (1.99 \times 10^{30} \text{ kg } M_\odot^{-1})^{-1/3}.$$

Then putting in the numbers:

$$R = \left(\frac{729}{32\pi^4} \right)^{1/3} \times \frac{(6.63 \times 10^{-34} \text{ J s})^2}{(5 \times 9.11 \times 10^{-31} \text{ kg})} \times \frac{(0.5)^{5/3} \times (1.99 \times 10^{30} \text{ kg})^{-1/3}}{(6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}) \times (1.67 \times 10^{-27} \text{ kg})^{5/3}} \times \left(\frac{M}{M_\odot} \right)^{-1/3}.$$

And so

$$R = 9.45 \times 10^6 \left(\frac{M}{M_{\odot}} \right)^{-1/3} \text{ m.}$$

Because $R_{\odot} = 6.96 \times 10^8 \text{ m}$, the white dwarf radius is

$$R_{\text{WD}} = \left(\frac{9.45 \times 10^6 \text{ m}}{6.96 \times 10^8 \text{ m } R_{\odot}^{-1}} \right) \times \left(\frac{M_{\text{WD}}}{M_{\odot}} \right)^{-1/3}$$

$$R_{\text{WD}} = \frac{R_{\odot}}{74} \times \left(\frac{M_{\text{WD}}}{M_{\odot}} \right)^{-1/3}.$$

d. Because the radius of the Earth is $R_{\oplus} = 6.37 \times 10^6 \text{ m}$,

$$R_{\text{WD}} = \left(\frac{9.45 \times 10^6 \text{ m}}{6.37 \times 10^6 \text{ m } R_{\oplus}^{-1}} \right) \times \left(\frac{M_{\text{WD}}}{M_{\odot}} \right)^{-1/3}$$

$$R_{\text{WD}} = 1.5 R_{\oplus} \times \left(\frac{M_{\text{WD}}}{M_{\odot}} \right)^{-1/3}.$$

The result of the previous activity

$$R_{\text{WD}} = \frac{R_{\odot}}{74} \times \left(\frac{M_{\text{WD}}}{M_{\odot}} \right)^{-1/3} \quad (18)$$

is remarkable for two reasons. First, it shows that the radius of a white dwarf with the same mass as the Sun is two orders of magnitude smaller, comparable in radius to the Earth, as illustrated in Figure 3.

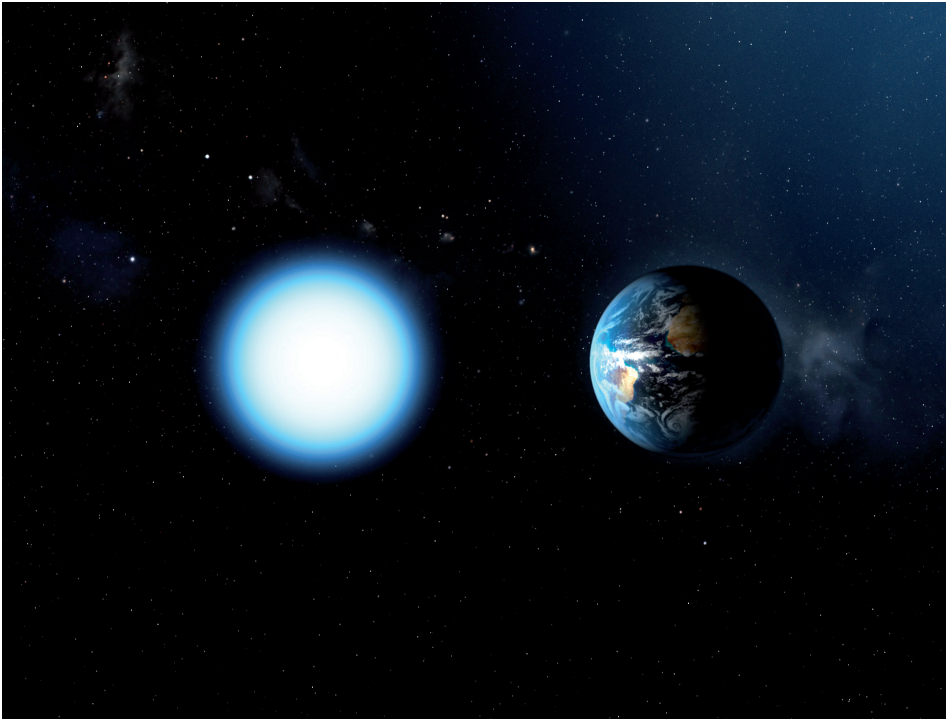


Figure 3 An artist's impression showing the relative sizes of the white dwarf Sirius B and the Earth.

Second, notice the mass-dependence of the radius: $R_{\text{WD}} \propto M_{\text{WD}}^{-1/3}$. Unlike low-mass main-sequence stars fusing hydrogen in helium, where $R_{\text{MS}} \propto M_{\text{MS}}^{2/5}$, note the important result that the sign of the exponent is *negative* for white dwarfs!

While the more massive a main-sequence star is, the *larger* its radius, the more massive a white dwarf is, the *smaller* its radius!

The derivation above of the white dwarf mass-radius relationship was based on the assumption that the electrons are non-relativistic. We showed this to be incorrect as the mass increases and it would imply infinite density (zero radius) at the Chandrasekhar limit, $1.4 M_{\odot}$. There is another formula which gives a more accurate picture of the radius of white dwarfs, but it is an empirical formula, simply fitted to observed data. This formula, derived by Michael Nauenberg in 1972, is

$$R_{\text{WD}} \approx 7.8 \times 10^6 \text{ m} \times \left[\left(\frac{1.4 M_{\odot}}{M_{\text{WD}}} \right)^{2/3} - \left(\frac{M_{\text{WD}}}{1.4 M_{\odot}} \right)^{2/3} \right]^{1/2}. \quad (19)$$

The formula derived in Equation 18, while unable to correctly describe white dwarfs near the Chandrasekhar limit, does encapsulate real physical ideas that are valid for low-mass stars. There is a place in science for both formulae!

4.4 White dwarf composition and cooling

The simplest type of white dwarf, resulting from the collapse of a star of very low main-sequence mass ($M_{\text{MS}} \lesssim 0.5 M_{\odot}$) in which no helium burning has occurred, would be composed entirely of helium. Such objects are referred to as **He white dwarfs**, with

masses $M_{\text{He,WD}} \lesssim 0.4 M_{\odot}$. However we are unlikely to find *isolated* He white dwarfs in the Galaxy for the following reason. The He white dwarfs form from stars with very low mass. Such stars have main-sequence lifetimes of $\sim 10^{11}$ years, or around 10 times the age of the Universe, so have not had time to evolve into white dwarfs yet. Any He white dwarfs that are seen must have formed as part of a binary star system that has undergone a phase of mass transfer which has altered the mass and composition of one or both of its components.

Low-mass stars, with $0.5 M_{\odot} \lesssim M_{\text{MS}} \lesssim 3 M_{\odot}$, undergo helium burning, producing carbon and oxygen in their cores. They lose some mass while on the AGB and so typically end their lives with masses of $\sim 0.4 - 0.8 M_{\odot}$. Even intermediate-mass stars, with

$3 M_{\odot} \lesssim M_{\text{MS}} \lesssim 8 M_{\odot}$, lose so much mass as giants that only $\lesssim 1.2 M_{\odot}$ remains to form the white dwarf. These carbon/oxygen cores are supported against further collapse by degenerate electrons, and hence do not attain the temperatures required to initiate carbon burning. The remnants of low- and intermediate-mass stars are therefore **CO white dwarfs**, with masses $0.4 M_{\odot} \lesssim M_{\text{CO,WD}} \lesssim 1.2 M_{\odot}$.

There is a narrow range of massive stars, with $8 M_{\odot} \lesssim M_{\text{MS}} \lesssim 11 M_{\odot}$, whose remaining cores are massive enough to collapse and ignite carbon (forming oxygen, neon and magnesium), but which ultimately fall below the Chandrasekhar limit owing to continued mass loss. When this carbon burning terminates, these stars will be supported by degenerate electrons again and will contract no further, leaving **ONeMg white dwarfs**, with masses typically in the range $1.2 M_{\odot} \lesssim M_{\text{ONeMg,WD}} \lesssim 1.4 M_{\odot}$.

Stars with $M_{\text{MS}} \gtrsim 11 M_{\odot}$ retain enough of their mass during mass loss that their cores continue to exceed the Chandrasekhar limit; at the end of each burning phase, their cores contract and heat up more. These stars complete all of the advanced burning phases, including silicon burning to form iron. Their endpoints will be explored in later sections of this course.

As there are no nuclear energy sources active in a white dwarf, its luminosity is due solely to the slow leakage of thermal energy into space, as radiation. It takes $\sim 10^9$ y for white dwarfs to fade to $L \sim 10^{-3} L_{\odot}$. As white dwarfs fade, they also cool. The luminosity of

white dwarfs is provided by thermal leakage rather than slow gravitational contraction because white dwarfs are degenerate, so their temperature and pressure are decoupled. As they cool, the pressure remains constant so the radius is unchanged. Hence there is *no* release of gravitational potential energy. As white dwarfs cool and fade in the H-R diagram, they do so along lines of constant radius (see Figure 1).

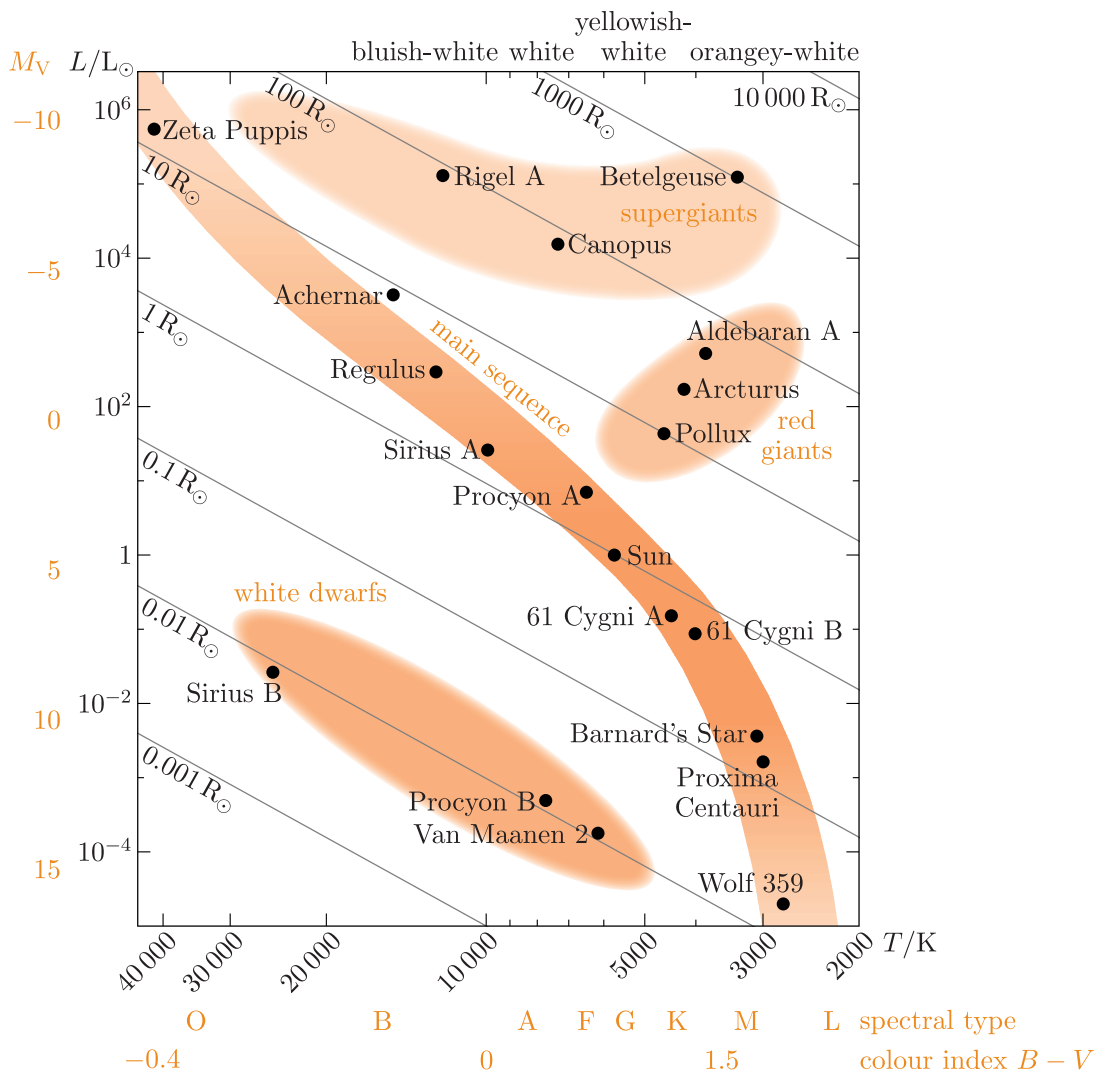


Figure 1 (repeated) A schematic H–R diagram indicating the locations of four regions where most stars are found. The vertical axis shows stellar luminosity in solar units or absolute visual magnitude; the horizontal axis shows surface temperature in kelvins, spectral type, or colour index defined as the difference between B and V magnitudes. (Notice that temperature increases to the *left*.) Diagonal lines of constant stellar radius are also shown and some notable named stars are indicated.

5 Supernovae

In this section, you'll see what happens to a star whose final core mass *exceeds* the Chandrasekhar limit, and subsequently, discover what remnant it may leave behind.

A star with a main-sequence mass $\gtrsim 11 M_{\odot}$ will complete all the stages of nuclear fusion

that are available. Silicon burning will result in a core composed mainly of iron-56, surrounded by concentric shells of silicon, oxygen, neon, carbon, helium and hydrogen. No energy can be released by the thermonuclear fusion of iron, so the core collapses and the degenerate electrons within it become more and more relativistic. When the mass of the core exceeds the Chandrasekhar limit (about $1.4 M_{\odot}$), the degenerate electrons are

no longer able to support the core, and a catastrophic collapse follows. The core will essentially collapse on a free-fall timescale, liberating gravitational potential energy.

- The free-fall timescale is given by $\tau_{\text{ff}} = (3\pi/(32G\rho))^{1/2}$. Calculate the free-fall

timescale for a stellar core with a density of $\rho = 10^{14} \text{ kg m}^{-3}$.

- Putting in the numbers to the equation above,

$$\tau_{\text{ff}} = \left(\frac{3 \times \pi}{32 \times 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \times 10^{14} \text{ kg m}^{-3}} \right)^{1/2} \approx 0.007 \text{ s}.$$

- The gravitational potential energy released in the collapse of a stellar core of mass M , from an initial radius R_1 to a final radius R_2 , is $E_g = (GM^2/R_2) - (GM^2/R_1)$.

Calculate the gravitational potential energy released when a stellar core of mass $1.4 M_{\odot}$ collapses to a radius of about 10 km.

- In this case, the final radius is very small, so $R_2 \ll R_1$, and we can neglect the initial gravitational potential energy as it is so small. Therefore the gravitational potential energy released is $E_g \approx GM^2/R_2$. So the energy released by the collapse of a $1.4 M_{\odot}$ stellar core is

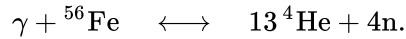
$$E_g \approx \frac{6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2} \times (1.4 \times 1.99 \times 10^{30} \text{ kg})^2}{10^4 \text{ m}} \approx 5 \times 10^{46} \text{ J}.$$

As the previous two questions show, the collapse of the core is *very* rapid and it liberates a *vast* amount of energy.

The initiation of exothermic (energy-liberating) fusion reactions provides pressure *support* for stars during their long-lasting burning phases. However, the initiation of *endothermic* (energy-absorbing) reactions draws kinetic energy out of the material and hence eliminates the pressure support. There are two processes that can absorb energy in the collapsing core: **photodisintegration** of nuclei by high-energy gamma rays and **electron capture** processes. In the first, the energy is used to unbind the nuclei, while in the second, energy is converted into the kinetic energy of neutrinos, which stream out of the star largely unhindered. Let's consider each of these two processes in more detail.

5.1 Nuclear photodisintegration

There are many ways in which high-energy gamma rays can disintegrate nuclei of iron-56, but as an illustration of the physics involved, and the amount of energy that may be absorbed by this photodisintegration process, consider the situation in which a nucleus of iron-56 is broken down into helium nuclei:



The amount of energy absorbed by this process in the forward direction is $\Delta Q = -124.4 \text{ MeV}$. At the temperature and density of a stellar core, the result is that about *three-quarters* of the iron nuclei are dissociated.

Activity 7

If each iron nucleus absorbs 124.4 MeV of energy by photodisintegration, and three-quarters of a core of mass $1.4 M_{\odot}$ is dissociated in this way, calculate the total energy absorbed by this process. (The mass of an iron-56 nucleus is $56u$, where $u = 1.66 \times 10^{-27} \text{ kg}$ is the atomic mass unit.)

Answer

The number of particles N in a sample of material is given by the total mass of the sample M divided by the individual masses of the particles m , i.e. $N=M/m$. Three-quarters of the core mass is $M = (3/4) \times 1.4 M_{\odot}$, so the number of iron-56 nuclei is

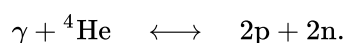
$$N = (3/4) \times 1.4 M_{\odot} / (56u)$$

$$N = \frac{1.05 \times 1.99 \times 10^{30} \text{ kg}}{56 \times 1.66 \times 10^{-27} \text{ kg}}$$

$$N = 2.25 \times 10^{55} \text{ nuclei.}$$

Each nucleus absorbs 124.4 MeV of energy, which is equivalent to $124.4 \times 10^6 \text{ eV} \times 1.60 \times 10^{-19} \text{ J eV}^{-1} = 1.99 \times 10^{-11} \text{ J}$. So the core absorbs $2.25 \times 10^{55} \times 1.99 \times 10^{-11} \text{ J} = 4.5 \times 10^{44} \text{ J}$ via the photodisintegration of iron-56 nuclei.

At still higher temperatures, helium nuclei will also undergo photodisintegration as follows:



The energy absorbed by this process in the forward direction is $\Delta Q = -28.3 \text{ MeV}$. At the temperature and density of a stellar core, the result is that about *half* of the helium-4 nuclei are dissociated.

Activity 8

If each helium-4 nucleus absorbs 28.3 MeV of energy by photodisintegration, and half of the core of mass $1.4 M_{\odot}$ is dissociated in this way, calculate the total energy absorbed by this process. (The mass of a helium-4 nucleus is $4u$, where $u = 1.66 \times 10^{-27}$ kg is the atomic mass unit.)

.....

Discussion

The number of particles is $N = M/m$. Half the core mass is $M = (1/2) \times 1.4 M_{\odot}$, so the number of helium-4 nuclei is

$$N = (1/2) \times 1.4 M_{\odot} / (4u)$$

$$N = \frac{0.7 \times 1.99 \times 10^{30} \text{ kg}}{4 \times 1.66 \times 10^{-27} \text{ kg}}$$

$$N = 2.10 \times 10^{56} \text{ nuclei.}$$

Each nucleus absorbs 28.3 MeV of energy, which is equivalent to $28.3 \times 10^6 \text{ eV} \times 1.60 \times 10^{-19} \text{ J eV}^{-1} = 4.53 \times 10^{-12} \text{ J}$. So the core absorbs $2.10 \times 10^{56} \times 4.53 \times 10^{-12} \text{ J} = 9.5 \times 10^{44} \text{ J}$ via the photodisintegration of helium-4 nuclei.

Therefore, the total amount of energy absorbed by nuclear photodisintegration is $4.5 \times 10^{44} \text{ J}$ from the dissociation of iron-56 plus $9.5 \times 10^{44} \text{ J}$ from the dissociation of helium-4, or about $1.4 \times 10^{45} \text{ J}$ in total.

5.2 Electron capture

The second mechanism by which energy is absorbed in the iron core of a star is that of electron capture, allowing energy to be carried away by **neutrinos**. The conversion of protons (in nuclei) to neutrons by electron capture is possible if the gas is sufficiently dense for degenerate electrons to have an energy greater than the 1.29 MeV mass-energy excess of neutrons relative to protons ($m_n c^2 = 939.56 \text{ MeV}$, while

$m_p c^2 = 938.27 \text{ MeV}$). Actually, the energy excess required for electron capture by bound

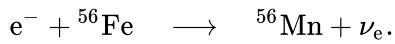
nuclear protons, rather than free protons, is usually somewhat higher and depends on the nucleus, but is nevertheless usually just a few MeV.

The general reaction may be written as



and is sometimes referred to as **neutronisation**.

At densities above about $10^{12} \text{ kg m}^{-3}$, iron-56 nuclei will capture electrons in the following reaction which produces a neutron-rich manganese nucleus:



where ν_e is a neutrino. Following this, further electron-capture reactions occur in

successive nuclei, converting more and more protons into neutrons.

Because electrons are rapidly used up by these reactions, the pressure support provided by degenerate electrons quickly disappears, and the core collapses rapidly, as noted earlier. The neutrinos produced by the electron-capture reactions carry away most of the energy. Although the passage of the neutrinos is hindered by the high densities in the core, which raises the likelihood of interactions, all of the neutrinos are still able to escape within a few seconds.

Activity 9

If each neutrino produced in the reactions described above carries away 10 MeV of energy, how much energy (in joules) is removed by neutrinos if the whole core (comprising $1.4 M_{\odot}$ of iron-56) undergoes neutronisation? (The mass of an iron-56 nucleus is $56u$, where $u = 1.66 \times 10^{-27} \text{ kg}$ is the atomic mass unit.)

Discussion

The number of nucleons contained in the stellar core is $1.4 M_{\odot}/u$ where u is the atomic mass unit. Because an iron-56 nucleus contains 26 protons (and 30 neutrons), the number of protons in the core is $(26/56) \times 1.4 M_{\odot}/u$. This is evaluated as

$$\frac{(26/56) \times 1.4 \times 1.99 \times 10^{30} \text{ kg}}{1.66 \times 10^{-27} \text{ kg}} \approx 7.8 \times 10^{56}$$

Assuming charge neutrality, there will also be $\approx 7.8 \times 10^{56}$ electrons. If every proton and electron undergoes neutronisation, this will produce $\approx 7.8 \times 10^{56}$ neutrinos.

If each neutrino carries away 10 MeV of energy, then the amount of energy removed by each neutrino in joules is $10 \times 10^6 \text{ eV} \times 1.60 \times 10^{-19} \text{ J eV}^{-1} = 1.6 \times 10^{-12} \text{ J}$.

So the total amount of energy removed is $1.6 \times 10^{-12} \text{ J} \times 7.8 \times 10^{56} = 1.2 \times 10^{45} \text{ J}$.

As the result of the previous activity shows, neutronisation, like nuclear photodisintegration, can also remove $\sim 10^{45} \text{ J}$ of energy from the collapsing stellar core within a few seconds!

5.3 Supernovae explosions

Over its lifetime, a high-mass star is supported against collapse by the thermal energy derived from converting hydrogen to iron, via intermediate species. Some of this energy is radiated away by the surface of the star over its long lifetime. Then, at the end of its life, it reverses that entire process by breaking apart the iron nuclei into their constituents. At the end of its life, a high-mass star will emit a similar amount of energy in a few seconds to that which it has emitted over its entire main-sequence lifetime!

Therefore, in a stellar core composed largely of iron, photodisintegration and neutronisation reactions trigger a rapid collapse, which will continue until the core reaches a density comparable to that of an atomic nucleus.

- Given that a nucleus of mass number A has a radius $R \approx 1.2 \times 10^{-15} \text{ m} \times A^{1/3}$, what is its density?
- The density $\rho = M/V$ where the mass $M = Au$ and the volume $V = 4\pi R^3/3$. Hence

$$\rho = 3Au / (4\pi \times (1.2 \times 10^{-15} \text{ m})^3 \times A) \text{ and therefore}$$

$$\rho \approx \frac{3 \times 1.66 \times 10^{-27} \text{ kg}}{4\pi \times (1.2 \times 10^{-15} \text{ m})^3} \approx 2 \times 10^{17} \text{ kg m}^{-3}$$

This is equivalent to around 200 million tonnes per cubic centimetre! When the core approaches nuclear densities, nuclear forces resist any further compression and the collapsing core will rebound. This sends a shockwave through the infalling material, which causes much of the stellar envelope to be ejected. This is a **supernova** explosion (see Figure 4).

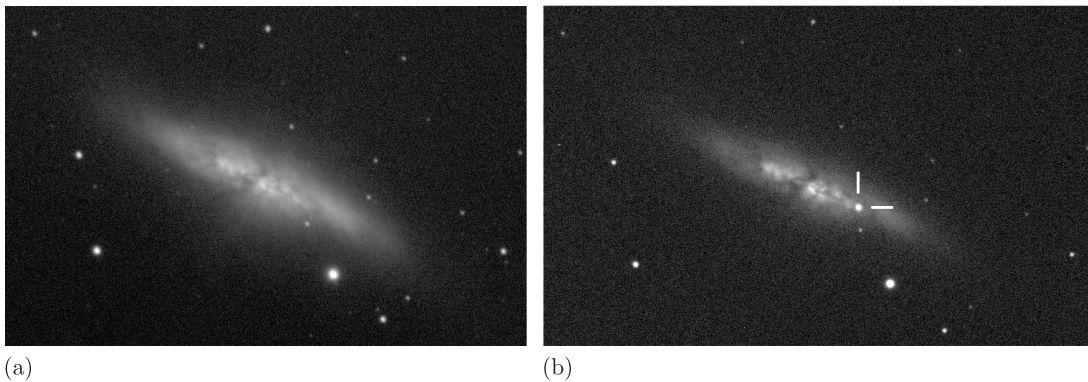


Figure 4 Two images of the same galaxy taken 41 days apart, with the location of a supernova highlighted by the white lines in the right-hand image.

Over the course of a year or so, the optical light output of a supernova is typically about 10^{42} J. However, it can also be observed that a further 10^{44} J of energy is carried away as the kinetic energy of the exploding debris, at velocities of tens of thousands of kilometres per second, to form a supernova remnant. These are certainly vast amounts of energy, but it turns out that they do not tell the whole story of the energy release during a supernova.

As you saw earlier, the gravitational potential energy released in the collapse of a stellar core is $\approx 5 \times 10^{46}$ J. This is 100 times greater than the observed energy carried away by the expanding debris, and it is at least 10 times the energy required to photodisintegrate the iron core, and 10 times the energy that can be carried away by neutronisation. So where does all the gravitational potential energy released by a supernova actually go?

In fact, there is an intermediate stage between the iron core and the production of a compact remnant. This takes the form of a hot, dense plasma of neutrons, protons, electrons, neutrinos and photons. At a temperature of 10^{11} K and a density of 10^{14} kg m⁻³ this plasma is opaque to electromagnetic radiation, but not to neutrinos; it is believed that neutrino/antineutrino pairs are produced in the plasma, and carry away most of the gravitational binding energy of the collapsing core.

- Quantify the ways in which energy is carried away from a supernova.
 - Photons provide the *least* effective means of removing the energy: only $\sim 10^{42}$ J over the first year or so after the explosion. The expansion of the ejected material carries 100 times as much as kinetic energy, $\sim 10^{44}$ J. However, the binding energy of the neutron star is $\approx 5 \times 10^{46}$ J, so much more energy must be removed than is carried by the photons and kinetic energy. The majority is thought to be carried away by neutrinos.
-

6 Neutron stars

The mass range of progenitor stars that collapse to form neutron stars is uncertain, but it is possible that all single progenitor stars with main-sequence masses in the range $M_{\text{MS}} \approx 11 - 25 M_{\odot}$ will do so. However, in some binary systems, neutron stars can also

form from stars that were initially even more massive than $25 M_{\odot}$, where mass transfer

between the two stars influences their evolution significantly. Whatever their origin, a newly formed neutron star will have a temperature of 10^{11} – 10^{12} K, but it will cool quickly to around 10^9 K on a timescale of a day, and to around 10^8 K within a hundred years.

Neutron stars are commonly observed as rapidly spinning radio pulsars. The youngest of these, such as the Crab pulsar which formed only ~ 1000 years ago, are seen to still reside in the centre of an expanding supernova remnant. Some so-called millisecond radio pulsars are observed in binary systems with a white dwarf or non-pulsar neutron star companion. Other neutron stars are seen in binary systems with a main-sequence or supergiant companion star from which they accrete large amounts of material. The accreting matter gives rise to strong X-ray emission, and such objects are known as X-ray binary stars.

The neutron stars in radio pulsars and X-ray binaries are measured to have masses in the range $M_{\text{NS}} \approx 1.2 - 2.2 M_{\odot}$, contained within a sphere of radius $R_{\text{NS}} \approx 10 - 15$ km;

hence they have a density of $10^{17} - 10^{18} \text{ kg m}^{-3}$. These extreme temperatures and densities give rise to some extraordinary consequences for the neutron star's properties.

6.1 Neutron star composition

For normal matter, the most stable configurations, with the lowest binding energy per nucleon, are nuclei around iron-56. However, when the density approaches $10^{14} \text{ kg m}^{-3}$,

the most stable nuclei are actually far more neutron-rich, such as nickel-78 and iron-76. At still higher densities, above $4 \times 10^{14} \text{ kg m}^{-3}$, a phenomenon called **neutron drip** occurs,

whereby neutrons 'leak out' of nuclei and an equilibrium mixture of nuclei, neutrons and electrons exists. When the density exceeds that of nuclear matter (at about $2 \times 10^{17} \text{ kg m}^{-3}$), the nuclei begin to merge and a dense gas of protons, neutrons and

electrons is produced. Under these conditions there are complicated and uncertain interactions between nuclei, so the equation of state for neutron stars is not known.

Despite this, a reasonable idea of the composition of a neutron star can be gained by considering a crude model in which the neutron star is treated as an ideal gas of degenerate electrons, protons and neutrons. In this situation, the normal beta decay of free neutrons is blocked because of the Pauli exclusion principle: there are no available quantum states left for the protons and electrons to occupy, so the neutrons are not able to decay. In particular, neutrons cannot decay if their Fermi energy is less than the sum of the Fermi energies of the electrons and protons, but they can decay if their Fermi energy is greater than that of the protons plus the electrons. An equilibrium will therefore exist when

$$E_{\text{F}}(\text{n}) = E_{\text{F}}(\text{p}) + E_{\text{F}}(\text{e}).$$

The neutrons and protons can be considered to be non-relativistic; hence their Fermi energies are given in each case by the sum of a rest-mass energy term and a kinetic energy term:

$$\text{neutrons : } E_F(n) = m_n c^2 + \frac{p_F(n)^2}{2m_n}$$

$$\text{protons : } E_F(p) = m_p c^2 + \frac{p_F(p)^2}{2m_p},$$

where m_n and m_p are the masses of the neutron and proton, and $p_F(n)$ and $p_F(p)$ are their Fermi momenta. Conversely, the electrons are much less massive than the protons and neutrons, so they can be considered ultra-relativistic. Their Fermi energy is simply

$$\text{electrons : } E_F(e) = p_F(e)c,$$

where $p_F(e)$ is the electron's Fermi momentum.

Now, because the Fermi momentum is $p_F = (3n/8\pi)^{1/3}h$ (Equation 8), we can write the equilibrium condition as

$$m_n c^2 + \left(\frac{3n_n}{8\pi}\right)^{2/3} \frac{h^2}{2m_n} = m_p c^2 + \left(\frac{3n_p}{8\pi}\right)^{2/3} \frac{h^2}{2m_p} + \left(\frac{3n_e}{8\pi}\right)^{1/3} hc.$$

Because the gas will be electrically neutral, the number densities of the protons and electrons must be equal: $n_p = n_e$. Furthermore, the mass-energy difference between the neutron and proton is $m_n c^2 - m_p c^2 = 1.29 \text{ MeV}$, so we can rearrange this as

$$\left(\frac{3n_p}{8\pi}\right)^{2/3} \frac{h^2}{2m_p} + \left(\frac{3n_p}{8\pi}\right)^{1/3} hc - \left(\frac{3n_n}{8\pi}\right)^{2/3} \frac{h^2}{2m_n} = 1.29 \text{ MeV}.$$

This equation can be solved numerically to find the ratio of neutrons to protons (n_n / n_p) at a given density. For instance, for a typical neutron star density of $2.5 \times 10^{17} \text{ kg m}^{-3}$, the

equation predicts a neutron-to-proton ratio of about 200 to 1. So, as long as there is one proton and one electron for every 200 neutrons, the neutrons are prevented from decaying, and the neutron star remains supported, essentially by the pressure of degenerate neutrons. Whereas white dwarfs are supported by *electron* degeneracy pressure, neutron stars are supported against further collapse by **neutron degeneracy pressure**.

Activity 10

Suppose that the neutron star is so dense that the protons and neutrons (as well as the electrons) are ultra-relativistic. What is the ratio of neutrons to protons in this case?

Discussion

We again start by writing $E_F(n) = E_F(p) + E_F(e)$, but now we have $E_F = p_F c$ for *each* type of particle where $p_F = (3n/8\pi)^{1/3} h$. So we can write the equilibrium condition as

$$\left(\frac{3n_n}{8\pi}\right)^{1/3} hc = \left(\frac{3n_p}{8\pi}\right)^{1/3} hc + \left(\frac{3n_e}{8\pi}\right)^{1/3} hc$$

and because $n_p = n_e$ in a neutral gas, this reduces to $n_n^{1/3} = 2n_p^{1/3}$ or

$n_n/n_p = 2^3 = 8$. Thus, even in the ultra-relativistic case, neutrons far outnumber protons, by a factor of 8 to 1.

6.2 The radius and mass of neutron stars

When deriving equations to describe neutron stars, we should be wary of oversimplifying the assumptions about conditions inside them. In reality, Newtonian gravity should be replaced by Einstein's general relativistic treatment, because for a neutron star of mass M_{NS} and radius R_{NS} , a neutron's gravitational potential energy, $GM_{NS}m_n/R_{NS}$, is

comparable to its rest-mass energy, $m_n c^2$.

The momenta of neutrons also approach the relativistic limits, requiring special relativity. Nevertheless, the non-relativistic approximation can lead to useful insights into the size of the star, and an understanding of the dominant physics. In particular, it is interesting to compare calculations of the radius of a white dwarf, supported by degenerate electrons, and a neutron star, supported by degenerate neutrons.

It turns out that the radius of a degenerate star scales according to the mass of the particle whose degenerate pressure supports the star. (The mass of the particle also determines its de Broglie wavelength, so you could also say that the de Broglie wavelength determines the radius of the star.) Because the pressure in a neutron star is provided by neutrons that are almost 2000 times heavier than an electron, neutron stars are a factor of almost 2000 smaller in radius. That is, while white dwarfs are comparable in size to the Earth, neutron stars have radii of only a few km! The typical masses and radii of different types of compact object are shown in Figure 5.

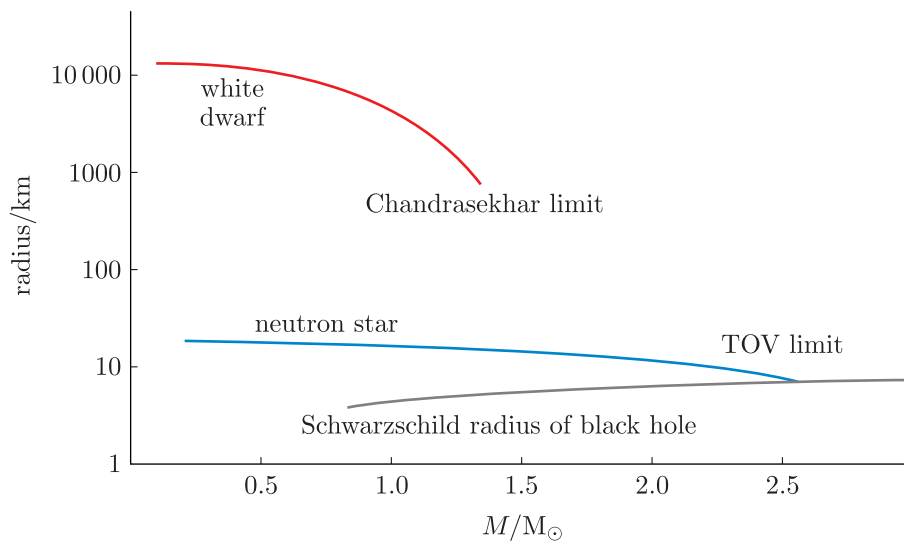


Figure 5 The mass-radius relationship for degenerate stellar remnants.

In principle, neutron stars have a minimum stable mass of $M_{\text{NS}} \approx 0.1 M_{\odot}$, although in practice the minimum mass observed is around $1.2 M_{\odot}$. An absolute upper mass limit of $M_{\text{NS}} \approx 2.9 M_{\odot}$ is set by the requirement that the sound speed inside a neutron star is less than the speed of light. However, it is more difficult to specify the actual upper mass limit for neutron stars than for white dwarfs for a couple of reasons.

First, interactions between neutrons are significant. They repel one another at close separations ($< 1.4 \times 10^{-15} \text{ m}$), which makes them less compressible, but their energies are also so high that they produce baryons, containing one or more strange quarks, and pions, which reduce the pressure and make them more compressible. The net effect is to enable neutron stars that are more massive than one might expect on the basis of simpler assumptions.

Second, the gravitational fields are so strong that Einstein's theory of gravity – general relativity – must be used instead of Newtonian gravity and, under general relativity, gravity also depends on pressure. Whereas, in a normal star, internal pressure resists gravity, under general relativity it strengthens the gravity, which reduces the maximum mass that will be stable.

The competing mechanisms of these two effects mean that it is extremely difficult to calculate the actual upper mass limit for neutron stars, and no consensus exists on the matter. Various calculations and theories suggest that the maximum mass is $M_{\text{TOV}} \approx 2.2 - 2.5 M_{\odot}$; this is referred to as the **Tolman–Oppenheimer–Volkoff (TOV)**

limit.

7 Black holes

Although this course is about white dwarfs and neutron stars, we cannot end without a brief mention of **black holes**. Black holes are the end point of a star's evolution if the mass of the degenerate stellar core exceeds the TOV limit. They have infinite density and are characterised by their **Schwarzschild radius**. This radius marks the event horizon of the black hole, within which the escape velocity exceeds the speed of light, meaning that nothing can escape from it or be observed within it. For a non-rotating black hole of mass M_{BH} , the Schwarzschild radius $R_{\text{Sch}} = 2GM_{\text{BH}}/c^2$, which is ~ 3 km per solar mass.

If the main-sequence mass of a star is in the range $M_{\text{MS}} \approx 16 - 25 M_{\odot}$, it *may* be that neutron degeneracy pressure is insufficient to halt the core collapse, and a black hole forms directly (instead of a neutron star) with no accompanying supernova explosion. Such an event may result in the formation of a black hole of mass $M_{\text{BH}} \approx 5 - 8 M_{\odot}$. The

apparent absence of compact objects between the upper mass limit observed for a neutron star ($\sim 2.2 M_{\odot}$) and the lower mass limit observed for a black hole ($\sim 5 M_{\odot}$) is referred to as the **mass gap**. However, recent observations of gravitational wave sources have found hints of objects in this mass range, thereby challenging our understanding of their nature and formation.

Single stars with main-sequence masses $M_{\text{MS}} \gtrsim 25 M_{\odot}$ will generally undergo a core-collapse supernova and produce a black hole with a mass of $M_{\text{BH}} \gtrsim 10 M_{\odot}$. At the lower end of the mass range, the black hole will form by fallback of material onto an initially created neutron star, while for stellar masses of $M_{\text{MS}} \gtrsim 40 M_{\odot}$, the black hole forms directly.

8 Quiz

Answer the following questions in order to test your understanding of the key ideas that you have been learning about.

In the following questions you may use the following values of physical constants, as necessary:

$$G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$$

$$h = 6.63 \times 10^{-34} \text{ J s}$$

$$k_B = 1.38 \times 10^{-23} \text{ J K}^{-1}$$

$$c = 3.00 \times 10^8 \text{ m s}^{-1}$$

$$m_e = 9.11 \times 10^{-31} \text{ kg}$$

$$m_H = 1.67 \times 10^{-27} \text{ kg}$$

$$u = 1.66 \times 10^{-27} \text{ kg}$$

Question 1

Which of the following statements about stellar evolution are *true*?

- ☐ Red giant stars are undergoing fusion of helium into carbon and oxygen.
- ☐ On the main sequence, stars convert hydrogen into helium.
- ☐ Stars spend less of their lives as red giants than they do on the main sequence.
- ☐ Supergiant stars have higher luminosities than red giant stars.
- ☐ More massive stars have shorter lives than less massive stars.
- ☐ High mass stars are more common than low mass stars.

Answer

The first five statements are all true. The last one is false: high mass stars are much rarer than low mass stars.

Question 2

A stellar core has a density of $3.0 \times 10^7 \text{ kg m}^{-3}$ and a composition comprising fully ionised helium-4, carbon-12 and oxygen-16 with relative amounts $X_{\text{He}} = 0.40$, $X_{\text{C}} = 0.30$ and $X_{\text{O}} = 0.30$. What is the limiting temperature, below which the electrons in the core would be degenerate?

- ☐ 110 million K
- ☐ 240 million K
- ☐ 320 million K
- ☐ 550 million K
- ☐ 630 million K

.....

Answer

Since helium-4, carbon-12 and oxygen-16 all have $Y_e = 0.5$ electrons per nucleon, the electron number density is simply

$$n_e = \frac{Y_e \rho}{m_H} = \frac{0.5 \times 3.0 \times 10^7 \text{ kg m}^{-3}}{1.67 \times 10^{-27} \text{ kg}} = 8.98 \times 10^{33} \text{ m}^{-3}$$

The limiting temperature below which electrons in the core would be degenerate is

$$T < \frac{h^2 n_e^{2/3}}{2\pi m_e k_B}$$

Putting in the numbers gives

$$T < \frac{(6.63 \times 10^{-34} \text{ J s})^2 \times (8.98 \times 10^{33} \text{ m}^{-3})^{2/3}}{2\pi \times 9.11 \times 10^{-31} \text{ kg} \times 1.38 \times 10^{-23} \text{ J K}^{-1}}$$

$$T < 240 \text{ million K}$$

Question 3

Match the following equations of state with the correct dependence on particle number density and temperature.

ultra-relativistic degenerate gas

non-relativistic degenerate gas

ideal gas

Match each of the items above to an item below.

$$P \propto n^{4/3}$$

$$P \propto n^{5/3}$$

$$P \propto nT$$

.....

Answer

The equations of state for degenerate matter do not depend on temperature; they only depend on the particle density to some power. The equation of state for an ideal gas depends on particle density *and* temperature.

Question 4

A stellar core has a density of $6.9 \times 10^8 \text{ kg m}^{-3}$ and a composition comprising fully ionised helium-4, carbon-12 and oxygen-16 with relative amounts $X_{\text{He}} = 0.70$, $X_{\text{C}} = 0.15$ and $X_{\text{O}} = 0.15$. What is the Fermi kinetic energy of the electrons as a percentage of the electron rest-mass energy, assuming them to be non-relativistic?

- ☐ 13%
- ☐ 18%
- ☐ 25%
- ☐ 37%
- ☐ 52%

Answer

Since helium-4, carbon-12 and oxygen-16 all have $Y_e = 0.5$ electrons per nucleon, the electron number density is simply

$$n_e = \frac{Y_e \rho}{m_H} = \frac{0.5 \times 6.9 \times 10^8 \text{ kg m}^{-3}}{1.67 \times 10^{-27} \text{ kg}} = 2.07 \times 10^{35} \text{ m}^{-3}$$

The Fermi kinetic energy can be written as

$$E_F = \left(\frac{3n_e}{8\pi} \right)^{2/3} \times \frac{h^2}{2m_e}$$

Putting in the numbers gives

$$E_F = \left(\frac{3 \times 2.07 \times 10^{35} \text{ m}^{-3}}{8\pi} \right)^{2/3} \times \frac{(6.63 \times 10^{-34} \text{ J s})^2}{2 \times 9.11 \times 10^{-31} \text{ kg}}$$

Therefore $E_F = 2.05 \times 10^{-14} \text{ J}$ which is about 128 keV. Since the rest-mass energy of an electron is 511 keV, this means that E_F is about 25% of the rest mass energy. (So the assumption that the electrons are non-relativistic may not be appropriate.)

Question 5

According to Nauenberg's formula (Equation 19), what is the radius of a white dwarf (in Earth radii) whose mass is $1.0 M_{\odot}$? (The radius of the Earth is $R_{\oplus} = 6370 \text{ km}$.)

- ☐ $0.25 R_{\oplus}$
- ☐ $0.36 R_{\oplus}$
- ☐ $0.58 R_{\oplus}$
- ☐ $0.82 R_{\oplus}$
- ☐ $0.93 R_{\oplus}$

Answer

The radius is given by

$$R_{\text{WD}} \approx 7.8 \times 10^6 \text{ m} \times \left[\left(\frac{1.4 M_{\odot}}{M_{\text{WD}}} \right)^{2/3} - \left(\frac{M_{\text{WD}}}{1.4 M_{\odot}} \right)^{2/3} \right]^{1/2}.$$

Putting in the numbers gives

$$R_{\text{WD}} \approx 7.8 \times 10^6 \text{ m} \times \left[\left(\frac{1.4}{1.0} \right)^{2/3} - \left(\frac{1.0}{1.4} \right)^{2/3} \right]^{1/2}.$$

So $R_{\text{WD}} = 5.25 \times 10^6 \text{ m}$ which is equivalent to $0.82 R_{\oplus}$.

Question 6

Match the following compact remnants with the progenitor single star main-sequence mass.

He white dwarf

CO white dwarf

neutron star

Match each of the items above to an item below.

$M_{\text{MS}} = 0.15 M_{\odot}$

$M_{\text{MS}} = 1.5 M_{\odot}$

$M_{\text{MS}} = 15 M_{\odot}$

Answer

Main-sequence stars with masses below $0.5 M_{\odot}$ will form He white dwarfs, those with masses in the range $0.5\text{--}8 M_{\odot}$ will form CO white dwarfs, and those with masses in the range $11\text{--}16 M_{\odot}$ (or possibly up to $25 M_{\odot}$) will form neutron stars.

Question 7

The core of a massive star on the verge of collapse has a mass of $1.6 M_{\odot}$ and is comprised of 20% helium-4 nuclei and 80% iron-56 nuclei, by mass. Assume (somewhat artificially) that all of the helium in the core undergoes photodisintegration, absorbing 28.3 MeV for each nucleus disintegrated; all of the iron in the core experiences electron capture, releasing a neutrino with energy 10 MeV for each neutronisation.

How much energy in total would be removed from the collapsing core by the combined effects of photodisintegration and neutronisation?

You may assume $m_{\text{He}} = 6.6447 \times 10^{-27} \text{ kg}$ and $m_{\text{Fe}} = 9.3480 \times 10^{-26} \text{ kg}$. A nucleus of helium-4 contains two protons and two neutrons; a nucleus of iron-56 contains 26 protons and 30 neutrons.

- ☐ $1.4 \times 10^{45} \text{ J}$
- ☐ $1.6 \times 10^{45} \text{ J}$
- ☐ $1.8 \times 10^{45} \text{ J}$
- ☐ $1.4 \times 10^{46} \text{ J}$
- ☐ $1.6 \times 10^{46} \text{ J}$
- ☐ $1.8 \times 10^{46} \text{ J}$

Answer

First consider the helium nuclei. The number of helium-4 nuclei in the core is

$$N_{\text{He}} = \frac{(20/100) \times 1.6 \times 1.99 \times 10^{30} \text{ kg}}{6.6447 \times 10^{-27} \text{ kg}} = 9.58 \times 10^{55}.$$

The total energy absorbed by the helium-4 nuclei is therefore

$$E_{\text{He}} = 9.58 \times 10^{55} \times 28.3 \text{ MeV} \times 1.60 \times 10^{-13} \text{ J MeV}^{-1} = 4.34 \times 10^{44} \text{ J}$$

Now consider the iron nuclei. The number of iron-56 nuclei in the core is

$$N_{\text{Fe}} = \frac{(80/100) \times 1.6 \times 1.99 \times 10^{30} \text{ kg}}{9.3480 \times 10^{-26} \text{ kg}} = 2.72 \times 10^{55}.$$

Note that each nucleus will contain 26 protons. Now assuming that each proton is converted into a neutron by electron capture, emitting a neutrino with energy 10 MeV, the energy removed by electron capture is

$$E_{\text{Fe}} = 2.72 \times 10^{55} \times 26 \times 10 \text{ MeV} \times 1.60 \times 10^{-13} \text{ J MeV}^{-1} = 1.13 \times 10^{45} \text{ J}$$

So the total energy removed is $E = E_{\text{He}} + E_{\text{Fe}} = 1.6 \times 10^{45} \text{ J}$

Question 8

Which of the following statements about white dwarfs and neutron stars are *true*?

- ☐ White dwarfs are always less massive than neutron stars.
- ☐ White dwarfs are smaller than neutron stars.
- ☐ White dwarfs are more dense than neutron stars.
- ☐ White dwarfs are supported by electron degeneracy pressure.
- ☐ Neutron stars are supported by proton degeneracy pressure.
- ☐ ONeMg white dwarfs are more common than CO white dwarfs.
- ☐ Neutron stars can have masses up to 3 times that of the Sun.

.....

Answer

Only the fourth statement is true. The most massive white dwarfs can be more massive than the least massive neutron stars; white dwarfs are always larger than neutron stars and always less dense too. Neutron stars are supported by neutron degeneracy pressure because neutrons far outnumber protons in their composition. CO white dwarfs are more common than ONeMg white dwarfs because they form from lower mass progenitor stars which are more common than higher mass progenitor stars. The maximum neutron star mass is no more than about 2.5 times the mass of the Sun.

Conclusion

1. Stars spend most of their lives fusing hydrogen into helium in the cores, sitting on the **main sequence** of the **Hertzsprung–Russell diagram**. When hydrogen in the core is depleted, low-mass ($M_{\text{MS}} < 3 M_{\odot}$) and intermediate-mass ($M_{\text{MS}} = 3\text{--}8 M_{\odot}$) stars evolve into **red giants** and undergo helium fusion, producing carbon and oxygen in their cores. Massive stars ($M_{\text{MS}} > 8 M_{\odot}$) undergo further fusion reactions, evolving as **supergiant stars**, and those with the highest masses ($M_{\text{MS}} > 11 M_{\odot}$) can eventually develop an iron core.
2. As asymptotic giant branch (AGB) stars become very luminous, a strong stellar wind removes most of the hydrogen envelope. The underlying helium-rich star gets smaller and hotter, developing a fast, radiation-driven wind. Bipolar outflows and other asymmetries are shaped by orbiting planetary- or stellar-mass companions of the AGB star. Once the central star's surface temperature reaches $T \sim 10^4$ K, it ionises the ejected envelope, which is then seen as an expanding **planetary nebula**. The star's H- and He-burning shells are extinguished, leaving a degenerate CO core with $T \sim 10^5$ K, which subsequently cools and fades to produce a **white dwarf**.
3. Degeneracy can be described in three equivalent ways:
 - (i) when the separation of particles is less than the **de Broglie wavelength** for their momentum, $l < \lambda_{\text{dB}} = h/p = h/(3mk_{\text{B}}T)^{1/2}$ (Equation 3).
 - (ii) when the number of particles per unit volume is higher than the number of available quantum states at their energy, given by the **quantum concentration**, $n > n_{\text{Q}} = (2\pi mk_{\text{B}}T/h^2)^{3/2}$ (Equation 4).
 - (iii) when the temperature of the particles is less than the limiting value given by $T < h^2 n^{2/3} / (2\pi mk_{\text{B}})$ (Equation 5).
4. Electrons, protons and neutrons are **fermions**. The **Pauli exclusion principle** dictates that no more than one identical fermion can occupy a given quantum state. So, at most two fermions (with spins $+1/2$ and $-1/2$, respectively) can occupy the same quantum state. In a cold electron gas, the energy of the most energetic degenerate electron is called the **Fermi energy**. The Fermi kinetic energy is $E_{\text{F}} = p_{\text{F}}^2 / 2m_{\text{e}}$ (Equation 6), and its momentum $p_{\text{F}} = (3n_{\text{e}}/8\pi)^{1/3} h$ (Equation 8) is called the **Fermi momentum**.
5. Whereas the **equation of state** for an ideal gas, $P = nk_{\text{B}}T$, depends on the number density of particles n and the temperature T , for a degenerate gas it depends only on the number density: $P_{\text{NR}} = K_{\text{NR}} n_{\text{e}}^{5/3}$ (Equation 10) or $P_{\text{UR}} = K_{\text{UR}} n_{\text{e}}^{4/3}$ (Equation 12), where K_{NR} and K_{UR} are constants for non-relativistic and ultra-relativistic conditions, respectively. This decoupling of pressure and temperature in a degenerate gas disables thermostatic regulation. An ideal gas responds to an increased temperature by increasing pressure and hence expanding and cooling slightly; in a degenerate gas, a thermonuclear runaway can develop.
6. The number density n of some type of particle can be written in terms of its mass fraction X , mass m and the gas density ρ , as $n = \rho X / m$. Hence, for electrons, $n_{\text{e}} = \rho Y_{\text{e}} / m_{\text{H}}$ (Equation 13), where Y_{e} is the **number of electrons per nucleon**. For pure hydrogen, $Y_{\text{e}} = 1$, whereas for helium-4, carbon-12 and oxygen-16 (and hence for all types of white dwarf), $Y_{\text{e}} = 0.5$.

7. White dwarfs are supported against further collapse by **electron degeneracy pressure**. When degenerate electrons provide the pressure support of a star, the Fermi energy, E_F , may be expressed in terms of the mass. For non-relativistic degenerate electrons, $E_F \propto M^{4/3}$ (Equation 17), so degenerate electrons become more relativistic in more-massive white dwarfs. For ultra-relativistic degenerate electrons, the core density approaches infinity as the mass increases towards the **Chandrasekhar limit**, $\sim 1.4 M_\odot$.
8. For non-relativistic degenerate electrons, a white dwarf mass-radius relationship can be derived: $R_{WD} \approx (R_\odot/74)(M_{WD}/M_\odot)^{-1/3}$ (Equation 18). This is comparable to the radius of the Earth and implies that more-massive white dwarfs have *smaller* radii.
9. Different types (He, CO and ONeMg) of white dwarf (WD) are the end points of low- and intermediate-mass main-sequence (MS) stars of different masses. They fade and cool along lines of constant radius in the H–R diagram.

White dwarfs

$M_{MS} / M_\odot$	Final core fusion	$M_{WD} / M_\odot$	WD composition
< 0.5	H burning	< 0.4	He white dwarf
0.5–8	He burning	0.4–1.2	CO white dwarf
8–11	C burning	1.2–1.4	ONeMg white dwarf

10. Stars with $M_{MS} > 11 M_\odot$ form an iron core supported by ultra-relativistic degenerate electrons. When the Chandrasekhar limit is reached, the electrons can no longer support the star. Nuclear **photodisintegration** by thermal photons and **electron capture** by nuclear protons (**neutronisation**) remove energy so efficiently that they send the core into free fall. The total amount of energy lost in a few seconds is comparable to the energy previously liberated via nuclear burning over the star's entire main-sequence lifetime!

(i) Photodisintegration of iron proceeds as $\gamma + {}^{56}\text{Fe} \rightarrow 13 {}^4\text{He} + 4\text{n}$, and then

$\gamma + {}^4\text{He} \rightarrow 2\text{p} + 2\text{n}$. This can break down about 75% of the iron core,

followed by 50% of the helium, and thus can absorb of order $\sim 10^{45}$ J of energy.

(ii) Neutronisation is the conversion of nuclear protons into neutrons via electron capture: $e^- + \text{p} \rightarrow \text{n} + \nu_e$; the neutrinos produced also carry away of order $\sim 10^{45}$ J of energy.

11. The collapse of a stellar core halts when the density reaches that of nuclear matter, and then rebounds. This sends a shock wave through the rest of the star that partially reverses the collapse, leading to the ejection of the outer layers as a **supernova**. Its typical luminous energy is $\sim 10^{42}$ J, and the kinetic energy of the ejecta is $\sim 10^{44}$ J. The gravitational binding energy released ($\sim 5 \times 10^{46}$ J) is 10 times more than the energy required to photodisintegrate the iron core or that is lost by neutronisation; it is also 100 times more than the kinetic energy of the ejecta. Thus most of the

liberated energy is probably lost in an intense burst of neutrinos from the resulting **neutron star**.

12. When the density of a neutron star's core reaches $\sim 4 \times 10^{14} \text{ kg m}^{-3}$, neutrons drip from the nuclei and once the density exceeds that of normal nuclear matter, $\sim 2 \times 10^{17} \text{ kg m}^{-3}$, nuclei merge into a dense gas of electrons, protons and neutrons. Neutron stars survive because the beta decay of free neutrons is blocked by the Pauli exclusion principle and so neutrons greatly outnumber protons, typically with $n_n \sim 200 n_p$. They are supported against further collapse by **neutron degeneracy pressure**.
13. Neutron stars have masses in the range $M_{\text{NS}} \sim 1.2\text{--}2.2 M_{\odot}$ and radii $R_{\text{NS}} \sim 10\text{--}15 \text{ km}$. Their maximum mass – the **Tolman–Oppenheimer–Volkoff (TOV) limit** – corresponds to the neutrons becoming ultra-relativistic, but is difficult to calculate because of neutron interactions and the need to use general relativity. They form from single stars with main-sequence masses $M_{\text{MS}} \sim 11\text{--}16 M_{\odot}$.
14. Single stars with $M_{\text{MS}} \sim 16\text{--}25 M_{\odot}$ may form neutron stars too or they may form low-mass ($M_{\text{BH}} \sim 5\text{--}8 M_{\odot}$) **black holes** instead, without a supernova. An apparent dearth of compact objects between the most massive neutron stars and the least massive black holes is referred to as the **mass gap**. Stars with initial masses $M_{\text{MS}} > 25 M_{\odot}$ may also form neutron stars if they are in a binary system, because mass transfer between the two stars can significantly influence their evolution. Otherwise, such massive stars will form black holes with masses $M_{\text{BH}} > 10 M_{\odot}$ and Schwarzschild radii (which mark the event horizon of the black hole) of $R_{\text{Sch}} \sim 3 \text{ km}$ per solar mass.

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Glossary

asymptotic giant branch

(AGB) The region of the **Hertzsprung–Russell diagram** occupied by stars that are powered by a mixture of shell-hydrogen burning and shell-helium-burning, with a carbon and oxygen core. Shell-helium burning causes the outer envelope of the star to expand and cool further, so that stars lie above and to the right of the red giant stars in the Hertzsprung–Russell diagram in this phase.

black hole

In the general theory of relativity, an object whose density is so high that nothing that comes within a certain distance of it, crossing a boundary called the event horizon, can subsequently emerge. This includes light (photons) and other massless particles, giving the black hole its name.

brown dwarf

Objects similar to stars, but with a temperature too low for hydrogen burning to begin. Brown dwarfs contract from protostars in the same way as stars, but their low mass means that electron degeneracy sets in before the temperature required for thermonuclear reactions is attained.

Chandrasekhar limit

The theoretical upper limit to the mass of a white dwarf, about $1.4 M_{\odot}$, also called the Chandrasekhar mass. Contrast with **Tolman–Oppenheimer–Volkoff (TOV) limit**.

Chandrasekhar mass

The maximum stellar mass that can be supported by electron degeneracy pressure: hence an upper limit on the mass of a white dwarf. The limit arises when the electrons at the centre of the star become fully relativistic. The value of the Chandrasekhar mass depends on the model used for the internal structure of the white dwarf and on its composition, but is around $1.4 M_{\odot}$.

chemical potential

A quantity used to describe the energy that each particle in an ensemble brings to the total.

CO white dwarf

A white dwarf composed chiefly of carbon and oxygen, formed from the core of a star whose initial mass was between about $0.5 M_{\odot}$ and $8 M_{\odot}$. They will have masses from $0.5 M_{\odot}$ to $1.2 M_{\odot}$.

de Broglie wavelength

According to quantum mechanics, any force-free particle can be described by a sinusoidal de Broglie wave. The wavelength of the wave is known as the de Broglie wavelength λ_{dB} and is related to the particle's momentum p by $\lambda_{dB} = h/p$ where h is the Planck constant.

degeneracy

The phenomenon whereby more than one quantum state is associated with a particular energy level in a given system. Any energy level that corresponds to more than one quantum state is said to be degenerate.

degeneracy pressure

A pressure that arises in degenerate matter as a result of the Pauli exclusion principle, which forbids two fermions in a system from occupying the same quantum state. This means that a degenerate system always has a non-zero kinetic energy density and therefore a non-zero pressure, independent of its temperature.

degenerate matter

Matter that is so dense that the laws of quantum mechanics must be used to describe the behaviour of the particles that it consists of. The critical density for degeneracy is the quantum concentration: if the density exceeds this value then the system will be degenerate.

electron

A type of elementary particle with charge -1.602×10^{-19} C, mass 9.109×10^{-31} kg (or about $0.511 \text{ MeV } c^{-2}$) and spin $1/2$. The electron is a stable lepton. As far as is known, it has no internal structure, and is therefore regarded as a truly fundamental particle. Electrons are constituents of all atoms.

electron capture

A form of beta decay in which a nucleus absorbs one of its own electrons, causing a proton to become a neutron and a neutrino to be emitted.

electron degeneracy

A condition in which the quantum nature of electrons cannot be ignored. If electrons are packed so densely that their separations are comparable to their de Broglie wavelength, then the Pauli exclusion principle prohibits the overlapping of electrons with the same energy.

electron degeneracy pressure

Pressure due to degenerate electrons.

equation of state

An equation that relates the variables of pressure, volume and temperature for a macroscopic system in an equilibrium state. (1) For an ideal gas, the equation of state is $PV = Nk_B T$ or $P = nk_B T$, where k_B is the Boltzmann constant. (2) For non-

relativistic degenerate electrons, $P_{NR} = K_{NR} n_e^{5/3}$ where $K_{NR} = \frac{h^2}{5m_e} \left(\frac{3}{8\pi} \right)^{2/3}$. (3)

For ultra-relativistic degenerate electrons, $P_{UR} = K_{UR} n_e^{4/3}$

where $K_{UR} = \frac{hc}{4} \left(\frac{3}{8\pi} \right)^{1/3}$.

Fermi energy

The energy of the highest occupied state is called the Fermi energy, E_F . A degenerate gas may be regarded as very cold, the available energy levels being much higher than those corresponding to the temperature of the material. For this reason, the Fermi energy is useful in considering the state of degenerate matter in the cores of low-mass giant stars or in compact stellar remnants.

Fermi kinetic energy

The kinetic energy $p_F^2/(2m)$ (in the non-relativistic case) of a particle whose momentum is the Fermi momentum p_F .

Fermi momentum

The momentum magnitude p_F of a particle whose energy is the Fermi energy E_F

where $E_F = p_F^2/(2m)$ (in the non-relativistic case).

fermion

A subatomic particle with spin $\pm 1/2$. Electrons, protons and neutrons are all fermions.

free-fall time

The time it would take for a body to collapse under the influence of gravity alone.

gravitational potential energy

The potential energy of a particle or body that arises from its interaction with other particles or bodies via the (conservative) gravitational force. The gravitational potential energy of a test mass m , placed at a point with gravitational potential Φ due to a central

body of mass M , is given by $E_g = m\Phi = -GMm/r$, where M is the mass of the central body.

helium burning

A nuclear fusion process in which helium nuclei react with one another to produce carbon, via the triple-alpha process. The process is initiated at temperatures of around 10^8 K, and can, for a time, be the dominant means of energy production in the cores of post-main-sequence stars.

helium flash

A runaway thermonuclear reaction that takes place in the centres of low-mass ($< 2.25 M_{\odot}$) stars when they initiate helium burning. Because the electrons in the centres of these stars are degenerate when helium burning starts, and because the electron pressure in the centre is dominant, the cores do not respond to the increased energy supply by expanding.

Hertzsprung–Russell diagram

(H-R diagram) A graph showing the relationship between temperature and luminosity for a group of stars.

He white dwarf

A helium white dwarf would form from a main-sequence star whose mass is less than $0.5 M_{\odot}$. Such a star will have a main-sequence lifetime of order 100 billion years, hence no such stars have yet evolved to become white dwarfs.

hydrogen burning

A nuclear fusion process in which hydrogen nuclei react with one another (possibly with the involvement of other nuclei) to produce helium. The process occurs in the cores of main-sequence stars, where it is the dominant form of energy production. It can also be responsible for shell burning in post-main-sequence stars.

hydrostatic equilibrium

A situation in which the forces acting on a fluid (normally gravitational forces) are balanced by the internal pressure of the fluid (including thermal, degeneracy and radiation pressure), so that the fluid neither collapses nor expands.

main sequence

A well-defined region in the Hertzsprung–Russell diagram, showing a positive correlation between temperature and luminosity, in which stars in their (core) hydrogen-burning phase can be found. Protostars evolve onto the main sequence and spend most of their lifetime as stars there until core hydrogen burning ceases and they evolve towards the red giant branch.

mass fraction

The number fraction weighted by the mass of each type of particle. The mass fraction X of a particular type of particle in a sample may be expressed as $X = mn/\rho$, where m is the mass per particle, n is the number density, and ρ is the overall mass density of the sample.

mass gap

The apparent absence of compact objects between the upper mass limit for neutron stars and the observed lower mass limit for black holes.

neutrino

A spin 1/2 uncharged lepton of very small mass. Three different types of neutrino are known: electron neutrino, muon neutrino and tau neutrino.

neutron

An uncharged elementary particle with mass 1.675×10^{-27} kg (about 0.1% greater than that of the proton), and spin $1/2$. Neutrons are baryons, composed of up and down quarks, and (when free) are unstable, with a mean lifetime of about 15 minutes.

neutron degeneracy

A condition in which the quantum nature of neutrons cannot be ignored. If neutrons are packed so densely that their separations are comparable to their de Broglie wavelength, then the Pauli exclusion principle prohibits the overlapping of neutrons with the same energy.

neutron degeneracy pressure

Pressure due to degenerate neutrons.

neutron drip

Under conditions of extremely high density ($\sim 10^{14}$ kg m⁻³), the most stable nuclei are those with large neutron excesses compared to normal. At still higher densities, ($\sim 4 \times 10^{14}$ kg m⁻³), free neutrons ‘drip’ from the nuclei, giving rise to a sea of nuclei, electrons and free neutrons.

neutronisation

The fusion of an electron with a proton in a nucleus to produce a neutron, thus reducing the atomic number Z of the nucleus by 1 and increasing the neutron number N by 1. Neutronisation occurs in the high density of compact stellar remnants, and produces neutron-rich isotopes prior to neutron drip occurring.

neutron star

A stellar-mass object supported against gravitational collapse by the degeneracy pressure of neutrons. A neutron star is expected to form in the final stages of the life of a massive star; at extremely high densities and pressures it becomes energetically favourable for protons and electrons to merge (inverse beta decay).

nucleon

A term used to mean either a proton or a neutron.

number density

The number of particles per unit volume, usually represented by the symbol n . The SI unit of number density is m⁻³.

number of electrons per nucleon

The quantity Y_e used to define the electron number density as $n_e = \rho Y_e / m_H$. For pure hydrogen it is equal to 1, whereas for helium-4, carbon-12 and oxygen-16 it is equal to 0.5.

ONeMg white dwarf

A white dwarf composed chiefly of oxygen, neon and magnesium, formed from the core of a star whose initial mass was between about $8 M_\odot$ and $11 M_\odot$. ONeMg white dwarfs will have masses in the range about 1.2 – $1.4 M_\odot$.

Pauli exclusion principle

A principle asserting that no two fermions can occupy the same quantum state.

photodisintegration

The breakup of a nucleus into small parts due to the absorption of a photon – typically a gamma ray – whose energy exceeds the binding energy of the compound nucleus.

planetary nebula

A shell of material with a mass of a few tenths of a solar mass that has been ejected from a star in the late red giant phase of its evolution. The gas is photoionised by the remaining hot material of the star (possibly by now a white dwarf). The majority of planetary nebulae are asymmetric, with a small fraction being largely spherical or elliptical.

polytropic stellar model

A model for the structure of a star in which the pressure and density are related by $P \propto \rho^\gamma$ everywhere in the star, where γ is the adiabatic index.

proton

A type of elementary particle found in the nucleus of every atom. Protons carry a positive charge $+e = 1.602 \times 10^{-19}$ C and have a mass of

1.673×10^{-27} kg = 938.3 MeV c^{-2} . The mass of a proton is about 0.1% less than the mass of a neutron and the two particles have similar sizes (about 10^{-15} m).

quantum concentration

The maximum number density of particles in some material if it is to remain non-degenerate. That is, if the actual number density $n > n_Q$, degeneracy will have set in.

Quantum concentration is defined as $n_{Q,NR} = (2\pi m k_B T / h^2)^{3/2}$ for non-relativistic particles, and $n_{Q,UR} = 8\pi(k_B T / hc)^3$ for ultra-relativistic particles.

red giant

A star which has ceased to burn hydrogen in its core and started shell hydrogen burning. Shell hydrogen burning causes the outer layers of the star to expand, making the star more luminous but reducing its surface temperature. Consequently, red giants occupy a region of the Hertzsprung–Russell diagram above and to the right of their original position on the main sequence.

red giant branch

(RGB) The region of the Hertzsprung–Russell diagram in which red giant stars are found.

Schwarzschild radius

The characteristic radius ($R_S = 2GM/c^2$) in the general relativistic treatment of an isolated non-rotating spherically symmetric body of mass M . It is about $3 \text{ km } M_\odot^{-1}$. A body all of whose mass lies within its Schwarzschild radius is a black hole.

spin

An intrinsic form of angular momentum carried by elementary particles and also by composite particles such as hadrons, nuclei and atoms. Electrons, protons and neutrons are fermions and each have spin of either $+1/2$ or $-1/2$.

supergiant star

An evolved massive star that has left the main sequence. Supergiants have a high luminosity and a range of temperatures, appearing as cool red supergiants and hot blue supergiants.

supergiant branch

The region of the Hertzsprung–Russell diagram in which supergiant stars are found.

supernova

A star whose luminosity temporarily increases to very high values before fading rapidly on a timescale of weeks to months. Energy is emitted in the form of light but also as a burst of neutrinos and as kinetic energy of ejected material. There are several ways in which this can happen.

Tolman–Oppenheimer–Volkoff (TOV) limit

The theoretical upper limit to the mass of a neutron star, about $2.2 M_{\odot}$. Contrast with Chandrasekhar limit.

white dwarf

A stellar-mass compact object, with a mass below the Chandrasekhar mass ($1.4 M_{\odot}$) supported against gravitational collapse by the degeneracy pressure of electrons. White dwarfs are the final products of the evolution of low-mass stars, after thermonuclear reactions have ceased and the outer regions of the star have been lost in stellar winds or as a planetary nebula.