Senior secondary

Maths: Study units 1-6

Scholar study workbook

Keeping Girls in School scholarship programme
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‘Keeping Girls in School’ Scholarship Programme

MSCE Resources: 2014–15

Contents

MSCE M1:
Numeracy and probability

What you are studying and why

Subject: Mathematics Unit M1

At the end of this unit you should be able to:

1. practise problem solving and understand concept mapping
2. revise and understand more about proportion and progressions in numeracy
3. revise and understand the meaning of terms used in probability
4. construct a probability tree diagram and use this to calculate probability of single and multiple events

Introduction

Welcome to the first maths unit. This is the first step to studying maths for your MSCE and to practise problem solving. We hope you will develop a good knowledge of maths for yourself and also for your work in a primary school. As you grow in confidence and knowledge in maths then you will be in good position to help others by showing how enthusiastic and interested you are about maths.

At the end of this unit is a glossary. This consists of a list of important mathematical words and their meanings. You may like to make a note of these as you progress through the unit.

Activity 1

Looking ahead to maths topics

Work with a fellow Scholar, or your Tutor to look ahead to the maths exam papers and try to:

- find out the topics needed on an MSCE maths paper (you will not need to study all of the topics in the same detail but you do need to know what to expect)

You may also find it useful to ask your Tutor for a copy of the specification for the MSCE. This document lists all the topics that could come up in the examinations.

All topics within mathematics are interrelated so a particular topic could be tested in more than one type of question.

Proportion

We say that two quantities are in **direct** proportion if as one increases the other also increases at the same rate. Here is an example.

The cost of loaves of bread depends on the number you buy. The more loaves you buy the more you pay.

The ratio of the number of loaves to the cost is the same

$$\frac{\text{number of loaves}}{\text{cost of loaves}} = \frac{1 \text{ loaf}}{100 \text{ kwacha}} = \frac{2 \text{ loaves}}{200 \text{ kwacha}} = \frac{4 \text{ loaves}}{400 \text{ kwacha}}$$

The number of loaves is directly proportional to the cost and there is a symbol that is used for 'is proportional to' so we do not have to write it out each time. The symbol is $\propto$ and we write

"the number of loaves $\propto$ the price paid".

If the number of loaves is n and the price is p for each loaf then $n \propto p$ and in our example $n/p = 1/3$ or $n = kp$ (where k is a constant).

Two quantities are in **inverse** proportion if as one increases the other decreases at the same rate. Here is an example.

The faster a cyclist pedals the shorter time it takes to reach a certain place.

You will remember that Distance = Speed × Time and so if the Distance stays the same then Speed × Time = constant. As the speed increases the time decreases and so speed and time are in inverse proportion. We can write $S \times T = k$ giving $S = k\ 1/T$ or $S \propto 1/T$.

Activity 2

Try to think of examples of both direct and inverse proportion for yourself before you go on to answer the MSCE questions below.

The questions in the MSCE papers are usually fairly clear and simple like question 1.) below. Sometimes there is a question where there is more than one quantity, perhaps one direct proportion and another inverse proportion like question 2.) below. Look carefully at the answer given. You will see that you just combine the different proportions.

You may also be asked a question like this:

"A variable y obeys the equation $y = kx + c$, where k and c are constants."

In this question you would be given values so that you obtain two simultaneous equations. You will revise these in the next unit.

MSCE questions

1. **Paper I 2008**. Given that $q \propto \sqrt{p}$ and $p = 4$ when $q = 3$ find the value of p when $q = 15$.
2. **Paper II 2008**. x varies directly as y and inversely as the square of n. If $x = 15$, $y = 24$ and $n = 4$, calculate the value of n when $x = 8$ and $y = 20$.

Do try these questions yourself before you look at our solutions below.

1. $q \propto \sqrt{p}$ so $q = k\sqrt{p}$ (where k is a constant)

   Substituting p = 4 and q = 3 gives: $3 = k\sqrt{4}$

   $k = 3/2$

   When q = 15: $15 = 3/2\ \sqrt{p}$

   $\sqrt{p} = 10$

   $p = 100$

2. $x \propto \dfrac{y}{n^2}$ so $x = k \times \dfrac{y}{n^2}$ (where k is a constant)

   Substituting x = 15, y = 24 and n = 4 gives: $15 = k \times \dfrac{24}{4^2}$

   $15 \times \dfrac{16}{24} = k$

   $k = 10$

When x = 8 and y = 20: $8 = 10 \times \frac{20}{n^2}$

$n^2 = 25$

$n = 5$

Arithmetic and geometric progressions (APs and GPs)

An **Arithmetic progression** (AP) is formed by repeatedly adding the same number to each term. This number is called the common difference (d). The first term is a.

a, a + d, a + 2d, a + 3d, a + 4d, and so on.

Note carefully that the nth term is

$a + (n - 1)\,d$.

You often have to find the common difference when answering questions about APs.

You may want to make a note of the property of the sum of *n* terms which is

$n/2(\,2a + (n - 1)d)$.

Check in your textbook to see how to prove that this expression always gives you the sum of an arithmetic progression to *n* terms. When you have checked you have understood the proof think about how you could remember it.

MSCE question

1. **Paper I 2009.** The 9^{th} term of an arithmetic progression, *y*, *y* + 4, *y* + 8, ... is 3.

   Find the value of *y*.

   Common difference (d) = 4

   First term (a) = y

   9th term = y + 8d

   So: $y + 8 \times 4 = 3$

   $y = -32 + 3 = -29$

A **Geometric Progression** (GP) progresses by multiplying each term by the same number, called the common ratio, each time. If the first term is *a* and the common ratio is *r* then the GP is a, ar, ar^2, ar^3, ar^4, and so on. The nth term would be ar^{n-1}.

The sum of *n* terms of a GP is $\frac{a(r^n - 1)}{n - 1}$

Do check that you know how to prove this by looking up your textbook, and again look at how you can remember this expression.

MSCE question

Paper II 2008. The second term of a geometric progression is $^-6$ and the fourth term is $^-54$. Calculate the common ratio given that it is negative.

2^{nd} term = ar^1 So, $ar = -6$

4^{th} term = ar^3 So, $ar^3 = -54$

Dividing:

$$\frac{ar^2}{ar} = \frac{^-54}{^-6} \Rightarrow r^2 = 9, \text{ so } r = -3 \text{ (we are told that r is negative)}$$

Probability

A chance experiment is one in which the result is not certain. The result of such a chance experiment is called an **outcome**.

Consider tossing a coin many times and checking the results. At each toss there is an outcome of this experiment: either a head or a tail comes up.

An event is any collection of outcomes. In this experiment, one event is the collection of both outcomes i.e. a head or a tail coming up.

There are many experiments you can try using coins, dice, playing cards, coloured balls or even just stones or pieces of paper marked with letters or numbers. When carrying out chance experiments you should make a table of the results as you go along. Use tally marks so it is easy to count. In the tossing coin experiment draw a two-column table and mark how many heads and how many tails come up like this.

Heads	Tails
~~IIII~~, ~~IIII~~, ~~IIII~~, ~~IIII~~, III	~~IIII~~, ~~IIII~~, ~~IIII~~, ~~IIII~~,III

In the experiment of tossing the coin many times shown in this table you will see that the number of outcomes for each of the two events, a head or a tail, are much the same or in mathematical language the events are **equally likely**.

Probability is the likelihood of an event occurring.

The experimental probability = $\dfrac{\text{number of outcomes of an event}}{\text{the total number of outcomes}}$

The more times that an experiment is carried out, the closer the experimental possibility becomes to the theoretical probability.

When there are several equally likely outcomes of an event we define

the theoretical **probability** of an event = $\dfrac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$

Mathematically for an event A this can be written as

$$P(A) = \frac{\text{number of outcomes of A}}{\text{total number of outcomes}}$$

Theoretically, the probability of getting a head is ½ and the probability of getting a tail is also ½.

We write this as P(H) = ½ and P(T) = ½

Probability is a fraction and it is always less than 1.

You can use a number line from 0 to 1 to show probability.

If something is very likely to happen it would be placed near 1 on the number line and if something is very unlikely to happen it would be placed near 0.

The probability that it will rain in July in Malawi would be near 0 on the number line but the probability that it will rain in January would be placed near the 1.

Activity 3

Draw a number line marked 0 at one end and 1 at the other. Write very likely in the correct place and also add unlikely. Now mark on the chance of getting a head when tossing a coin and add marks for rain in July and the sun rising. Try and add three more examples of different probabilities you can think of.

Here are three other questions for you to try yourself:

1. What is the probability of getting a six when you throw a dice?
2. What are the probabilities of picking the letter L from the word MALAWI and of picking the letter A?
3. What is the probability of picking a king from a pack of 52 cards?

Check you understood and got these answers:

1. P(six) = 1/6
2. P(L) = 1/6 and P(A) = 2/6 = 1/3
3. P(king) = 4/52 or 1/13

Probability notation

If we consider an event A then A′ is the event that A does not happen. A′ will be the collection of outcomes which do not belong to A. A′ is called the **complement** of A.

Since it is certain that either A happens or it does not happen P(A) + P(A′) = 1.

For example P(rain) + P(no rain) = 1

Here is another example taken from a past MSCE paper.

If in a certain experiment the probability of an event A happening is 0.3 what is the probability that A does not happen?

The answer is 0.7 as P(A′)= 1 – P(A)

MSCE question

Paper II 2008. The probability of a bus arriving early at a depot is 1/10 and arriving late is 3/5. If 400 buses are expected at the depot during the day, calculate the number of buses that are likely to arrive in time.

Do try this yourself before you look at the answer below.

ANS: The bus can arrive early, late or on time. The probability of it arriving on time is

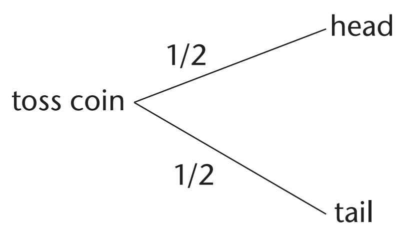
1 – (time arriving late) – (time arriving early) = 1 – 1/10 – 3/5 = 3/10

If there are 400 buses then 3/10 of these bases arrive on time which is 120 buses.

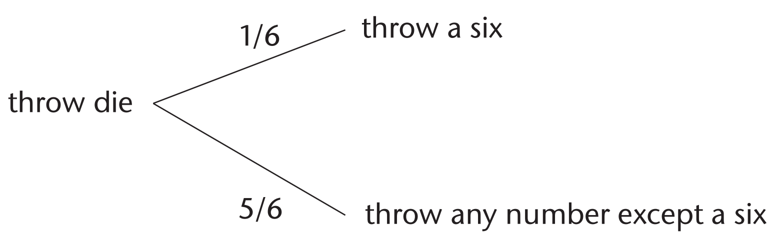
Tree diagrams

Tree diagrams can help you to calculate probabilities more easily.

For the coin each probability is ½ and the diagram looks like this:



For the die the diagram looks like this:



Try this for yourself by drawing a tree diagram for finding the probability of picking the letters L and A from the word MALAWI.

These diagrams are called **tree diagrams** because they look like the branches of a tree and they are used to help to understand and to solve problems in probability.

In each of these examples only one thing happened. We say that there was a **single event** – tossing the coin to get a head, throwing the die to get a six and picking a card to get a king. Now let's think about going a bit further and asking what happens when there is more than one happening and there are many or **multiple events.**

A coin is tossed twice. What are the outcomes and what is the probability of getting:

(a) a head each time?

(b) at least one head?

There are four outcomes when a coin is tossed twice and they are **equally likely.** They are:

head, head (HH)

head, tail (HT)

tail, head (TH)

tail, tail (TT)

Since they are all equally likely each has probability ¼. This can be written mathematically as:

P(HH) = 1/4

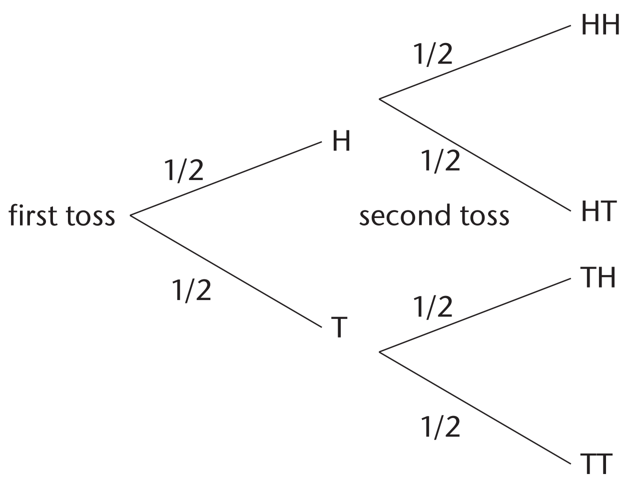
P(HT) = 1/4

P(TH) = 1/4

P(TT) = 1/4

Here is a tree diagram which shows this clearly.

Since at each toss the probability of getting a head or a tail is ½ at the second toss the probability of each event is ½ × ½ = ¼.



Answers:

(a) The probability of getting a head each time is ½ × ½ = ¼ i.e. P(HH) = ¼

(b) There are three ways of getting at least one head: HH; HT; TH and each has probability ¼ and so the probability of getting at least one head is

P(HH) + P(HT) + P(TH) = ¼ + ¼ + ¼ = 3/4

Here is another example.

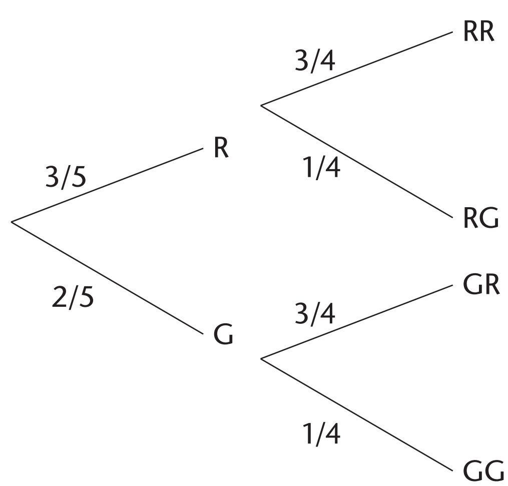
If a bag has three red balls and 2 green balls in it and you pick a ball out and then another without replacing the first what is the probability that you pick:

a) two green balls?

b) at least one red ball?

On the first pick P(red) = 3/5 and P(green) = 2/5. At the second pick there are still 3 red balls but only 1 green ball left and only 4 balls left altogether since there is no replacement.

Here is the tree diagram:



a) The probability of getting 2 green balls is $2/5 \times 1/4 = 2/20 = 1/10$

b) The probability of getting at least one red ball can be calculated in two ways. It is either:

(i) the sum of the probabilities of R,R + R,G + G,R = $3/5 \times 3/4 + 3/5 \times 1/4 + 2/5 \times 3/4 =$
$9/20 + 3/20 + 6/20 = 18/20 = 9/10$

or (ii) the complement of GG P(GG') which is $1 - 1/10 = 9/10$ (because the only event with no red ball is GG).

Draw a tree diagram showing the probabilities of what happens when a die is thrown twice.

Use this to find the probability of:

a) throwing a six twice

b) throwing a six followed by an odd number.

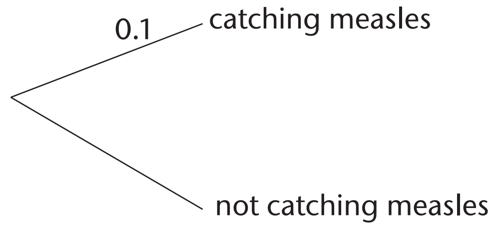
Sometimes, with multiple events, the outcome of one event affects the outcome of the other. This is known as **conditional probability**. An example of this would be taking a counter out of a bag, not replacing it and then taking another counter out of the bag. Or, given a set of 10 cards, taking a card out and then taking out another without replacing the first card. The first probability here would be calculated 'out of 10' but the second probability would be calculated 'out of 9'.

Tree diagrams are always used for conditional probabilities. Look out for these by reading any 'tree diagram question' very carefully.

Practice questions

(2 hours)

1. This tree diagram shows the number of students who catch measles over some unspecified time interval according to the Government statistics.



   a) What is P(not catching measles)?

   b) In a school of 800 students how many can be expected to get measles?

2. A box contains 3 red balls and 2 white balls. If two balls are picked without replacement, use the tree diagram to find the probability that:

   a) 1 ball is red

   b) both balls are white

   c) at least one ball is white.

3. Given that d α t. If d = 20 when t = 2½, find the:
   (a) relationship between d and t.
   (b) value of d when t = 3

4. The table below shows the results of an experiment in a Physical Science practical lesson.

Length of wire (cm)	15		45	60
Resistance (ohms)	5	10	15	

Find the:
   (a) constant of proportionality
   (b) resistance when the length of the wire is 60 cm
   (c) length when the resistance is 10 ohms.

5. x varies directly as y and inversely as z. x = 9 when y = 3 and z = 6. Find:
   (a) the relationship between x, y and z
   (b) z when x = 36 and y = 8.
6. The nth term of an AP is 12 – 4n. Find the common difference.
7. Write down the next two terms in an AP: 3, 6, 9, ... , ...
8. The nth term of an AP is $3 \times 2^{n-1}$. Find the 5^{th} term.
9. A number is drawn at random from the set: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Find the probability that the number will be:
   (a) prime
   (b) < 5
   (c) prime and < 5
   (d) prime or < 5.
10. The probability of a day being hot is 2/5. The probability of a person wearing a jersey when the day is hot is 1/7 and the probability of a person wearing a jersey when the day is cold is 5/7.

    Draw a tree diagram to represent the information above and use it to calculate the probability of a person wearing a jersey.
11. The probability of a family giving birth to a child is 0.5. If there are 3 children in a family, calculate the probability that:
   (a) they are all the same sex
   (b) there are two boys and one girl
   (c) the first born is a girl and the last born is a boy.

How am I doing?

This section is a study tool.

Now look back over this unit and be honest about what was difficult.

Later use it to discuss with your tutor any extra help you need.

Before the exam use this tool to revise.

	☺ **Easy** (Tick this box if you feel confident that you **understand this section well**)	**Fine** (Tick this box if you still **need a little work** on this section)	☹ **Difficult** (Tick this box if you still **need a lot of work** on this section)
Proportion			
Progression			
Probability			
Tree diagrams			

Notes on what to do next:

Signed (by Scholar): .... Date: ....

Signed (by Tutor): .... Date: ....

Answers to practice questions

1. (a) P(not catching measles) = 1 - P(catching measles)

   = 1 - 0.1

   = 0.9

   (b) 0.1 x 800 = 80 students

2. Initially: P(red) = 3/5 and P(white).

   (a) P(1 red) = P(RW) + P(WR)

   = (3/5 x 2/4) + (2/5 x 3/4)

   = 3/10 + 3/10

   = 6/10 = 3/5

   (b) P(WW) = 2/5 x 1/4 = 1/10

   (c) P(at least one ball is white) = 1 - P(no whites)

   = 1 - P(RR)

   = 1 - (3/5 x 2/4)

   = 1 - 3/10 = 7/10

3. (a) $d \propto t$ so $d = kt$ (where k is a constant)

   Substituting d = 20 and t = 2½:

   $20 = k \times 2\frac{1}{2}$

   $k = \dfrac{20}{2\frac{1}{2}} = 8$

   So: $d = 8t$

   (b) When t = 3d = 8 × 3 = 24

4. (a) From the table:

   length of wire (l) = 15 when resistance (r) = 5 and also (l) = 45 when (r) = 15.

   This implies that $(l) \propto (r)$

   l = k r (i.e. they are in direct proportion)

   k = 3

   (b) When l = 60, 60 = 3r, r = 20 ohms

   (c) When r = 10, l = 3 × 10 = 30 cm

5. (a) $x \propto \frac{y}{z}$ so $x = k \times \frac{y}{z}$ (where k is a constant)

Substituting x = 9, y = 3 and z = 6:

$9 = k \times \frac{3}{6}$

k = 18

$\Rightarrow x = \frac{18y}{z}$

(b) When x = 36 and y = 8

$36 = \frac{18 \text{ x } 8}{z}$

36z = 144

z = 4

6. nth term = 12 – 4n

Substituting the 1st term: 12 – 4 = 8

Substituting the 2nd term: 12 – 8 = 4

This means that the common difference (d) = - 4

7. 12 and 15 (common difference is 3).

8. nth term = $3 \times 2^{n-1}$. 5th term = $3 \times 2^4 = 3 \times 16 = 48$

9. (a) Prime numbers are numbers which have two factors only, one and the number itself. From this set, the four prime numbers are 2, 3, 5 and 7.

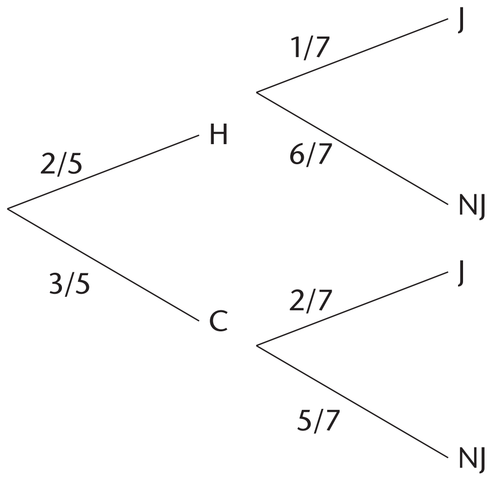
So P (prime) = 4/10 = 2/5.

(b) P (< 5) = 4/10 = 2/5 because there are 4 out of 10 numbers which are less than 5.

(c) P (prime and < 5) = 2/5 × 2/5 = 4/25.

(d) P (prime or < 5) = 2/5 + 2/5 = 4/5.

10.



KEY:

H = Hot

C = Cold

J = Jersey

NJ = No jersey

P (wearing Jersey) = 2/5 × 1/7 + 3/5 × 5/7 which simplifies to 17/35

11. The probability of a boy or a girl is always 0.5 so there is no need to draw a tree diagram.

    (a) P (BBB or GGG) = $0.5^3 + 0.5^3 = 0.250$

    (b) P (BBG) + P (BGB) + P (GBB) = $0.5^3 + 0.5^3 + 0.5^3 = 0.375$

    (c) P (GGB) + P (GBB) = $0.5^3 + 0.5^3 = 0.250$

Glossary

Arithmetic progression (AP)	3,5,7,9 ... is an example of an arithmetic progression. As you move from one term to the next term, you are always adding on the same amount (+2).
Common difference (d)	The constant amount that you add on when moving from one term to the next. The nth term of an AP = a + (n - 1)d.
Common ratio (r)	The constant amount that you multiply by when moving from one term to the next. The nth term of a GP = ar^{n-1}.
Complement	The total opposite of something, in probability. For example, if A is the event happening, then the complement of A, written A’, is the event not happening. P(A) + P(A’) = 1
Conditional probability	When the outcome of one event affects the outcome of another.
Direct proportion	Two variables are in direct proportion if one is a constant multiple of the other. If one variable is multiplied by 3, then the other variable is also multiplied by 3. This means that y = 3x. In general terms, $y \propto x$ and y = kx (where k is the constant of proportionality).
Equally likely	When something has the same chance of happening as not happening, for example, getting a head or a tail when tossing a coin once.
Event	Something that you want to find the probability of. If you want to find ‘the probability of getting a tail when tossing a coin once’, the event is ‘getting a tail’.
Geometric Progression (GP)	3, 9, 27, 81 ... is an example of a geometric progression. As you move from one term to the next term, you are always multiplying by the same amount (3).
Inverse proportion	Two variables are in inverse proportion if as one variable increases, the other variable decreases inversely. If one variable is multiplied by 2, then the other variable is divided by 2. This means that y = 1/2 x. In general terms $y \propto 1/x$ and y = k/x (where k is the constant of proportionality).
Multiple events	When two independent events happen at the same time or successively.
Outcome	One of the possible results of an experiment. When tossing a coin, there are two possible outcomes: heads or tails.
Probability	An indication of how likely something is to happen. This can be described in words such as ‘certain’, ‘fairly likely’ or ‘impossible’. It can also be calculated in numerical form.
Tree diagram	A diagram used to calculate the probabilities of multiple or successive events.

MSCE M2:
Basic algebra and logarithms

What you are studying and why

Subject: Mathematics Unit M2

This unit is about basic algebra and logarithms. You are going to practise problem solving in one area of algebra.

At the end of this unit you should be able to:

1. calculate using indices (powers) and surds
2. state the rules of logarithms and use them in calculations
3. solve logarithmic equations.

Introduction

Algebra is about relationships between numbers. The good news is that when numbers change we can guess or predict what will happen especially when we know the rules.

This topic is important for other reasons than the exam. For example, in biology and other sciences, you need to know about numbers changing, about problem solving and estimating and checking solutions.

Before we begin the work on logarithms we must recap on what we know about indices or powers. The reason for this is that logarithms and powers are not really different. We are also going to recap on what you have learned before on surds. There are always questions on powers, surds and logarithms in the MSCE.

Powers

In the expression 2^5, two is referred to as the **base** and the small number 5 is the **index** or **power**.

$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$,

$(-3)^4 = -3 \times -3 \times -3 \times -3 = 81$

$5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

$16^{\frac{1}{2}} = \sqrt{16} = 4$

Here are some examples of powers for you to calculate:

1. 4^{-2} 2. $(\frac{3}{2})^3$ 3. $(-\frac{1}{4})^{-2}$ 4. $27^{\frac{1}{3}}$ 5. 23^0

You also learned the following three rules of indices:

1st rule: $a^m \times a^n = a^{m+n}$ An example is $2^3 \times 2^4 = 2^{3+4} = 2^7$

2nd rule: $a^m \div a^n = a^{m-n}$ An example is $3^6 \div 3^2 = 3^{6-2} = 3^4$

3rd rule: $(a^m)^n = a^{mn}$ An example is $(5^3)^2 = 5^{3\times 2} = 5^6$

You also learned that:

- $a^{-m} = \frac{1}{a^m}$
- $\sqrt[m]{a} = a^{\frac{1}{m}}$
- $a^0 = 1$
- $a^1 = a$

If you have forgotten these it is easy to check that they are true. Here is the first rule:

$a^m \times a^n = a \times a \times a \times \ldots$ (m times) $a \times a \times a \times \ldots$ (n times) $= a \times a \times a \times \ldots$ ($m + n$ times) $= a^{m+n}$

Try checking the second and third in a similar way by cancelling out as in the second and counting the as in the third.

Now we can work on more complicated examples using powers, simplifying expressions and solving equations.

Examples

1. $(10^3)^2 = 10^6 = 1\,000\,000$ (Using the third rule of indices)

Alternatively you can write

$(10^3)^2 = (10 \times 10 \times 10)^2 = (10 \times 10 \times 10) \times (10 \times 10 \times 10) = 1000 \times 1000 = 1\,000\,000$

2. $2^3 \times 4^{-2} = 2 \times 2 \times 2 \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{2}$
3. $16^{-\frac{5}{4}} = (16^{\frac{1}{4}})^{-5} = 2^{-5} = \frac{1}{32}$
4. $12a^3 \times 3a^{-2} = 36a^{3-2} = 36a$
5. Solve the equation $3^x \times 3^{x+1} = 9$

   $3^x \times 3^{x+1} = 9$

   $3^x \times 3^{x+1} = 3^2$

   So, simplifying $3^{(2x+1)} = 3^2$

   Equating indices gives: $2x + 1 = 2$

   $2x = 1$

   $x = \frac{1}{2}$

6. What is the power to which 5 must be raised to give 125?

The answer is 3 because $5 \times 5 \times 5 = 5^3 = 125$

Here are some for you to try. We have given the answers at the end of this unit.

Activity 1

1. Calculate or simplify the following:
   a) 2^{-5}
   b) $(-2)^{-3}$
   c) $16a^{\frac{3}{4}} \times 4a^{\frac{1}{4}}$
   d) $(16y)^{\frac{3}{4}} \times 4y^{\frac{1}{4}}$

2. Are the following statements true or false?
   a) $a^3 \times a^{\frac{1}{3}} = a$
   b) $x^0 = y^0$
   c) If $z = 16$ then $3z^{\frac{1}{2}} = 12$

3. Solve the equation $3^x = \frac{1}{27}$
4. What is the power to which 2 must be raised to give 32?

Surds

The roots of some numbers can be calculated to give rational answers.

A **rational number** is one that can be written as a fraction where both the numerator and the denominator are whole numbers.

Examples of these are $\sqrt{100} = 10$, $\sqrt{\frac{1}{4}} = \frac{1}{2}$ and $\sqrt[3]{27} = 3$.

Those roots which cannot be rewritten as rational numbers are called **surds**. Numbers like $\sqrt{2}$, $\sqrt{3}$ or $\sqrt[3]{20}$ cannot and they are called **irrational numbers**.

The majority of questions on surds at MSCE level involve fractions where the denominator contains a single square root or an expression containing square roots such as $\frac{2}{\sqrt{5}}$ or $\frac{4}{1 + \sqrt{5}}$.

Surds can be simplified by rationalising the denominator.

To simplify $\frac{\sqrt{500}}{\sqrt{5}}$ you need to multiply the numerator and the denominator by $\sqrt{5}$. This does not change the value of the expression because $\frac{\sqrt{5}}{\sqrt{5}}$ is equivalent to 1.

This gives $\frac{\sqrt{500} \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{\sqrt{2500}}{5} = \frac{50}{5} = 10$

Here are some more difficult examples on surds taken from past MSCE papers.

1. **2009 Paper 1.** Simplify $\frac{11}{5 - \sqrt{3}}$ leaving your answer with a rational denominator.
2. **2008 Paper 2.** Simplify $\frac{15\sqrt{6}}{\sqrt{98} - \sqrt{2}}$ to its simplest form.

There is a trick to simplify a surd like this which you will probably remember.

We use the algebraic identity called **the difference of two squares**

$$a^2 - b^2 = (a - b)(a + b)$$

We multiply both the numerator and the denominator of the surd by the factor which gives the squares when it is multiplied by the denominator.

We will revise this for you by giving the answers to these two questions.

Consider the first question. We do not want $5 - \sqrt{3}$ on the denominator.

However, if we multiply both the numerator and the denominator of the fraction by

$5 + \sqrt{3}$ we get $\frac{11(5+\sqrt{3})}{(5-\sqrt{3})(5+\sqrt{3})} = \frac{11(5+\sqrt{3})}{5^2-(\sqrt{3})^2} = \frac{11(5+\sqrt{3})}{25-3} = \frac{11(5+\sqrt{3})}{22} = \frac{5+\sqrt{3}}{2}$

Let us now consider the second question. First we will get rid of the square roots in the denominator by multiplying the numerator and denominator by $\sqrt{98} + \sqrt{2}$.

This will give us

$$\frac{15\sqrt{6}(\sqrt{98}+\sqrt{2})}{(\sqrt{98}-\sqrt{2})(\sqrt{98}+\sqrt{2})} = \frac{15\sqrt{6}\sqrt{98}+15\sqrt{6}\sqrt{2}}{(\sqrt{98})^2-(\sqrt{2})^2} = \frac{(15\sqrt{2}\sqrt{3}\sqrt{98})+(15\sqrt{2}\sqrt{3}\sqrt{2})}{98-2}$$

$$= \frac{(15\sqrt{2}\sqrt{3}\sqrt{2}\sqrt{7}\sqrt{7})+(15\sqrt{2}\sqrt{3}\sqrt{2})}{96} = \frac{(15 \times 2 \times 7 \times \sqrt{3})+(15 \times 2\sqrt{3})}{96} = \frac{210\sqrt{3}+30\sqrt{3}}{96}$$

$$= \frac{240\sqrt{3}}{96} = \frac{5\sqrt{3}}{2} \text{ (answer)}$$

Try the following activity. The answers are at the end of this unit.

Activity 2

1. Simplify a) $\frac{\sqrt{72}}{\sqrt{18}}$ b) $\frac{\sqrt{625}}{\sqrt{25}}$ c) $\frac{\sqrt{12} \times \sqrt{24}}{\sqrt{3} \times \sqrt{48}}$
2. Rationalise each of the following surds
   a) $\frac{1}{\sqrt{2}+\sqrt{3}}$ b) $\frac{\sqrt{3}}{2\sqrt{3}-1}$ c) $\frac{4\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$
3. Write $\frac{1}{2\sqrt{5}-\sqrt{7}}$ in the form $a\sqrt{b} + c\sqrt{d}$

Logarithms

In the question at the end of the section on powers and indices, you were asked 'What is the power to which 2 must be raised to give 32?' We think your answer was 5 because $2^5 = 32$.

Here are some similar questions for you to try.

What is the power to which 7 must be raised to give 49? The answer is 2 because $7^2 = 49$.

What is the power to which 3 must be raised to give 81? The answer is 4 because $3^4 = 81$.

Mathematicians call these powers **logarithms**. (Do you remember we said at the beginning that logarithms were really powers?)

If $2^5 = 32$, then 5 is the logarithm of 32, to the base 2. We write this as $5 = \log_2 32$.

If $7^2 = 49$, then 2 is the logarithm of 49, to the base 7. It is written as $2 = \log_7 49$.

If $3^4 = 81$, then 4 is the logarithm of 81, to base 3. It is written $4 = \log_3 81$.

It is important to realise that a logarithm is a power.

Logarithms began long ago when a mathematician called John Napier saw that by using powers and turning large numbers into powers he could multiply large numbers easily. Here is an example of what he did.

He took two large numbers like 416 and 1432 and changed them into powers. Then he used the first rule above to multiply them by adding the powers. Then he turned the powers back into whole numbers like this:

$416 \times 1432 = 10^{2.6191} \times 10^{3.1559} = 10^{2.6191+3.1559} = 10^{5.7750} = 595\ 712$

Of course it does not have to be base 10, any base will work but we have shown you what Napier did using base 10.

How did Napier know that $10^{2.6191}$ was equal to 416? He created a list of powers of ten and what they were equal to by a process of repeated approximation until he got a value that was close enough. For example, you know already that $10^2 = 100$ and $10^3 = 1000$. So the power to which 10 has to be raised to give 416 must be somewhere between 2 and 3. So, $416 = 4.16 \times 100 = 10^x \times 10^2 = 10^{2+x}$.

We know that the square root of ten, $10^{\frac{1}{2}} \approx 3.162$, so $10^2 \times 10^{\frac{1}{2}} = 10^{2\frac{1}{2}} = 10^{2.5}$ would give us $100 \times 3.162 = 316.2$, which is not quite enough, but it's closer.

By continuing like this, we can get a power of ten that gives us very nearly the exact answer of 416.

Napier was the first to call these powers logarithms. You may have seen log tables at the back of old textbooks. Now as we have handheld calculators, we no longer need Napier's tables of logarithms to calculate large numbers but we still use logarithms because many natural happenings grow or decline in a logarithmic way. For example, populations grow logarithmically until they run out of food or space. Our ears, which have to deal with sounds that can vary from being very faint to very loud, respond to the loudness in a logarithmic way (the signals they send to the brain change in proportion to the logarithm of the sound intensity). For that reason, the measure of the loudness of sounds is in the decibel scale, which is logarithmic.

Let us repeat what we have said.

If $416 = 10^{2.6191}$ then 2.6191 is the logarithm of 416 to the base 10 and we normally shorten logarithm to 'log' and place the base beside it like this and write $2.6191 = \log_{10}416$.

Writing this in mathematical language for any number we can define a logarithm as follows:

If $b = a^x$ then x is the logarithm of b to the base a.
If $a^x = b$ then $x = \log_a b$

This converts powers into logarithms.

Sometimes we want to work in the opposite direction and so we have:

If x is the logarithm of b to the base a then $b = a^x$.

If $x = \log_a b$ then $b = a^x$

This converts logarithms into powers.

You must remember these two definitions and we recommend that you write them down in your notebook. We have given you the explanation above so you remember that a logarithm is a power and so obeys the rules of powers.

We can change or convert from one form into the other.

Here is an example of changing from powers form to logarithmic form:

$9^2 = 81$ so $\log_9 81 = 2$

Here is an example of changing from logarithmic form to powers form:

$\log_{27} 9 =$ so $27\ = 9$

Examples:

As $1000 = 10^3$ then $3 = \log_{10} 1000$

As $36 = 6^2$ then $2 = \log_6 36$

As $32 = 2^5$ then $5 = \log_2 32$

The value of $\log_7 49$ is 2 as $7^2 = 49$

The value of $\log_5 625$ is 4 as $5^4 = 625$

Here is an exercise for you to try. The answers are at the end of the unit.

Activity 3

1. Write the following in logarithmic form:

   a) $10^6 = 1\,000\,000$ b) $32^{\frac{1}{5}} = 2$ c) $9^{-3} = \frac{1}{729}$

2. Write in power form:

   a) $\log_{10} 10 = 1$ b) $\log_5 25 = 2$ c) $\log_a c = b$ d) $\log_3 \frac{1}{3} = -1$

3. Find the value of the following expressions:

   a) $\log_2 16$ b) $\log_2 \frac{1}{16}$ c) $\log_5 5^{-5}$ d) $\log_3 \frac{1}{3}$ e) $\log_5 5$

The rules of logs are similar to the rules of powers. Here they are

1st rule: $\log_a xy = \log_a x + \log_a y$

2nd rule: $\log_a x/y = \log_a x - \log_a y$

3rd rule: $\log_a x^y = y\log_a x$

We will show you the proof of the first rule that $\log_a xy = \log_a x + \log_a y$.

Let $u = \log_a x$ and $v = \log_a y$ then by converting from logs into powers we have

$a^u = x$ and $a^v = y$

$xy = a^u \times a^v = a^{u+v}$ (the first rule for powers)

$u + v = \log_a xy$ by converting from powers into logs

but $u + v = \log_a x + \log_a y$ and so $\log_a xy = \log_a x + \log_a y$

Here is an example using the first rule:

$\log_{10} 12 = \log_{10} (3 \times 4) = \log_{10} 3 + \log_{10} 4$

We would like you to try to prove the second and third rules yourself in the same way.

There are five other important facts about logarithms which you should remember.

1. $\log_a a = 1$

Example: If $\log_2 2 = x$, then $2^x = 2$. So x must equal 1.

2. $\log_a 1 = 0$

Example: If $\log_{10} 1 = x$, then $10^x = 1$. So x must equal 0. (Anything to the power of 0 has a value of 1.)

3. $\log_a 0$ has no meaning

Example: If $\log_{10} 0 = x$, then $10^x = 0$. There are no values for x. The graph of a logarithm does not cross the y axis.

4. $\log_a b$ does not exist if $b < 0$ and $a > 0$

Example: If $\log_{10}(-100) = x$, then $10^x = -100$. There are no values for x. The graph of a logarithm with these conditions does not include negative values on the x axis.

5. $\log_1 a$ has no meaning

Example: If $\log_1 9 = x$, then $1^x = 9$. There are no values for x. Logarithms cannot be of base 1.

Using the rules and the other facts we have found we can simplify expressions and solve equations using logarithms.

Look at these examples and then try the following activity.

Examples

Read each through carefully. Then cover it up and try to do it on your own. When you are happy about this, try Activity 4.

1. Simplify the following

a) $2\log_2 a + 5\log_2 b - 3\log_2 c = \log_2 a^2 + \log_2 b^5 - \log_2 c^3 = \log_2 \frac{a^2b^5}{c^3}$

b) $\log_7 12 - \log_7 98 + \log_7 \frac{1}{42} = \log_7 \frac{12}{(98 \times 42)} = \log_7 \frac{1}{(49 \times 7)}$

$= \log_7 \frac{1}{7^3} = \log_7 7^{-3} = -3$

c) $\frac{\log_{10}1000}{\log_{10}100} = \frac{\log_{10}10^3}{\log_{10}10^2} = \frac{3\log_{10}10}{2\log_{10}10} = \frac{3}{2}$

2. Solve the following equations

a) Find x if $\log_x 5 + \log_x 2 = 1$

$\log_x(5 \times 2) = \log_x 10 = 1$ which gives $x^1 = 10$

Answer is $x = 10$.

b) Find x if $\log_5(q + 8) - \log_5 q = 1$

$\log_5 \frac{(q+8)}{q} = 1$

$\frac{(q+8)}{q} = 5^1 = 5$

$q + 8 = 5q$

$8 = 4q$

$q = 2$

c) Find p if $\log_2(p + 2) + \log_2(p - 2) = 5$

$\log_2((p + 2)(p - 2)) = 5$

$\log_2(p^2 - 4) = 5$

$2^5 = p^2 - 4$

$32 = p^2 - 4$

$p^2 = 36$

Answer is $p = 6$ or -6.

$p = 6$ gives $\log_2 8 + \log_2 4 = \log_2 8 \times 4 = \log_2 32 = 5$ which is true

$p = -6$ gives $\log_2 -4 + \log_2 -8$ which is meaningless because we cannot find the log of a negative number. The only solution is $p = 6$.

Here is an activity for you to try. The answers are at the end of the unit.

Activity 4

1. Find the value of the following using any method

   a) $\log_5 125$ b) $\log_3 729$ c) $\log_3 \frac{1}{9}$

2. Find the value of $\log_2 16 + \log_3 \frac{1}{81} + \log_6 36 - \log_5 \frac{1}{25}$

3. Solve for x

   a) $\log_5 x = 2$ b) $\log_4 x = 3$ c) $\log_{10} x = -3$

4. Expand the following into separate logarithms

   a) $\log_5 2uv$ b) $\log_5 \frac{2}{pq}$ c) $\log_5 pq^2$

5. Find p if $\frac{1}{3}\log_2 p = \log_2(3p + 2)^{\frac{1}{3}}$

6. Solve for x the equation $\log_5 625 = 6x + 1$

Note. It is important that you know the rules of logarithms and are aware of the mistakes often made. Here again are the rules.

$\log ab = \log a + \log b$

$\log a/b = \log a - \log b$

$\log a^b = b \log a$

Practice questions

(2 hours)

1. Given that $\log_{10}2 = 0.3010$, $\log_{10}3 = 0.4771$

   Find:

   a) $\log_{10}16$

   b) $\log_{10}5$

   c) $\log_{10}50$

   d) $2 + \log_{10}6$

2. Simplify:

   a) $\log_{10}64 - \log_{10}8$

   b) $\dfrac{\log 16}{\log 2}$

3. Solve the equation: $\log_{10}16 + \log_{10}y = 1$

4. Find the value of x if $\frac{1}{2}\log_3 x = \log_3(6 - x)^{\frac{1}{2}}$

5. Simplify:

   a) $\dfrac{\log 16}{\log 4}$

   b) $\log_6 4 + \log_6 9$.

6. Evaluate $\log_7 49$.

7. Solve the equation $\log_5 25^x = 3$.

How am I doing?

	☺ **Easy** (Tick this box if you feel confident that you **understand this section well**)	**Fine** (Tick this box if you still **need a little work** on this section)	☹ **Difficult** (Tick this box if you **still need a lot** of work on this section)
Calculate using indices (powers) and surds			
State the rules of logarithms and use them in calculations			
Solve logarithmic equations			

Notes on what to do next:

Signed (by Scholar): .... Date:..................

Signed (by Tutor): .... Date:..................

Answers

Activity 1

1. a) $2^{-5} = \frac{1}{32}$ b) $(-2)^{-3} = -\frac{1}{8}$ c) $16a^{\frac{3}{4}} \times 4a^{\frac{1}{4}} = 64a$

   d) $(16y)^{\frac{3}{4}} \times 4y^{\frac{1}{4}}$

   $= (\sqrt[4]{16})^3 \times 4 \times y^{\frac{3}{4}} \times y^{\frac{1}{4}}$

   $= 8 \times 4 \times y$

   $= 32y$

2. a) $a^3 \times a^{\frac{1}{3}} = a$ True

   b) $x^0 = y^0$ True (Anything to the power of 0 has a value of 1)

   c) $3z^{\frac{1}{2}} = 12$

   $z^{\frac{1}{2}} = 4$ (squaring both sides)

   $Z = 16$

   If $z = 16$ then $3z^{\frac{1}{2}} = 12$ True

3. $3^x = 1/27$ is the same as $3^x = 3^{-3}$ and so $x = -3$

4. 2 must be raised to the power of 5 because $32 = 2^5$

Activity 2

1. a) $\frac{\sqrt{72}}{\sqrt{18}} = 2$ b) $\frac{\sqrt{625}}{\sqrt{25}} = 5$ c) $\frac{\sqrt{12} \times \sqrt{24}}{\sqrt{3} \times \sqrt{48}} = \sqrt{2}$

2. a) $\frac{\sqrt{1}}{\sqrt{2} + \sqrt{3}} = \sqrt{3} - \sqrt{2}$ b) $\frac{\sqrt{3}}{2\sqrt{3} - 1} = \frac{6 + \sqrt{3}}{11}$ c) $\frac{4\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}} = \frac{23 - 5\sqrt{15}}{2}$

3. $\frac{1}{2\sqrt{5} - \sqrt{7}} = \frac{2\sqrt{5} + \sqrt{7}}{20 - 7} = \frac{2\sqrt{5} + \sqrt{7}}{13}$

Activity 3

1. $10^6 = 1{,}000{,}000$ can be written as $6 = \log_{10} 1{,}000{,}000$

   b) $32^{\frac{1}{5}} = 2$ as $\frac{1}{5} = \log_{32} 2$

   c) $9^{-3} = \frac{1}{729}$ as $-3 = \log_9 1/729$

   This can also be written as $3 = \log_9 729$ as $-3 = \log_9 1 - \log_9 729 = -\log_9 729$

2. a) $\log_{10} 10 = 1$ is $10^1 = 10$ b) $\log_5 25 = 2$ is $5^2 = 25$

   c) $\log_a c = b$ is $a^b = c$ d) $\log_3 \frac{1}{3} = -1$ is $3^{-1} = \frac{1}{3}$

3. a) $\log_2 16 = 4$ b) $\log_2 \frac{1}{16} = -4$ c) $\log_5 5^{-5} = -5$ d) $\log_3 \frac{1}{3} = -1$

   e) $\log_5 5 = 1$

Activity 4

1. a) $\log_5 125 = 3$ b) $\log_3 729 = 6$ c) $\log_3 \frac{1}{9} = -2$

2. $\log_2 16 + \log_3 \frac{1}{81} + \log_6 36 - \log_5 \frac{1}{25} = \log_2 2^4 + \log_3 3^4 + \log_6 6^2 - \log_5 5^2$

   $= 4 - 4 + 2 + 2 = 4$

3. a) $\log_5 x = 2$ gives $5^2 = x$; $x = 25$ b) $\log_4 x = 3$ gives $4^3 = x$; $x = 64$

   c) $\log_{10} x = -3$ gives $10^{-3} = x$; $x = \frac{1}{1000}$

4. a) $\log_5 2uv = \log_5 2 + \log_5 u + \log_5 v$ b) $\log_5 \frac{2}{pq} = \log_5 2 - \log_5 p - \log_5 q$

   c) $\log_9 pq^2 = \log_5 p + 2 \log_5 q$

5. $\frac{1}{3}\log_2 p = \log_2 (3p + 2)^{\frac{1}{3}}$

   $\log_2 p^{\frac{1}{3}} = \log_2 (3p + 2)^{\frac{1}{3}}$

   $= p = 3p + 2$ which gives $p = -1$

6. $\text{Log}_5 625 = 6x + 1$

   $5^{6x+1} = 625$

   $5^{6x+1} = 5^4$

   $= 6x + 1 = 4$, so $x = ½$

Answers to practice questions

1. a) $\log_{10} 16 = 4\log_{10} 2 = 4 \text{ x } 0.3010 = 1.204$

   b) $\log_{10} 5 = \log_{10} 10/2 = \log_{10} 10 - \log_{10} 2 = 1 - 0.3010 = 0.699$

   c) $\log_{10} 50 = \log_{10} 5 + \log_{10} 10 = 0.699 + 1 = 1.699$

   d) $2 + \log_{10} 6 = 2 + \log_{10} 2 + \log_{10} 3 = 2 + 0.3010 + 0.4771 = 2.7781$

2. a) $\log_{10} 64 - \log_{10} 8 = \log_{10} 64/8 = \log_{10} 8 = 3 \log_{10} 2 = 3 \text{ x } 0.3010 = 0.903$

   b) $\frac{\log 16}{\log 2} = \frac{4\log 2}{\log 2} = 4$

3. $\log_{10} 16 + \log_{10} y = 1$

   $\log_{10} 16y = \log_{10} 10$

   $= 16y = 10$ and $y = 5/8$

4. $\frac{1}{2}\log_3 x = \log_3 (6 - x)^{\frac{1}{2}}$

   $\log_3 x^{\frac{1}{2}} = \log_3 (6 - x)^{\frac{1}{2}}$

   $= x^{\frac{1}{2}} = (6 - x)^{\frac{1}{2}}$

   So $x = 6 - x$, which gives $x = 3$

5. a) $\frac{\log 16}{\log 4} = \frac{\log 4^2}{\log 4} = \frac{2\log 4}{\log 4} = 2$

   b) $\log_6 4 + \log_6 9 = \log_6 (4 \times 9) = \log_6 36 = \log_6 6^2 = 2\log_6 6 = 2(1) = 2$

   Remember: $\log_a a = 1$

6. $\log_7 49 = \log {}_7 7^2 = 2 \log_7 7 = 2\ (1) = 2$

7. $\log_5 25^x = 3$

$\log_5 5^{2x} = \log_5 5^3$

But $\log_5 5 = 1$

$=2x\ (1) = 3(1)$

$2x = 3$

$x = \frac{3}{2} = 1½$

or

$\log_5 25^x = 3$

$\log_5 5^{2x} = \log_5 5^3$

So, $5^{2x} = 5^3$

$=2x = 3$ (equating powers)

$x = 1½$

Glossary

Base	The base is a number that you are raising to a power. For example, in 2^3 the base is 2.
Index	The index or power tells you how many times a number is to be multiplied by itself. For example 2^3 means $2 \times 2 \times 2$. The index is 3, so three 2s are multiplied together.
Indices	The plural of 'index'.
Irrational number	A number which cannot be written as a fraction where both the numerator and the denominator are whole numbers
Logarithms	These are indices or powers. If $y = x^a$, then a is the logarithm of y to base x. This can also be written as $\log_x y = a$.
Rational number	A number that can be written as a fraction where both the numerator and the denominator are whole numbers.
Rationalising the denominator	Is the process used to remove surds from the denominator of a fraction.
Surd	A number containing an irrational root e.g. $\sqrt{5}$.

MSCE M3:
Algebra 2

What you are studying and why

Subject: Mathematics Unit M3

This is the second unit on algebra.

At the end of this unit you should be able to:

1. use common words used in algebra
2. factorise quadratic expressions
3. solve quadratic equations and simultaneous equations, one linear and one quadratic
4. use algebraic fractions in solving problems
5. change the subject of a formula.

Introduction

This unit develops your basic algebra. Algebra is used to solve problems in real life situations. It is useful not just for your Mathematics examination but also in other subjects, for example in science, agriculture, forestry and building.

Common words used in algebra

At the back of this unit is a glossary of common words. Take some time to read through these words and their meanings. You will meet them throughout this unit.

Factorising

When two or more brackets are multiplied together a single expression is formed. Factorising is the opposite or inverse of this 'multiplying out'. It is when an expression is factorised or rewritten back into two or more brackets that are to be multiplied together.

Example

Factorise $x^2 - 5x$

$x^2 - 5x = x(x - 5)$

x is a common factor and divides into x^2 and $5x$ exactly.

In the last unit, you learnt how to factorise expressions which are the differences of two squares.

Examples

Factorise the following:

1. $x^2 - y^2$

   $x^2 - y^2 = (x - y)(x + y)$

2. $4a^2 - 9b^2$

   $4a^2 - 9b^2 = (2a)^2 - (3b)^2 = (2a - 3b)(2a + 3b)$

Most quadratic expressions consist of three terms, like $x^2 - 5x + 6$.

A general quadratic expression in x, is written in the form $ax^2 + bx + c$, where a, b and c are constants.

If you are asked to factorise a quadratic expression then your task is to find ***if*** there are two linear factors that multiply together to equal this quadratic expression and then to determine what they are.

Here is a hint. Remember how you multiplied the two factors $(x + 2)(x + 4)$. You would have written:

$(x + 2)(x + 4)$

$= x^2 + 2x + 4x + 8$

$= x^2 + 6x + 8$

When you factorise you will go backwards. To factorise examples such as $x^2 + 6x + 8$, first consider the middle term $6x$ and the end term 8 and then find two numbers that will add together to give 6 and multiply to give 8. These numbers are 2 and 4 and so

$x^2 + 6x + 8$

$= x^2 + 2x + 4x + 8$ split the middle term

$= x(x + 2) + 4(x + 2)$ take common factors out of each pair of terms

$= (x + 2)(x + 4)$ $(x + 2)$ is a common factor

Hint: It is worth noting that:

(i) if the constant term, c, is positive, its two factors have the same sign, which is the sign of the middle term, b.

(ii) if the constant term, c, has a negative sign, its two factors differ in signs. The sign of the bigger factor is the same as the sign of the middle term, b.

Examples

1. $x^2 + 5x + 6$

   $= x^2 + 2x + 3x + 6$

   $= x(x + 2) + 3(x + 2)$

   $= (x + 2)(x + 3)$

   (Here $2 + 3 = 5$ and $2 \times 3 = 6$.)

2. $x^2 – 6x + 8$

   $= x(x – 4) – 2(x – 4)$

   $= (x – 4)(x – 2)$

   (Here $–4 + (–2) = –6$ and $–2 \times –4 = 8$.)

3. $x^2 – x – 6$ (Here we must find two numbers that will add up to –1 and multiply to give –6. They are –3 and + 2.)

   So $x^2 – x – 6$

   $= x^2 – 3x + 2x – 6$

   $= x(x – 3) + 2(x – 3)$

   $= (x – 3)(x + 2)$

4. $2x^2 + 7x + 3$ (This is more difficult but is solved in the same way. We must find two numbers that add up to 7 and multiply to give $2 \times 3 = 6$. They are 1 and 6.)

$2x^2 + 7x + 3$

$= 2x^2 + x + 6x + 3$

$= x(2x + 1) + 3(2x + 1)$

$= (2x + 1)(x + 3)$

Activity 1

Here are some quadratic expressions for you to factorise yourself. The answers are at the end of the unit. Be very careful with the signs!

Factorise the following quadratic expressions:

1. $x^2 - 5x + 6$
2. $x^2 + 3x + 2$
3. $x^2 + 9x + 14$
4. $x^2 - 3x + 2$
5. $x^2 + 4x - 5$
6. $x^2 - 6x$
7. $3x^2 + x - 2$

(After you have worked through some examples you may find you can factorise without writing in the middle step.)

Quadratic equations

$x^2 - 5x + 6 = 0$ is a quadratic equation.

In the activity you found that

$x^2 - 5x + 6 = (x - 3)(x - 2)$

So $(x - 3)(x - 2) = 0$

If any two numbers are multiplied together to give 0, then one of these numbers must be 0.

So either $x - 3 = 0$ or $x - 2 = 0$

$x = 3$ or $x = 2$

You can always check your answers by substituting these values back into the expression $x^2 - 5x + 6$.

$3^2 - 5(3) + 6 = 9 - 15 + 6 = 0$ ✓

$2^2 - 5(2) + 6 = 4 - 10 + 6 = 0$ ✓

Example

Solve the equation $3x^2 + x - 2 = 0$

$$3x^2 + x - 2 = 0$$

$$(3x - 2)(x + 1) = 0$$

So either $3x - 2 = 0$ or $x + 1 = 0$

$x = \frac{2}{3}$ or $x = -1$

Check

$$3(\tfrac{2}{3})^2 + \tfrac{2}{3} - 2 = \tfrac{4}{3} + \tfrac{2}{3} - 2 = 0 \checkmark$$

$$3(-1)^2 + (-1) - 2 = 3 - 1 - 2 = 0 \checkmark$$

If the equation is not in the form $ax^2 + bx + c = 0$ you must first rearrange it to be in this form. For example if you were asked to solve $5x^2 = 3x + 2$ you should first write it as $5x^2 - 3x - 2 = 0$ and continue as before.

Activity 2

Solve the following equations:

1. $x^2 + 5x + 6 = 0$
2. $x^2 - 5x - 6 = 0$
3. $x^2 + 5x - 6 = 0$
4. $x^2 + 9x + 14 = 0$
5. $x^2 - 9 = 0$
6. $2x^2 + 5x = 3$

This method of solving quadratic equations only works if you can factorise and sadly many equations do not factorise easily. Other methods of factorising quadratics are 'completing the square' and 'using the formula'.

Completing the square

We are beginning this section by revising what is a perfect square and how to make an expression like $ax^2 + bx$ into a perfect square.

A perfect square is the quadratic expression formed from multiplying one linear expression by itself.

$x^2 + 2x + 1$ is a perfect square as it is equivalent to $(x + 1)(x + 1)$ or $(x + 1)^2$

$x^2 + 10x + 25$ is a perfect square as it is equivalent to $(x + 5)(x + 5)$ or $(x + 5)^2$

Check the two statements above by multiplying out the brackets.

In general

$x^2 + 2bx + b^2$ is a perfect square as it is equivalent to $(x + b)(x + b)$ or $(x + b)^2$

Examples

1. What must be added to $x^2 + 8x$ to make it a perfect square?

   Comparing $x^2 + 8x$ with $x^2 + 2bx + b^2$ means that $2b = 8$ so $b = 4$

   So $\quad x^2 + 8x + ? = (x + 4)^2$

   Therefore, you will need to add $4^2 = 16$

2. What must be added to $x^2 - 24x$ to make it a perfect square?

   Comparing $x^2 - 24x$ with $x^2 + 2bx + b^2$ means that $2b = -24$, so $b = -12$

   So $\quad x^2 - 24x + ? = (x - 12)^2$

   Therefore, you will need to add $(-12)^2 = 144$

Now use what you have just revised to solve quadratic equations when you cannot factorise them. BUT always try to factorise first if you can.

These examples will show you how to do this.

Examples

1. Solve $x^2 + 2x = 5$ by completing the square

   Take the left hand side of the equation and complete the square by adding 1 (half of 2, the coefficient of x, squared).

   $x^2 + 2x + 1 = 5 + 1$

   Notice that 1 must also be added to the right hand side of the equation to balance it out.

   The left hand side can now be written in bracket form and solved from here.

   $(x + 1)^2 = 6$

   $x + 1 = \sqrt{6}$ or $-\sqrt{6}$ Square root both sides

   $x = -1 + \sqrt{6}$ or $-1 - \sqrt{6}$

   You can go on and find the decimal answers if the question asks for them.

2. Solve $3x^2 - 4x - 3 = 0$ by completing the square

   $3x^2 - 4x - 3 = 0$ Dividing by 3

   $x^2 - \frac{4}{3}x - 1 = 0$ Add 1 to both sides

   $x^2 - \frac{4}{3}x = 1$

   $x^2 - \frac{4}{3}x + (\frac{2}{3})^2 = 1 + (\frac{2}{3})^2$ Completing the square by adding $(\frac{2}{3})^2$ (half of $-\frac{4}{3}$, the coefficient of x, squared)

   $(x - \frac{2}{3})^2 = 1 + \frac{4}{9}$

   $(x - \frac{2}{3})^2 = \frac{13}{9}$

   $x - \frac{2}{3} = +\sqrt{\frac{13}{3}}$ or $-\sqrt{\frac{13}{3}}$ Square root both sides

   $x = \frac{2}{3} + \sqrt{\frac{13}{3}}$ or $\frac{2}{3} - \sqrt{\frac{13}{3}}$ Add $(\frac{2}{3})$ to both sides

You can use completing the square to find the formula for solving quadratic equations.

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

Note: in this formula the numerator $-b \pm \sqrt{(b^2 - 4ac)}$ is **ALL** divided by $2a$.

Use it to solve the two equations above.

1. Solve $x^2 + 2x = 5$

   Expressing the equation in the form $ax^2 + bx + c = 0$, the equation becomes

   $x^2 + 2x - 5 = 0$

   Here a = 1, b = 2 and c = –5

   The formula gives the solutions as

   $$x = \frac{-2 \pm \sqrt{(4 + 20)}}{2}$$
   $$= \frac{-2 \pm \sqrt{24}}{2}$$
   $$= \frac{2 \pm 2\sqrt{6}}{2}$$
   $= -1 \pm \sqrt{6}$ as before.

2. Solve $3x^2 - 4x - 3 = 0$

   Here a = 3, b = –4 and c = –3

   Solution is $x = \frac{4 \pm \sqrt{(16 + 36)}}{6} = \frac{4 \pm \sqrt{52}}{6} = \frac{4 \pm 2\sqrt{13}}{6} = \frac{2 \pm \sqrt{13}}{3}$ as before.

Try this example now taken from the MSCE. You should always try to factorise first and if it does not factorise then you can choose whether you will complete the square or use the formula.

Activity 3

1. 2008 Paper 1 Question 10

   Solve the equation $3(a + 1)^2 - 3 = 0$

Look at our answer at the end of the unit and compare it with yours.

Simultaneous equations

You already know how to solve two linear simultaneous equations. Sometimes you are given one linear and one quadratic equation to solve. You use the same substitution method as before.

Here is an example taken from 2009 Paper 2 Question 7b.

Solve the simultaneous equations:

$$x - 3y - 1 = 0 \quad \text{(i)}$$
$$x^2 - 8y^2 - y + 5 = 0 \quad \text{(ii)}$$

Substituting $x = 3y + 1$ from (i) into (ii) you have

$$(3y + 1)^2 - 8y^2 - y + 5 = 0$$
$$9y^2 + 6y + 1 - 8y^2 - y + 5 = 0$$
$$y^2 + 5y + 6 = 0$$
$$(y + 2)(y + 3) = 0$$

So either $(y + 2) = 0$ or $(y + 3) = 0$
And so either $y = -2$ or $y = -3$

From equation (i) $x - 3y - 1 = 0$ gives $x = 3y + 1$.

When $y = -2$, $x = 3 \times -2 + 1 = -6 + 1 = -5$

When $y = -3$, $x = 3 \times -3 + 1 = -9 + 1 = -8$

$x = 3y + 1$ and so when $y = -2$, $x = -5$ and when $y = -3$, $x = -8$

Solution is

$x = -5$ and $y = -2$

$x = -8$ and $y = -3$

Check in (ii) for $x = -5$, $y = -2$ then $25 - 32 + 2 + 5 = 0$ correct

$x = -8$, $y = -3$ then $64 - 72 + 3 + 5 = 0$ correct

Try this one yourself.

Activity 4

Solve the simultaneous equations:

1. $x + y = 7$

   $x^2 + y^2 = 29$

2. $x + 2y = 2$

   $x^2 + 2xy = 8$

Look at our answers at the end of the unit and compare them with yours.

Fractions in algebra

This section will help you to revise not only fractions in algebra but also those in arithmetic.

Do you remember in arithmetic that you can cancel in an expression like this $\frac{2}{5} \times \frac{15}{8}$

Here you can cancel the 2 and the 8 and also cancel the 5 and the 15 to give $\frac{1}{1} \times \frac{3}{4} = \frac{3}{4}$

but not in this $\frac{2}{5} + \frac{15}{8}$?

In this case you must find the LCM of 5 and 8 which is 40 and continue like this $\frac{2 \times 8 + 15 \times 5}{40}$

$= \frac{16 + 75}{40} = \frac{91}{40}$

It is exactly the same in algebra. You must be very careful with signs.

Fractions in algebra are simplified in the same way as in arithmetic. You add, subtract, multiply and divide fractions exactly as you would in arithmetic.

Here are some examples to study. You will see that they have been simplified exactly as if the letters were numbers.

1. $\frac{p^2}{pq} = \frac{p}{q}$ 2. $\frac{6bc}{9cd} = \frac{2b}{3d}$ 3. $x - \frac{x}{3} = \frac{3x - x}{3} = \frac{2x}{3}$

4. $x \div \frac{x}{y} = x \times \frac{y}{x} = y$ 5. $a^3 \div \frac{1}{a^2} = a^3 \times \frac{a^2}{1} = a^5$ 6. $\frac{ab}{cd} \times \frac{cd^2}{ab^2} = \frac{d}{b}$

Now revise addition and subtraction, multiplication and division and then solve some equations containing fractions.

For addition and subtraction you must find the lowest common multiple (LCM) and simplify.

Examples

1. Simplify $\frac{a-b}{ab} + \frac{b-c}{bc} + \frac{c-a}{ca}$

In this example the LCM is abc.

$$\frac{a-b}{ab} + \frac{b-c}{bc} + \frac{c-a}{ca}$$
$$= \frac{c(a-b) + a(b-c) + b(c-a)}{abc}$$
$$= \frac{ca - bc + ab - ac + bc - ab}{abc}$$
$$= 0$$

2. $\frac{4}{x-2} - \frac{1}{x+3} - \frac{5}{x^2+x-6}$

With this example you must first factorise $x^2 + x - 6$

$x^2 + x - 6 = (x - 2)(x + 3)$. The LCM is $(x - 2)(x + 3)$

$$\frac{4}{x-2} - \frac{1}{x+3} - \frac{5}{x^2+x-6} = \frac{4(x+3) - 1(x-2) - 5}{(x-2)(x+3)} = \frac{4x + 12 - x + 2 - 5}{(x-2)(x+3)}$$
$$= \frac{3x+9}{(x-2)(x+3)} = \frac{3(x+3)}{(x-2)(x+3)} = \frac{3}{x-2}$$

Now some examples of multiplying and dividing.

For multiplication and division you must factorise whenever possible.

Examples

1. $\frac{1}{2}x \times \frac{x^2-1}{x^2-x} = \frac{1}{2}x \times \frac{(x-1)(x+1)}{x(x-1)} = \frac{1}{2}(x+1)$

2. $\frac{a^2-2a-3}{a^2-6a+9} \div (a+1) = \frac{(a-3)(a+1)}{(a-3)(a-3)} \times \frac{1}{(a+1)} = \frac{1}{a-3}$

Before we ask you to do an exercise on fractions in algebra you will work with examples solving equations with fractions.

Examples

1. Solve the equation $\frac{x}{x-2} - \frac{3}{x+1} = 1$

Multiply each term on both sides by $(x - 2)(x + 1)$

$x(x + 1) - 3(x - 2) = (x - 2)(x + 1)$

$x^2 + x - 3x + 6 = x^2 - x - 2$

$-x = -8$

$x = 8$ is the answer (Check by substituting $x = 8$ in the equation to give $\frac{8}{6} - \frac{3}{9} = \frac{4}{3} - \frac{1}{3} = 1$ which is correct)

2. Here is a problem you can ask your friend to solve.

   I think of a number. I square it and then subtract 6 from it. If I then divide it by 5 my answer is 2. What is the number I thought of?

   Let the number be x then $\frac{x^2 - 6}{5} = 2$

   To solve this equation multiply both sides by 5 then:
   $x^2 - 6 = 10$
   $x^2 = 16$
   $x = +4$ or -4

Check to make sure both of these are correct.

Here is an exercise for you to try on fractions using questions from past MSCE papers. The answers are at the end of the unit.

Activity 5

1. 2008 Paper 1 Question 3

   Simplify $\frac{x^2 - 1}{x} \times \frac{x^2}{x - 1}$

2. 2008 Paper 2 Question 2

   Simplify $\frac{1}{x} - \frac{1}{x + 2y} - 1$

Look at our answers at the end of the unit and compare them with yours.

Changing the subject of a formula

Formulae are very important not only in mathematics but also in other subjects like science.

Here are some formulae you will recognise:

$d = v \times t$ (distance = speed (velocity) × time)

$A = l \times b$ (Area of a rectangle)

$C = 2\pi r$ (circumference of a circle)

The subject is the symbol on its own and conventionally it is put on the left side.

The subject of $d = v \times t$ is d which you can find if you know the time and speed. But if you want to know the speed and know the distance and time you need the formula to be written in another way – you need to change the subject from d to v.

You may need to use BODMAS.

B Remove brackets
O Order means simplify squares and cubes
D Divide
M Multiply
A Add
S Subtract

You must also remember that if you add, subtract, multiply or divide one side of an equation you must also do the same to the other side.

Examples

1. 2008 Paper 1 Question 4

Make t the subject of the formula $M = K + \frac{3y^2}{t}$

Answer:

$$M = K + \frac{3y^2}{t}$$

$$\frac{K + 3y^2}{t} = M$$ Subtract K from both sides to leave the t term on its own on the left hand side.

$$\frac{3y^2}{t} = M - K$$ Multiply both sides by t

$$3y^2 = t(M - K)$$ Divide both sides by the bracket (M – K)

$$\frac{3y^2 - K}{M} = t$$

So

$$t = \frac{3y^2}{M - K}$$

2. 2009 Paper 2 Question 4a

Given that $xy^3 = 5 - 2y^3$ make y the subject of the formula

Answer:

Step 1: Take all terms in y to the LHS.

$$xy^3 + 2y^3 = 5$$

Step 2: Factorise the LHS.

$$y^3(x + 2) = 5$$

Step 3: Divide both sides by $(x + 2)$.

$$y^3 = \frac{5}{(x + 2)}$$

Step 4: Take the cube root of both sides.

Answer $y = {}^3\sqrt{\left(\frac{5}{(x + 2)}\right)}$

Here are some for you to try. The answers are given at the end of the unit.

Activity 6

1. 2009 Paper 1 Question 18

   Make r the subject of the formula $P = m(\frac{r}{x})^3$.

2. Interpret the formula $A = \pi r^2$ and make r the subject.

3. If temperature in °F (Fahrenheit) is to be converted into °C (Celsius) then $F = 32 + \frac{9}{5}C$.

   Make C the subject of the formula.

   Find the value of C when i) F = 32 and ii) F = 212.

Look at our answers at the end of the unit and compare them with yours.

Practice questions

(2 hours)

1. Factorise the following:

   a) $x^2 - 2x - 35$

   b) $2x^2 - 5x + 2$.

2. Solve the following quadratic equation $4x^2 - 25 = 0$.

3. 2009 Paper 2 Question 11a

   Solve the equation $3x^2 - 7x - 2 = 0$ giving your answer to two significant figures.

4. Solve the simultaneous equations

   $4x - y = 7$

   $xy = 15$

5. 2009 Paper 1 Question 16

   Express $\frac{1}{x-1} + \frac{3y}{xy-y}$ as a single fraction.

6. Make A the subject of the formula $\sqrt{(2A)} = 4b$.

How am I doing?

	☺ **Easy** (Tick this box if you feel confident that you **understand this section well**)	**Fine** (Tick this box if you still **need a little work** on this section)	☹ **Difficult** (Tick this box if you still **need a lot of work** on this section)
Factorising quadratic expressions			
Solving quadratic equations			
Solving simultaneous equations			
Using fractions in algebra			
Changing the subject of a formula			

Notes on what to do next:

Signed (by Scholar): .... Date:..................

Signed (by Tutor): .... Date:..................

Answers

Activity 1

1. $x^2 - 5x + 6 = (x - 2)(x - 3)$
2. $x^2 + 3x + 2 = (x + 1)(x + 2)$
3. $x^2 + 9x + 14 = (x + 7)(x + 2)$
4. $x^2 - 3x + 2 = (x - 1)(x - 2)$
5. $x^2 + 4x - 5 = (x + 5)(x - 1)$
6. $x^2 - 6x = x(x - 6)$
7. $3x^2 + x - 2 = (3x - 2)(x + 1)$

Activity 2

1. $x^2 + 5x + 6 = 0$

   $(x + 2)(x + 3) = 0$

   $x + 2 = 0$ or $x + 3 = 0$

   $x = -2$ or -3

2. $x^2 - 5x - 6 = 0$

   $(x - 6)(x + 1) = 0$

   $x - 6 = 0$ or $x + 1 = 0$

   $x = 6$ or -1

3. $x^2 + 5x - 6 = 0$

   $(x + 6)(x - 1) = 0$

   $x + 6 = 0$ or $x - 1 = 0$

   $x = -6$ or 1

Check each.

4. $x^2 + 9x + 14 = 0$

   $(x + 7)(x + 2) = 0$

   $x + 7 = 0$ or $x + 2 = 0$

   $x = -7$ or $x = -2$

5. $x^2 - 9 = 0$

   $(x - 3)(x + 3) = 0$

   $x - 3 = 0$ or $x + 3 = 0$

   $x = 3$ or $x = -3$

6. $2x^2 + 5x = 3$

   $2x^2 + 5x - 3 = 0$

   $(2x - 1)(x + 3) = 0$

   $2x - 1 = 0$ or $x + 3 = 0$

   $x =$ or $x = -3$

Check each.

Activity 3

1. $3(a + 1)^2 - 3 = 0$

   $(a + 1)^2 = 1$

   $a + 1 = 1$ or -1

   $a = 0$ or $a = -2$

Check.

Activity 4

1. $x + y = 7$

   $x^2 + y^2 = 29$

   If $x + y = 7$ then $x = 7 - y$

   Substitute in $x^2 + y^2 = 29$ gives $(7 - y)^2 + y^2 = 29$

   $49 - 14y + y^2 + y^2 = 29$

   $2y^2 - 14y + 20 = 0$

$y^2 - 7y + 10 = 0$

$(y - 2)(y - 5) = 0$

$y - 2 = 0$ or $y - 5 = 0$

$y = 0$ or $y = 5$

$x = 7 - y$ so when $y = 2$, $x = 5$ and when $y = 5$, $x = 2$

Solution is $x = 5$, $y = 2$ and $x = 2$, $y = 5$

Checking $x + y = 5 + 2 = 7$

and $x^2 + y^2 = 4 + 25 = 29$

2. $x + 2y = 2$

$x^2 + 2xy = 8$

Substituting $x = 2 - 2y = 2(1 - y)$ gives $4(1 - y)^2 + 2.2(1 - y)y = 8$

Hence $(1 - 2y + y^2) + y - y^2 = 2$ or $-y = 2 - 1 = 1$ Hence $y = -1$ and $x = 4$

Solution is

x	4
y	-1

Checking $4 - 2 = 2$ and $16 - 8 = 8$

Activity 5

1. $\frac{x^2 - 1}{x} \times \frac{x^2}{x - 1} = \frac{(x - 1)(x + 1)x^2}{x(x - 1)} = (x + 1)x$

2. $\frac{1}{x} - \frac{1}{x + 2y} - 1 = \frac{(x + 2y) - x - x(x + 2y)}{x(x + 2y)} = \frac{x + 2y - x - x^2 - 2xy}{x(x + 2y)} = \frac{2y - 2xy - x^2}{x(x + 2y)}$

Activity 6

1. $m(\frac{r}{x})^3 = P$

$(\frac{r}{x})^3 = \frac{P}{m}$

$(\frac{r}{x}) = \sqrt[3]{(\frac{P}{m})}$

$r = x\sqrt[3]{(\frac{P}{m})}$

2. $A = \pi r^2$ is the formula for the area A of a circle of radius r

$A = \pi r^2$

$\pi r^2 = A$

$r^2 = \frac{A}{\pi}$

$r = \sqrt{\frac{A}{\pi}}$

3. $\mathrm{F} = 32 + \frac{9}{5}\mathrm{C}$

$\frac{9}{5}\mathrm{C} = \mathrm{F} - 32$

$9\mathrm{C} = 5(\mathrm{F} - 32)$

$\mathrm{C} = \frac{5(\mathrm{F} - 32)}{9}$

i) When F = 32, C = 0

ii) When F = 212, $\mathrm{C} = \frac{5(212 - 32)}{9} = \frac{5(180)}{9} = 100$

Answers to practice questions

1. a) $x^2 - 2x - 35 = (x - 7)(x + 5)$

   b) $2x^2 - 5x + 2 = (2x - 1)(x - 2)$

2. $4x^2 - 25 = 0$ gives $(2x - 5)(2x + 5) = 0$

   Hence either $2x - 5 = 0$ or $2x + 5 = 0$

   So $x = \frac{5}{2}$ or $-\frac{5}{2}$

3. $x = \frac{7 \pm \sqrt{(49 + 24)}}{6} = \frac{7 \pm \sqrt{73}}{6} = \frac{7 \pm 8.54}{6} = \frac{15.54}{6}$ or $\frac{-1.54}{6}$

   $x = 2.59$ or -0.26 correct to two significant figures.

4. $4x - y = 7$; $xy = 15$

   Substituting $y = \frac{15}{x}$ gives $4x - \frac{15}{x} = 7$

   Multiplying both sides by x gives $4x^2 - 15 = 7x$ or $4x^2 - 7x - 15 = 0$

   Solving this quadratic equation gives $(4x + 5)(x - 3) = 0$ which gives $x = -\frac{5}{4}$ or 3

   When $x = 3$, $y = \frac{15}{3} = 5$.

   When $x = -\frac{5}{4}$, $y = 15 \div (-\frac{5}{4}) = (-15 \times 4)/5 = -12$

5. $\frac{1}{x-1} + \frac{3y}{xy - y} = \frac{1}{x-1} + \frac{3y}{y(x-1)} = \frac{1}{x-1} + \frac{3}{x-1} = \frac{4}{x-1}$

6. $\sqrt{(2A)} = 4b$ Square both sides

   $2A = (4b)^2$

   $2A = 16b^2$ Divide both sides by 2

   $A = 8b^2$

Glossary

Variable	A symbol such as a or x which represents a number that can take different values.
Constant	A fixed number that never changes.
Formula	A formula shows the relationship between two or more variables, e.g. $C = 2\pi r$.
Expression	A mathematical statement including variables, e.g. $4xy + y^2$ or $5a + 4b + 3c$.
Term	Each part of an expression is a term, e.g. $5a + 4b + 3c$ has three terms.
Factor	When two or more quantities are multiplied together, each is called a factor of the product. In $5xy$ the factors are 5, x and y and in $(2x + 1)(x - 1)$, the factors are $(2x + 1)$ and $(x - 1)$.
Factorising	The process of rewriting an expression into two or more brackets, so that when multiplied together they give the original expression. Factorising is the opposite of expanding.
Coefficient	The number, with its sign that goes in front of a variable, e.g. 3 is the coefficient of $3x^2$.
Equation	A statement showing that two expressions are equal, e.g. $3x + 11 = 20 + 2x$.
Degree of an expression	The highest power of the variable in the equation, e.g. $x^3 + 5$ is of degree 3.
Linear expression or equation	An expression or equation with first degree terms only, e.g. $3x + 1 = 7$.
Quadratic expression or equation	An expression or equation, of degree 2, e.g. $x^2 + 3x - 6 = 0$
Perfect square	The quadratic expression formed from multiplying one linear expression by itself.

MSCE M4: Measuring Geometric Shapes and Solids

What you are studying and why

Subject: Mathematics Unit: M4

At the end of this unit you should be able to:

1. identify and sketch 3D shapes
2. find surface area and volume of 3D shapes
3. identify and calculate angles between lines, planes as well as lines and planes.

Revise

This unit builds on what you already know about geometry and mathematics all around you. It is about measuring geometric shapes and solids or in mathematical language we say mensuration.

You have already found the areas of common shapes and in this unit you extend that to finding the surface area of solids and volumes of solids. We encourage you to look at shapes all around you and to be curious about measurement and estimating. For example consider the shape of a pencil or material for dresses, or think about the differences in roof shapes of buildings. It is difficult to draw solid articles on paper and you have to practise this in order to answer questions in the MSCE on solids like cubes, cylinders and prisms.

We all live in a world of solids – a three-dimensional world – but we carry out our mathematics on two-dimensional pieces of paper! You have to be able to make accurate sketches which show this solid world.

Before we begin this unit make sure that you know the difference between the words *sketch* and *draw/construct* as used in the MSCE.

To **sketch** is to produce a rough diagram which you do with only a pencil and ruler and is done quickly.

To **draw or construct** is to produce an accurate diagram which is done using a pencil, ruler and compasses and is done carefully and takes longer.

The 3D shapes considered in this unit are cubes, cuboids, spheres, cylinders, cones, pyramids and prisms. For each of these it is important that you have many real objects to refer to. Here are some examples.

When considering

- cubes and cuboids look at ordinary boxes of different sizes
- cylinders look at tins
- spheres look at balls
- cones look at funnels, roofs of local maize silos.

A tent is a good example of a prism, which is a shape with parallel faces and therefore a constant cross-section; spheres and cubes are prisms as well. You may find pictures of the Egyptian pyramids which show rectangular pyramids.

Diagram 1 shows some other everyday examples.

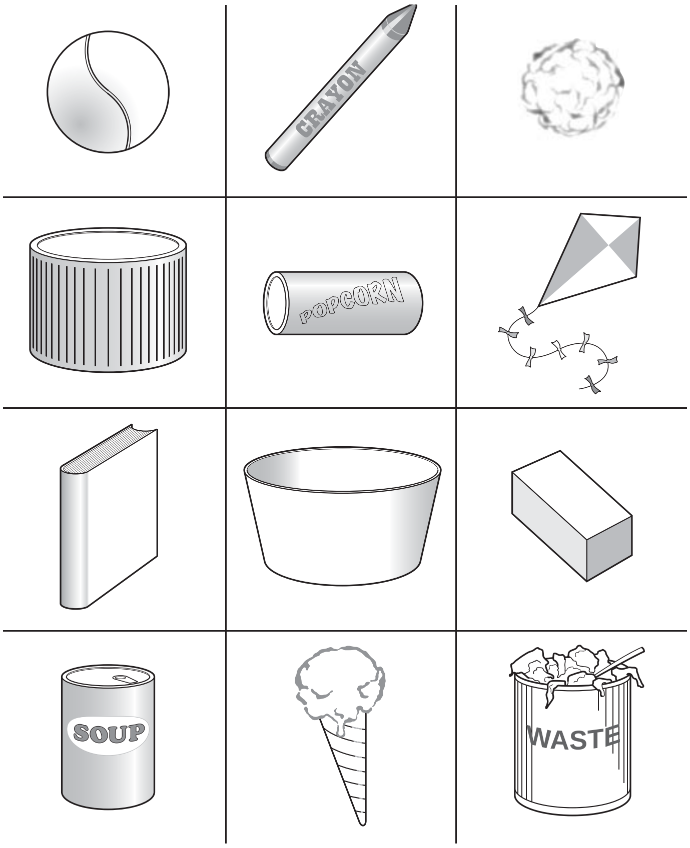


Diagram 1: Everyday examples of 3D shapes

Exercise 1

For the MSCE you must be able to identify and also sketch these quickly. Practise now by copying those in Diagram 2 and naming each. The answers are at the end of the unit.

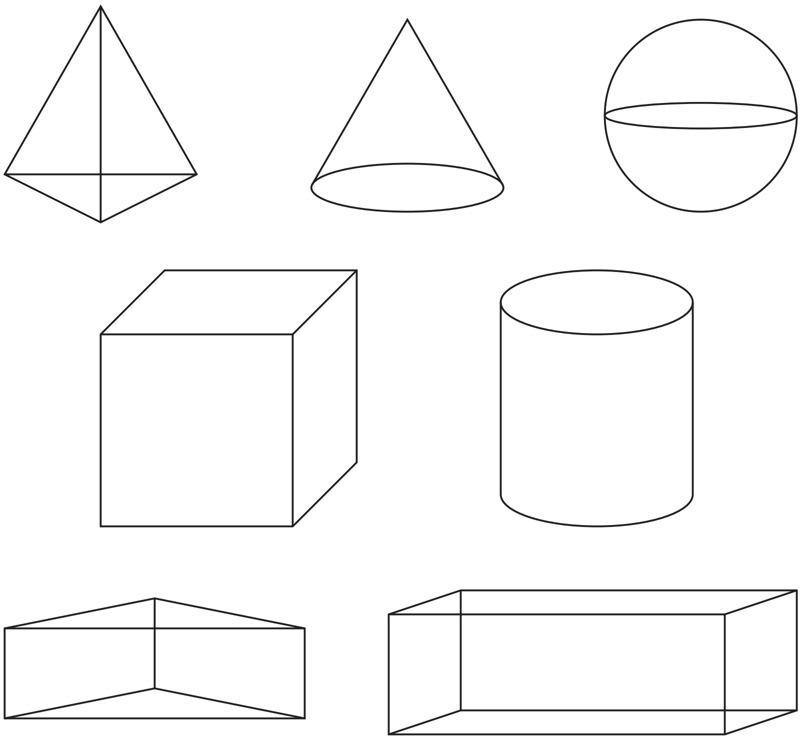


Diagram 2: Cubes, cuboids, spheres, cylinders, cones, pyramids and prisms

You will have made **nets** of cubes and other solids; a net is a pattern that you can cut and fold to make a model of a solid shape. Here are three for you to try. Perhaps you know others.

Exercise 2

Copy the first three nets and then cut them out and fold them. Name the solid you have created. Then roll iv) to make a cylinder.

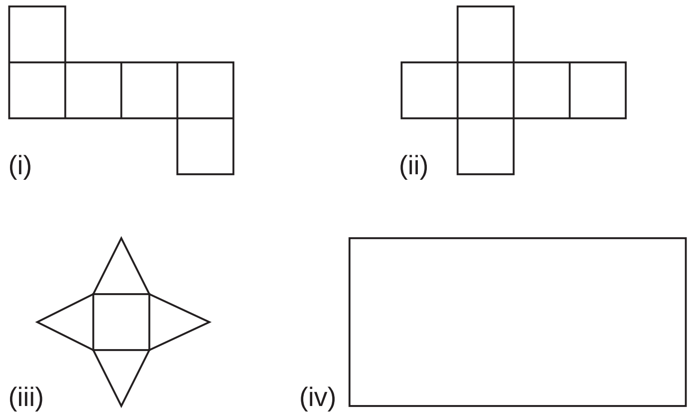


Diagram 3: Nets

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The answers are given at the end of the unit.

Finally before going on we would like you to revise the vocabulary connected with 3D solids. The three important ones are *vertex*, *edge* and *face* in Diagram 4.

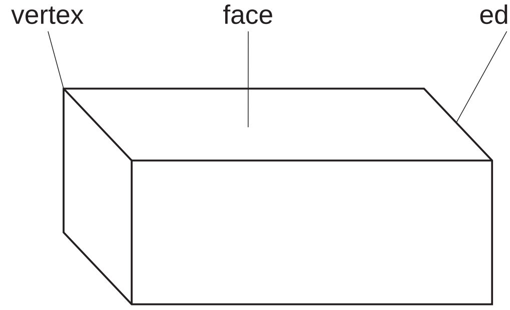

Diagram 4: Cuboid vertex, face and edge

Surface area of solids

To find the surface area of any 3D solid is to find the area of each of the faces of the solid and add those areas together.

We will quickly revise with you the area formulae which you should know.

Shape	**Area**
Rectangle of length a and breadth b	$a \times b$
Square of side a	a^2
Triangle of height h and base b	$\frac{1}{2}b \times h$
Parallelogram of base b and height h	$b \times h$
Trapezium of opposite sides a and b and height h	$\frac{1}{2}h(a + b)$
Circle of radius r	πr^2

For a cuboid with sides a, b and c we think it is easier to find the area of each of the rectangular faces. If you count the rectangular faces you find there are six in three sets of two with areas ab, bc and ca and so the surface area is $\mathbf{2(ab + bc + ca)}$.

Now we would like you to write down the surface area of a cube of side a.

Did you write $6a^2$?

For a cylinder of length a and radius of the two circular faces r you must find the area of the circles and the area of the side; although it goes around the side of the cylinder, this face is in fact a rectangle as you will remember from Exercise 2 part iv) when you were asked to roll a rectangle to make a cylinder.

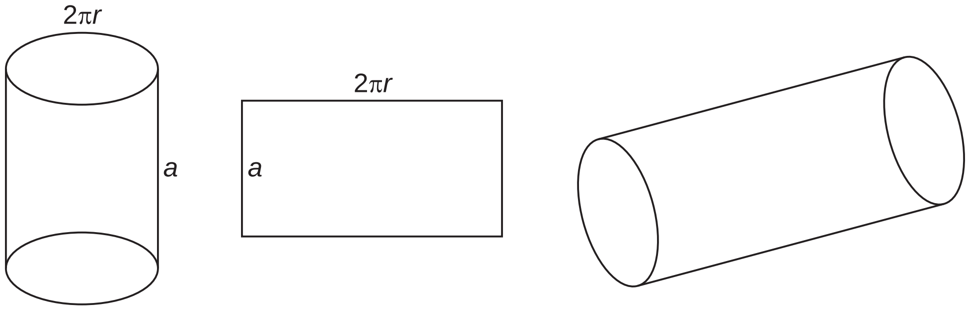

Diagram 5: Cylinder surface area

You will see that for the cylinder made by rolling the rectangle the distance round the circular face of the cylinder, $2\pi r$ (the circumference of the circle), is the length of the rectangle which has breadth equal to the length of the cylinder, a. The area of each circle is πr^2. The area of the two circular ends is therefore $\pi r^2 \times 2 = 2\pi r^2$. Therefore the total surface area of the circular cylinder is the sum of the areas of the circular ends ($2\pi r^2$) and the area of the rectangle $2\pi ra$.

The surface area of a cylinder is $2\pi r^2 + 2\pi ra = \mathbf{2\pi r(r + a)}$.

As in a prism, parallel faces are equal in area, to find the surface area of any prism you add twice the area of the parallel faces to the sum of the areas of each of the sides.

You find the surface area of a pyramid with either a triangular base or a rectangular base by finding the area of the base and the areas of each triangular side and adding them up.

Here are the formulae you must remember.

Solid	**Surface area**
Cuboid of sides a, b and c	$2(ab + bc + ca)$
Cube of side a	$6a^2$
Cylinder of length a and radius r	$\mathbf{2\pi r(r + a)}$
Regular triangular pyramid of slant height a and with all triangles the same	Area of base + $\frac{1}{2}$(perimeter of base $\times a$)
Square pyramid with base side a, slant height b and all triangles the same	$a^2 + 2ab$

Do not worry about the surface area of a sphere or a cone. If there is a question in the MSCE about these you will be given the formula to use.

Example

Diagram 6 shows an open trough in the shape of a prism.

Find its surface area to calculate the amount of material needed to make it.

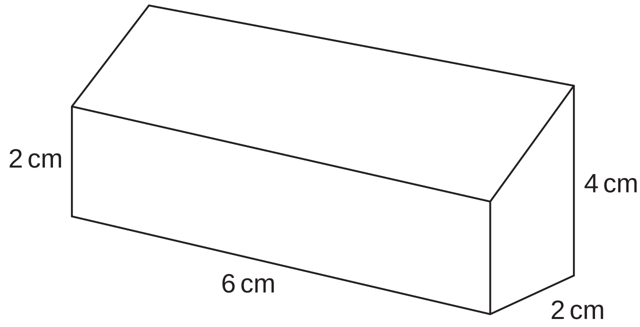

Diagram 6: Prism-shaped trough

The trough is a prism.

Two opposite ends of the prism are trapezium with area

$\frac{1}{2} \times 2(2 + 4)$ cm^2 = 6 cm^2 each.

The bottom is a rectangle of sides 2 cm × 6 cm of area 12 cm^2.

The front is a rectangle 6 cm × 2 cm of area 12 cm^2.

The back is a rectangle of sides 6 cm × 4 cm of area 24 cm^2.

(The top is open.)

Total surface area is 6 + 6 + 12 + 12 + 24 cm^2 = 60 cm^2.

Exercise 3

1) Find the surface area of a cube of sides 5 cm, 6 cm and 10 cm.
2) Find the surface area of a cylinder of height 5 cm and diameter 10 cm.
3) Find the surface area of a sphere of radius 4 cm. (The surface area is $4\pi r^2$.)
4) Find the surface area of a cone of height 4 cm and radius of base 3 cm.

(Surface area of a cone is given by $\pi r(r + \sqrt{(r^2 + h^2)})$.)

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The answers are given at the end of the unit.

Volume of Solids

You need to remember the formulae for the volume of cuboids, cones, cylinders and simple prisms.

Solid	**Volume**
Cuboid sides of length a, b, c	$a \times b \times c$
Cube side a	a^3
Cylinder length a and radius r	$\pi r^2 a$
Any prism	Area of base × length

There are volume formulae which you do not need to worry about and remember because they are always given if you need them in a question in the MSCE.

Example

1) 2008 Paper 1 Question 15

A cylindrical metal bar whose volume is 594 cm^3 is melted down and cast into a sphere. Calculate the radius of the sphere, leaving your answer correct to two decimal places.

(Volume of a sphere is $\pi \frac{4}{3} r^3$. Take $\pi = 3.142$.)

$\pi \frac{4}{3} r^3 = 594$

$r^3 = 594 \times 3 \div (4\pi) = 141.787$

$r = \sqrt[3]{(141.787)} = 5.21$

The radius of the sphere is 5.21 cm correct to two decimal places.

Exercise 4

1) 2008 Paper 2 Question 10a

Diagram 7 shows a cone placed on top of a cylinder. The height of the cone is 10 m and that of the cylinder is 12 m. The diameter of both cone and cylinder is 8 m. Calculate the total volume of the shape to two decimal places. (The volume of a cone is $\frac{1}{3}\pi r^2 h$. Take $\pi = 3.142$.)

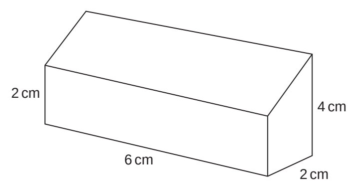

Diagram 7: Cylinder with cone on top

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2) 2009 Paper 1 Question 3

The volume of a pyramid is 60 cm^3 and its base area is 20 m^2. Calculate the height of the pyramid. (Volume of pyramid = $\frac{1}{3}$ base area $\times$ height.)

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The answers are given at the end of the unit.

Angles and planes in 3D

Any questions on this topic in the MSCE are very straightforward. However, in many cases, questions in this topic will require you to identify right-angled triangles and be able to use Pythagoras' Theorem (the square on the hypotenuse equals the sum of the square on the other two sides) and some trigonometry. It is therefore important that you identify the right-angled triangle containing the required angle.

Here is an example to illustrate how to find some simple angles between lines and planes and between two planes.

Example

In the cuboid ABCD:EFGH illustrated in Diagram 8, AB = 12 cm, BC = 9 cm and BF = 8 cm.

1) Find the length of AC.

2) Find the angles between

   i) the lines GA and AC

   ii) the line AF and the plane ABCD

   iii) the planes GAC and ABCD.

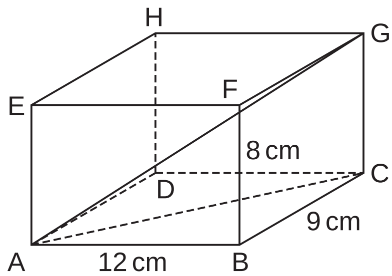

Diagram 8: Cuboid ABCD:EFGH

Answer:

1) By Pythagoras' Theorem $AC^2 = AB^2 + BC^2$

   Hence $AC = \sqrt{(12^2 + 9^2)}$ cm $= \sqrt{225}$ cm = 15 cm.

2) i) The angle between the lines GA and CA is the angle GAC. To find this angle you must look at triangle GAC. GC = FB = 8 cm and AC = 15 cm and angle ACG = 90°.

   Using trigonometry tan GAC = $\frac{8}{15}$ = 0.5333 and so angle GAC = 28° to the nearest degree.

   ii) The angle between a line and a plane is found by dropping a perpendicular from the line to the plane and finding the angle produced. The angle between AF and plane ABCD is the angle FAB and tan FAB = $\frac{8}{12}$ = 0.667 giving angle FAB = 34° to the nearest degree.

   iii) The angle between the plane ABCD and the plane GAC is a right angle because plane ABCD is horizontal and the plane GAC is vertical.

The angle between two planes is found by dropping perpendiculars from each plane and finding the angle between them.

Here is an example for you to try yourself taken from the 2009 paper.

Exercise 5

1) 2009 Paper 2 Question 10b

Diagram 9 shows a prism ABCDEF. AB = 30 m, BC = 10 m, CF = 5 m and angle BCF = 90°. Calculate

i) the volume of the prism

ii) angle FAC to the nearest degree.

(The diagram is not drawn to scale.)

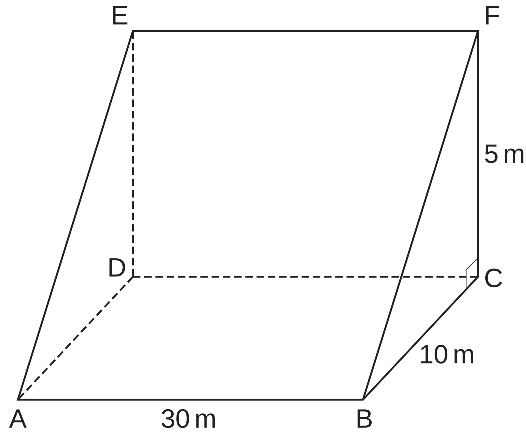

Diagram 9: Prism ABCDEF

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The answers are given at the end of the unit.

Questions to practise

(2 hours)

1) A tin of beans has a diameter of 7 cm and is 4.5 cm high. Work out the surface area of the tin and its volume.

2) A tent has height 2 m, breadth 2 m and length 3 m. Find its surface area and its volume.

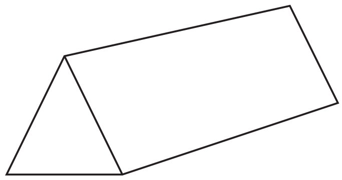


Diagram 10: Tent

3) The triangular pyramid NPKV in Diagram 11 has all its sides equal to 6 cm. M is the midpoint of NP.

   i) Find the length of VM and KM.

   ii) Is the angle between the planes VNP and KNP the angle VNK or angle VMK?

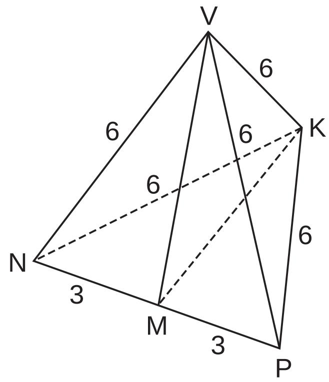


Diagram 11: Triangular pyramid NPKV

4) Diagram 12 represents a shed with a sloping roof, and dimensions as shown.

   i) Find the area of the end WXYZ.

   ii) Find the angle between the roof and the horizontal.

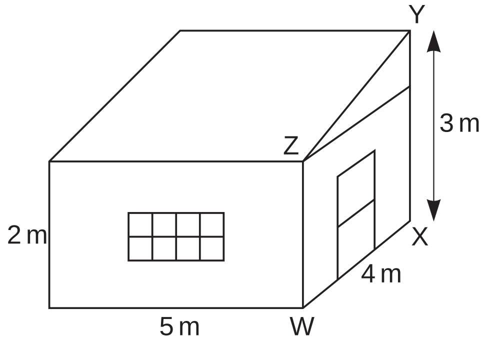


Diagram 12: Shed

5) A hollow metal sphere has outer and inner radii of 7 cm and 6 cm, respectively. Calculate the volume of metal required to make it. (The volume of a sphere is $V = \frac{4}{3}\pi r^3$.)

6) A cube of side 5 cm is painted on all sides. If it is sliced into 1 cm cubes how many of these cubes have exactly one side painted?

How am I doing?

This section is a study tool.

Now look back over this unit and be honest about what was difficult.

Later use it to discuss with your tutor any extra help you need.

Before the exam use this tool to revise.

	☺ **Easy** (Tick this box if you feel confident that you **understand this section well**)	**Fine** (Tick this box if you still **need a little work** on this section)	☹ **Difficult** (Tick this box if you **still need a lot** of work on this section)
Sketch common solids			
Calculate the surface area of common solids			
Calculate the volume of common solids			
Identify angles in commons solids			

Notes on what to do next:

Signed (by Scholar): .... Date:..................

Signed (by Tutor): .... Date:..................

Answers to Exercises

Answers for exercise 1

Starting at the left top corner and moving clockwise the solids are: a triangular pyramid; a cone; a sphere; a cube; a cylinder; a triangular prism; and a cuboid.

Answers for exercise 2

i) and ii) are cubes, iii) is a square pyramid and iv) is a cylinder.

Answers for exercise 3

1) Surface area of cuboid = $2(5 \times 6 + 6 \times 10 + 10 \times 5)$ cm^2 = 2×140 cm^2 = 280 cm^2.

2) Surface area of the cylinder = $2 \times 3.142 \times 5(5 + 5)$ cm^2 = 314.20 cm^2.

3) Surface area of sphere = $4 \times 3.142 \times 16 = 201.09$ cm^2.

4) Surface area of cone = $3.142 \times 3(3 + \sqrt{(9 + 16)}) = 3.142 \times 3(3 + 5) =$ 75.38 cm^2.

Answers for exercise 4

1) Volume of a cylinder is $\pi r^2 h$. The radius of both cone and cylinder is 4 cm.

   Let h_1 = height of cone and h_2 = height of cylinder

   Total volume = $\frac{1}{3}\pi r^2 h_1 + \pi r^2 h_2 = \pi r^2(\frac{1}{3}h_1 + h_2) = 3.142 \times 16(\frac{10}{3} + 12)$ cm^3

   Total volume = $3.142 \times 16 \times 15.333$ cm^3 = 770.82 cm^3.

2) Volume of pyramid = $\frac{1}{3}$ area of base $\times$ height

   $60 = \frac{1}{3} \times 20 \times h$

   $180 = 20 \times h$

   $h = 90$

   Height of pyramid = 90 cm.

Answers for exercise 5

i) Volume of prisms = 2 × area of base × height = 2 × area of triangle × height

   Area of triangle which has a right angle = $\frac{1}{2} \times 10 \times 5 = 25$ cm^2

   Volume of prism = 25×30 cm^3 = 750 cm^3.

ii) To find angle FAC find side AC in triangle FAC.

   tan FAC = $\frac{FC}{AC}$

   FC = 5; AC = $\sqrt{(10^2 + 30^2)}$ (Pythagoras' Theorem) = 31.62 cm

   tan FAC = $\frac{5}{31.62}$ = 0.1582 giving angle FAC = 9° to the nearest degree.

Answers for Questions to practise

1) i) The surface area of a cylinder is given by $2\pi r(r + h)$

Surface area of tin of beans = $2 \times 3.142(3.5 + 4.5)$ cm^2

= $2 \times 3.142 \times 3.5 \times 8$ cm^2 = 175.95 cm^2.

ii) The volume of a cylinder is given by $\pi r^2 h$

Volume of tin of beans = $3.142 \times 3.5 \times 3.5 \times 4.5$ cm^3 = 173.20 cm^3.

2) First find the slanting side using Pythagoras' Theorem. Edge = $\sqrt{(2^2 + 1^2)}$ = $\sqrt{5}$ = 2.24

(The base is a rectangle, sides 2 m by 3 m; the end is an isosceles triangle, height 2 m, base 2 m; the long edges are rectangles 3 m by $\sqrt{5}$ m.)

Surface area = area of base + 2 × area of triangle + 2 × area of sides

= $(2 \times 3 + 2 \times \frac{1}{2} \times 2 \times 2 + 2 \times 3 \times \sqrt{5})$ m^2 = $(6 + 4 + 13.4)$ m^2 = 23.4 m^2

Volume = area of triangle × length = $\frac{1}{2} \times 2 \times 2 \times 3$ m^3 = 6 m^3.

3) i) In triangle VNM angle VMN is a right angle and so by Pythagoras' Theorem

VM = $\sqrt{(VN^2 - NM^2)} = \sqrt{(36 - 9)} = \sqrt{27} = 3\sqrt{3}$

In triangle KMP angle KMP is a right angle and so by Pythagoras' Theorem

KM = $\sqrt{(KP^2 - MP^2)} = \sqrt{(36 - 9)} = \sqrt{27} = 3\sqrt{3}$

ii) The angle between planes VNP and KNP is angle VMK.

(The angle VNK is the angle between the line NV and the plane KNP.)

4) i) WXYZ is a trapezium. The area is $\frac{1}{2}(2 + 3) \times 4 = 10$ m^2.

ii) The angle between the roof and the horizontal is the angle whose tangent is $\frac{1}{4} = 0.25$; this angle is 14° to the nearest degree.

5) Volume of material needed is $\frac{4}{3} \times \pi\,(7^3 - 6^3)$ cm^3 = $(\frac{4}{3} \times \frac{22}{7} \times (343 - 216))$ cm^3 = 532 cm^3.

6) The 1 cm cubes on the edges of the big cube have either two or three sides painted. Only those not on the outer edges have just one side painted. These are on the sides of the big cube in squares of side 3 cm as there is one taken off at each edge, 5 – 2 = 3. There are 3 × 3 = 9 on each side of these squares and as there are six sides of the big cube and so **9** × **6** = **54** cubes with only one side painted white altogether.

MSCE M5: Statistics

What you are studying and why

Subject: Mathematics Unit M5

This unit is on statistics.

At the end of this unit you should be able to:

1. group and illustrate data
2. draw and use frequency polygons
3. calculate the mean of ungrouped data
4. define and calculate variance and standard deviation of ungrouped data.

Introduction

There is always a question on statistics in the MSCE paper. Statistics is used when you want to make sense of large amounts of data or information. It is used all around you, especially in things like newspapers and reports. People use different types of statistics within reports to back up their own ideas. They often include graphs or charts in order to do this.

Can you write down the names of some different types of graphs or charts? Write them below. If you have seen them used, write down where you saw them.

- ....
- ....
- ....
- ....

There are many different types but you are going to revise five of these. They are pictograms, pie charts, line graphs, bar graphs and histograms.

Pictograms

Here is a pictogram showing the different animals kept by 20 families in a village.

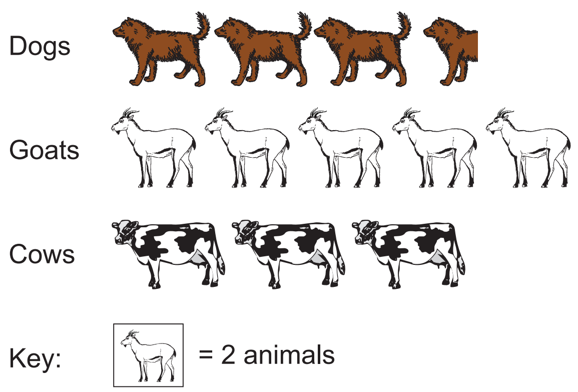


Figure 1

What information can you get from this pictogram?

....

....

This is called **interpreting** the pictogram.

What is the importance of the key?

....

Add up the total number of animals in the pictogram. Why do they not add up to 20?

....

Pie charts

A pie chart clearly shows how something is divided up. It is not so useful for finding accurate amounts.

Here is a question from the MSCE Paper 1 in 2008.

> *In a survey conducted at Chiutsa village, 20 people responded YES, 30 responded NO and 10 responded DON'T KNOW.*
>
> *Draw a pie chart to represent the information.*

Step 1
Find the total number of people who took part in the survey.
There are 60 people altogether.

Step 2
You have to divide your circle into three sections; one for YES, one for NO and one for DON'T KNOW. The size of these sections is **in proportion** to the number of people. There are 360° in a circle and a total of 60 people in the survey. How many degrees are represented by 1 person?

$360 \div 60 = 6$

One person is represented by 6°.

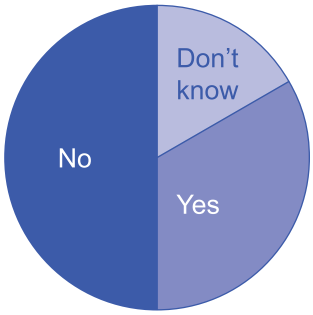
Step 3
Complete the following:

20 people voted YES	These are represented by 20 × 6° = 120°
30 people voted NO	These are represented by .....× 6° = 180°
.... people voted DON'T KNOW	These are represented by .....× 6° = ...°

Step 4
Before you draw the pie chart, can you think of a way of checking your answers?

....

Draw the pie chart.



You may need to revise how to use a protractor. Discuss this with a fellow scholar.

Remember that there are two scales on a protractor.

Line graphs

Line graphs are formed from the plotting of matching values as small crosses. These points are joined using straight lines. Line graphs are only used for **continuous data**. Discuss what is meant by continuous data.

It is unlikely that you will have to draw a line graph in the examination but you will have to interpret one.

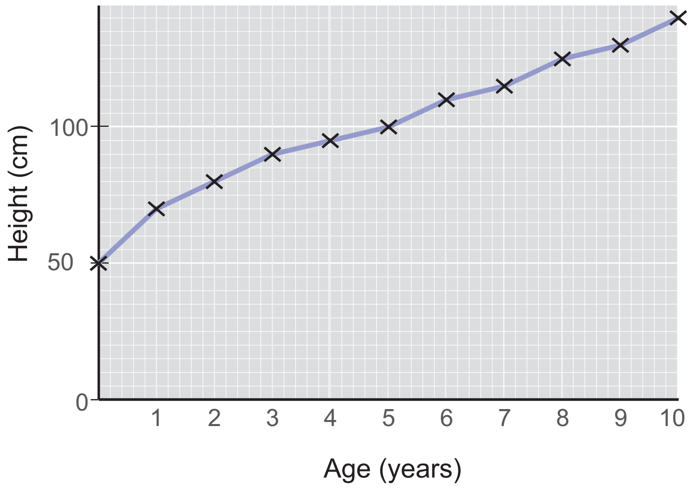


Figure 2: Line graph to show a girl's height in cm from birth until age 10

Answer these questions:

1. By how much did the girl's height increase between her birth and when she was 10?

   ....

2. Between what ages was the greatest annual increase in height?

   ....

3. When the girl's height was 85cm what was her age (in years)?

   ....

Answers:

1. 140 - 50 = 90
2. Between 0 and 1 year. The graph is the steepest at this point.
3. She was 2 ½ years old.

Bar graphs and histograms

Bar graphs or charts can only be used for **discrete data**. You discussed the meaning of continuous data earlier. Now discuss what the difference between continuous and discrete data is?

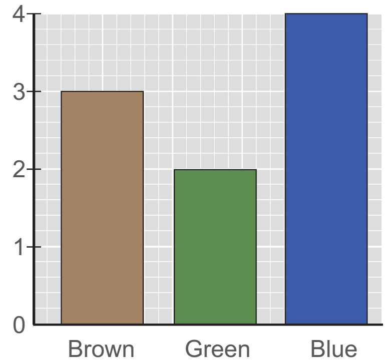


Figure 3: Bar graph to show the favourite colours for a group of children

The vertical axis shows the number of children. This is often called the **frequency**.

Answer these questions:

1. How many prefer blue? ....
2. How many more prefer blue to brown? ....
3. Which is the least popular colour? ....

Answers:

1. 4
2. 4 - 3 = 1
3. Green

Activity 1

1. Given the following figures, draw a pie chart to show the different languages spoken in Malawi:

   Chichewa 60%; Chinyanja 15%; Chiyai 10%; Chitumbuka 10%; Others 5%.

   Note that in this question, you are not given the number of people. You do know that all the people are included, so 100% represents all the people.

2. Using the line graph below, which shows the population of Malawi in thousands from 1961–2003, find:

   (a) the population in 2003

   (b) the growth in population from 2001–2003

   (c) the total growth from 1961–2003.

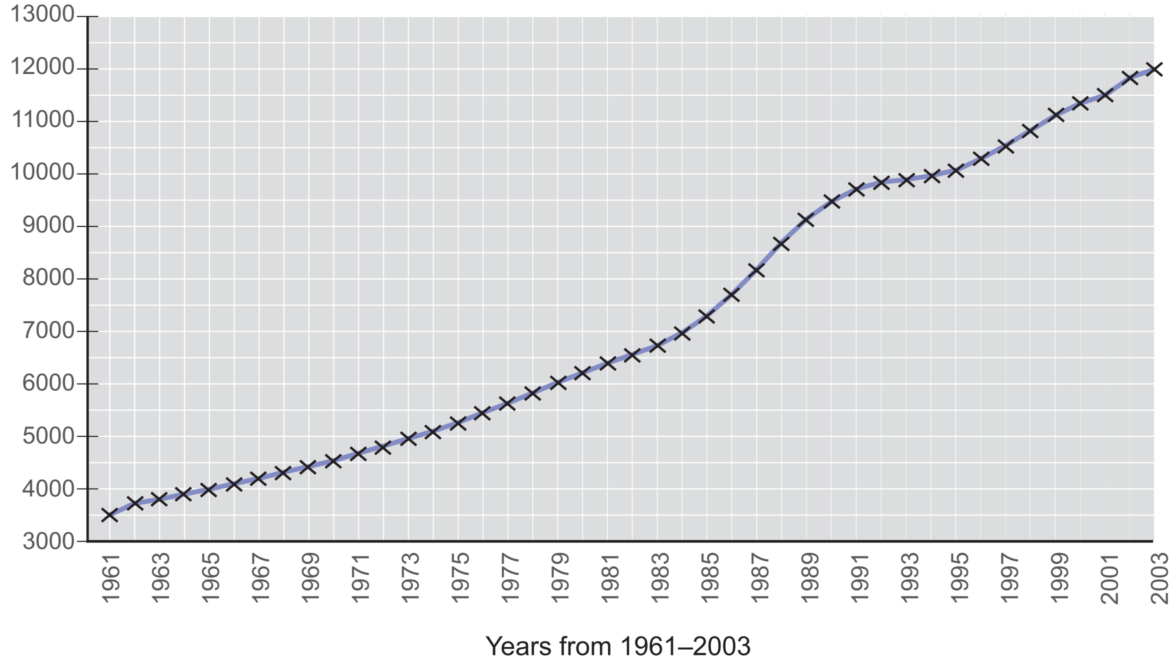


Figure 4

Answers:

1. Find the angle related to each language in the pie chart:

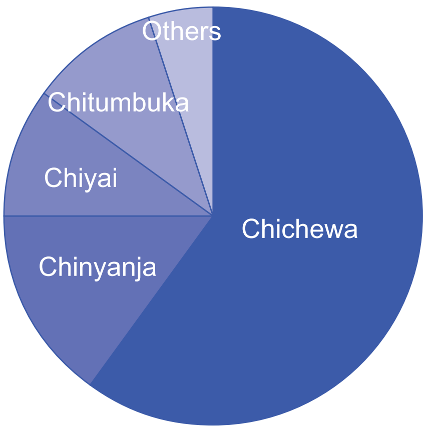
Chichewa = $\frac{60 \times 360}{100} = 216°$

Chinyanja = $\frac{15 \times 360}{100} = 54°$

Chiyai = $\frac{10 \times 360}{100} = 36°$

Chitumbuka = $\frac{10 \times 360}{100} = 36°$

Others = $\frac{5 \times 360}{100} = 18°$



Pie chart to show different languages spoken in Malawi

2. Note that the vertical axis shows the population in thousands.
   (a) The population in 2003 was 12,000,000.
   (b) The growth in population from 2001–2003 = 12,000,000 - 11,500,000 = 500,000.
   (c) The total growth from 1961–2003 = 12,000,000 - 3,500,000 = 8,500,000.

Grouped data

Statistics often deals with large numbers. When you work with a large amount of data, it is useful to group the data into **classes**. This makes graphs and charts easier to draw and calculations easier to do.

You will begin by looking at an example that has only one value in each class. This means that the data is discrete.

Example 1

This list shows the scores of 100 candidates in a test:

4	1	2	3	6	6	2	1	3	5	5	1	3	6	2	4	3	2	6	2
6	3	4	2	4	5	3	5	4	4	5	4	6	4	1	2	4	3	5	3
4	2	6	4	3	4	1	4	2	4	6	5	3	4	4	2	3	3	2	5
5	2	4	3	4	3	4	1	3	4	5	6	3	2	6	4	3	5	5	2
4	5	5	4	3	5	5	5	4	5	5	4	5	4	2	3	5	2	4	4

How would you find out how many candidates got a score of 1, 2 ... ? Could you use tally marks?

The chart below shows the counting with tally marks. Candidates have been tallied according to their score in the test. Each time a number occurs a mark is put next to that number. The fifth mark is used to bundle the previous four occurrences together, so that they can be counted in fives.

Score	No. of candidates	No. of candidates
1	卌 |	6
2	卌 卌 卌 |	16
3	卌 卌 卌 ||||	19
4	卌 卌 卌 卌 卌 |||	28
5	卌 卌 卌 卌 |	21
6	卌 卌	10

These values can be rewritten into a table.

Score (x)	Frequency (f)
1	6
2	16
3	19
4	28
5	21
6	10

'Number of candidates' has been changed to 'Frequency'. This is called a **frequency table**. For a score of 5, the frequency is 21. Write down what this means.

....

....

You should have written something similar to:

21 candidates got a score of 5 in their test.

A graph can be drawn from this frequency table.

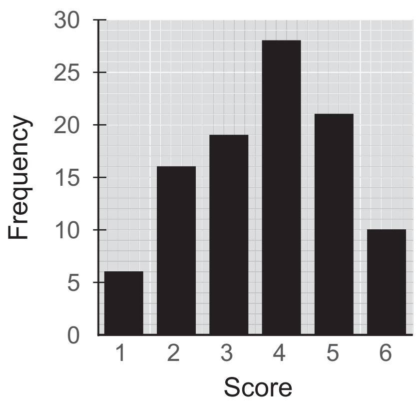


Figure 5: Bar graph to show the candidates' scores

The gaps between the bars show that the data is discrete. Half marks cannot be awarded; the marks must be whole numbers and there are no other values in between these.

Example 2

This example involves continuous data.

The time taken in minutes (to the nearest minute) for 100 students to get to school on a certain day is given below.

20	37	10	25	32	29	42	24	28	48	34	12	16	22	33	40	35
27	31	34	20	40	11	17	8	28	32	31	18	23	16	32	45	9
37	30	12	50	39	40	38	17	27	47	26	6	45	27	44	23	24
17	14	31	29	22	28	18	41	44	38	19	25	36	29	36	20	28
24	21	13	35	33	27	20	28	35	29	30	28	14	29	15	16	21
19	26	25	25	30	24	22	41	48	44	42	33	21	23	26		

Why is time continuous data?

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....

Time is continuous data because there are always more accurate values that lie between two values:

10.5 lies between 10 and 11 minutes

10.55 lies between 10.5 and 10.6 minutes

10.552 lies between 10.55 and 10.56 minutes.

Values for time can be measured more and more accurately but, of course, this depends on what you are using to measure it.

The times above range from 6 minutes to 50 minutes. Why wouldn't you use the numbers 1, 2, 3, ... 50?

....

Instead, then, you need to group them into suitably sized, equal groups or classes. Discuss the grouping with fellow scholars.

It is usual to form between 5 and 10 groups. Any more, or fewer, groups than this and the shape of the data on the graph is lost. The groups must never overlap.

Copy and complete the following frequency table.

Time (min)	**Tally**	**Frequency**
1–5		
6–10		
11–15		
16–20		
21–25		
26–30		
31–35		
36–40		
41–45		
46–50		

If you add up the numbers in the frequency column you should get 100. You can use this as a check.

The classes have upper and lower bounds or boundaries, e.g. the class 11–15 has upper and lower bounds of 10.5 and 15.5. This means that all values in between these bounds lie in this class. The lower bound, 10.5, is also included but not the upper bound, 15.5. Any value minutely less than 15.5 would be in the class but 15.5 itself will be included in the next class, 16–20. This means that there are no overlaps.

Time (min)	No. of students (frequency)
1–5	0
6–10	4
11–15	7
16–20	14
21–25	17
26–30	21
31–35	14
36–40	10
41–45	9
46–50	4

Complete the following statements:

There are ..... students in the class 6–10. This means that ..... students took between ..... minutes and ..... minutes to get to school.

The .... or width of any class is 5 minutes. For example:

the width of the 11–15 class $= (15 - 11) + 1$

$= 4 + 1$

$= 5$ wide.

It is not just the difference between the two numbers. You have to add on 1.

The **mid-point** or class centre of the class 11–15 is 13.

$$\frac{11 + 15}{2} = \frac{26}{2} = 13$$

Write down the mid-points of all the classes.

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....

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The frequency distribution can be drawn from the frequency table.

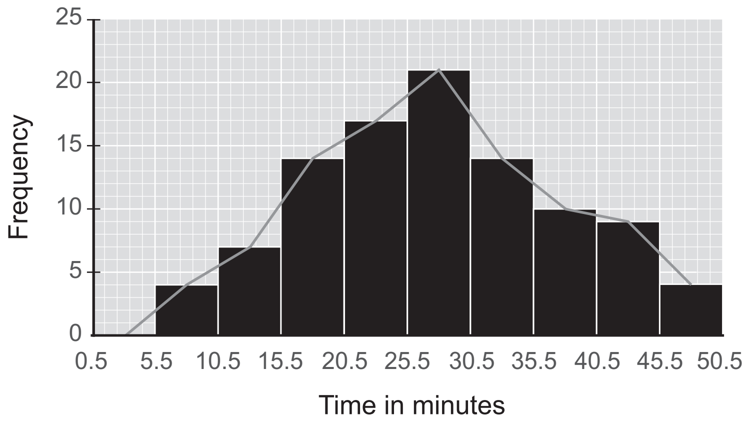

Figure 6

Notice that:

- the bar chart is formed of bars or columns with no gaps in between them. This is because the data is continuous.
- When you have equal class intervals, you can also call this type of bar chart a **histogram.**
- The edges of the bars are labelled with the lower/upper bounds.
- The vertical axis is labelled frequency.
- The line joining the mid-points of each class is called a **frequency polygon.**

Activity 2

1. This bar graph shows the number of peas in a pod for a given sample.

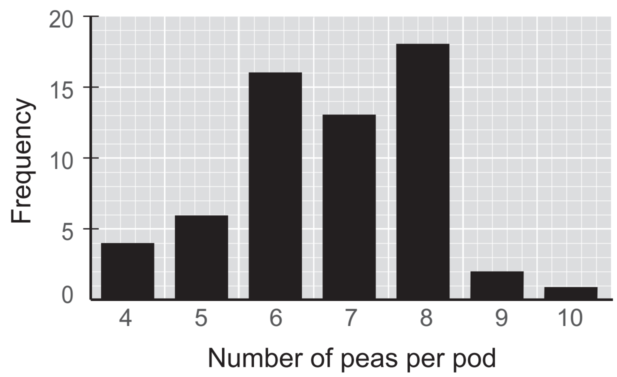

Figure 7

   (a) Make a frequency table from this graph.

   (b) Find how many pods were in the sample.

2. MSCE Paper 1 2009 Q19

   The table below shows the mid-point of marks scored by a group of 50 students in a test. Using a scale of 2cm to represent 1 unit on the horizontal axis and 2cm to represent 2 units on the vertical axis, draw a frequency polygon to represent this information.

Mid-point mark	12	13	14	15	16	17	18	19
Number of students	2	3	6	12	10	8	6	3

Answers:

1. (a)

No of peas in each pod (x)	**Frequency of pods (f)**
4	4
5	6
6	16
7	13
8	18
9	2
10	1

   (b) The number of pods in the sample is the sum of the frequency column. The sum of the frequencies is 60, so there were 60 pods.

2. Frequency polygon to show the distribution of marks in a test

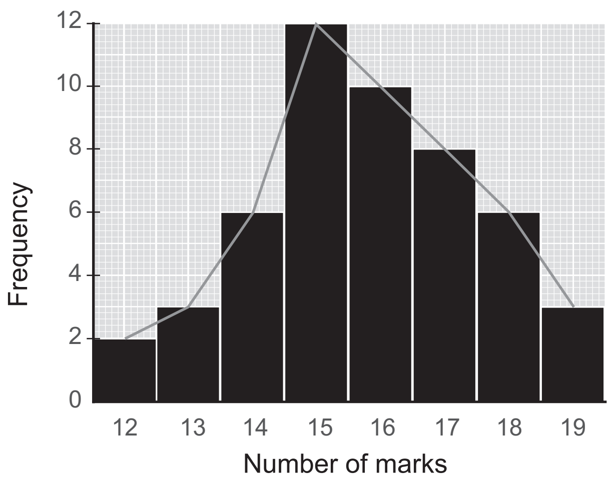

Figure 8

The mean

The mean is one of the three types of averages. Name the other two.

....

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The mean is the most useful one as it is used to analyse data.

To find the mean of the following 6 marks: 8, 6, 9, 5, 3 and 5;

- add them all up
- then divide the number by 6, because there are 6 marks.

$$\text{Mean} = \frac{8+6+9+5+3+5}{6} = \frac{36}{6} = 6$$

If we know that all other numbers lie around a certain value, you can use a shorter and easier method to find the mean.

Example 1

The marks of 15 students are 76, 74, 73, 72, 71, 71, 70, 70, 70, 69, 68, 68, 66, 64, 63.

Look at the numbers. They are all around 70. This is called the **estimated mean.**

This method involves finding the **deviation** from 70. If you deviate from something, you move away from it. The deviation is the amount that the number is away from 70.

76 is 6 larger than 70, so the deviation is +6.

63 is 7 smaller than 70, so the deviation is -7.

To find the average or mean deviation:

- add up all the deviations
- then divide by 15, because there are 15 students.

$$\text{Mean deviation} = \frac{6+4+3+2+1+1+0+0+0+(-1)+(-2)+(-2)+(-4)+(-6)+(-7)}{15}$$

$$= \frac{-5}{15}$$

$$= -0.33 \text{ (to 2dp)}$$

$$\text{Mean} = \text{estimated mean} + \text{mean deviation}$$

$$= 70 - 0.33 = 69.67 \text{ (to 2dp)}$$

Our estimate of 70 was very good.

You could check this answer by using the old method of adding up all the 15 marks and then dividing by 15. Only do this if you have the time. You will get the same answer but you will also realise how much quicker and easier the new method is because you are dealing with smaller numbers.

Now let us consider even more numbers and use a frequency table. In a test for 100 students the following marks were obtained:

Marks (x)	Frequency (f)
1	1
2	4
3	6
4	9
5	12
6	16
7	20
8	15
9	12
10	5
	100

Find the mean using:

(a) method 1, without using an estimated mean

(b) method 2, using an estimated mean.

Method 1

With a frequency table, you need to use the following formula:

Mean ($\bar{x}$) = $\frac{\Sigma fx}{\Sigma f}$

'Σ' is a greek capital 'S' so think of this sign as saying 'sum'. 'Sum' means add up.

So:

$\bar{x} = \frac{\text{The sum of the fx column}}{\text{The sum of the x column}}$

The fx column is found by multiplying the two columns together.

Marks (x)	Frequency (f)	fx
1	1	1
2	4	8
3	6	18
4	9	36
5	12	60
6	16	96
7	20	140
8	15	120
9	12	108
10	5	50
	$\Sigma f = 100$	$\Sigma fx = 637$

So: $\bar{x} = \frac{\Sigma fx}{\Sigma f}$

$= \frac{637}{100}$

Method 2

Look at the numbers in the frequency column. What would be a good estimate for your mean? Why?

....

....

The highest frequencies are for 6 and 7 marks, so either of these is a good choice for the estimated mean. Taking 6 as the estimated mean, fill in the deviation column below.

Marks (x)	Frequency (f)	Deviation (d)	fd
1	1		
2	4		
3	6		
4	9		
5	12	-1	-12
6	16	0	
7	20		
8	15	2	
9	12		
10	5		
	100		

Think about the table. In the last example, using an estimated mean, you found the mean deviation by adding up the mean deviation and dividing by the total number of numbers.

What is different in this example?

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You should have said something about the frequency column. For example, there isn't a single person with a mark of 5, there are 12 people. So there would be 12 lots of deviations of -1 from this row of the table. This is shown in the last column which is labelled 'fd'.

fd means f × d. All values in this column are found from multiplying together the frequency and the deviation. Complete the last column and add it up.

You should have found the total of all deviations to be 37. There are 100 students altogether.

The mean deviation $= \frac{37}{100}$

$= 0.37$

The mean = estimated mean + mean deviation

$= 6 + 0.37$

$= 6.37$

Here is the correct table in case you need to check where you have gone wrong.

Marks (x)	Frequency (f)	Deviation (d)	fd
1	1	-5	-5
2	4	-4	-16
3	6	-3	-18
4	9	-2	-18
5	12	-1	-12
6	16	0	0
7	20	1	20
8	15	2	30
9	12	3	36
10	5	4	20
	100		106 - 69 = 37

An extra example has been included below. If you are still unsure of what you are doing, try this. If you are happy to move on to the activity without doing this, then do so.

Example 2

A survey was done in Blantyre. It was to find the number of wage earners in different families. The results are shown in the table below. Find the mean number of wage earners in a family.

Wage earners	f	d	fd
0	347		
1	532		
2	1769		
3	280		
4	187		
5	50		
6	20		
	3185		

1. Take the estimated mean to be 2. Complete the rest of the table. Then find the mean.

   The correct table is shown below. Cover this up and check your table when you have finished.

Wage earners	f	d	fd
0	347	-2	-694
1	532	-1	-532
2	1769	0	0
3	280	1	280
4	187	2	374
5	50	3	150
6	20	4	80
	3185		884 - 1226 = -342

The survey asked 3185 families.

Mean deviation $= \dfrac{-342}{3185}$

Mean $= 2 - \dfrac{342}{3185} = 2 - 0.1 = 1.9$

As the number of people must be a whole number, the guess of 2 in each family is correct.

Activity 3

1. Find the mean number of peas in each pod from Question 1 in Activity 2.
2. In the end-of-year History exam, the mean mark scored by the 30 students in Form 11A was 55% and the mean mark scored by the 25 students in Form 11B was 44%. What was the mean mark of all the students?

Answers:

1. From the frequency table created for Question 1 in Activity 2, choose an estimated/assumed mean and find the deviations. For example, using an estimated mean of 7:

x	f	d	fd
4	4	-3	-12
5	6	-2	-12
6	16	-1	-16
7	13	0	0
8	18	1	18
9	2	2	4
10	1	3	3
	$\Sigma f = 60$		$\Sigma fd = -15$

Mean deviation $= \frac{\Sigma fd}{\Sigma f}$

$= \frac{-15}{60}$

$= -0.25$

Mean = 7 - 0.25 = 6.75

2. Total number of students = 30 + 25 = 55

   Total number of marks (%) from Form 11A = 30 x 55 = 1650

   Total number of marks (%) from Form 11B = 25 x 44 = 1100

   Total number of marks from all 55 students = 2750

   Mean (%) $= \frac{\text{total number of marks}}{\text{total number of students}}$

   $= \frac{2750}{55}\%$

   $= 50\%$

Variance and standard deviation

The tables below show the average rainfall in Lilongwe and in Edinburgh, Scotland, in 2009.

Rainfall in Lilongwe in 2009

Month	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**	**Year**
Precipitation (mm)	208	199	131	39	7	1	1	2	3	8	66	178	843

Mean rainfall for Lilongwe is 70.25mm

Rainfall in Edinburgh in 2009

Month	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**	**Year**
Precipitation (mm)	57	42	51	41	51	51	57	65	67	65	63	58	668

Mean rainfall for Edinburgh is 55.66mm

You can see that the mean rainfall for Lilongwe is higher than that for Edinburgh. Compare the tables. What else could you say about the rainfall for each of the cities?

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There are many things that you could have said. Some examples are:

- In May to October, the rainfall in Lilongwe is much smaller than the rainfall in Edinburgh.
- The rainfall in Edinburgh seems to be all round about the same number.
- The rainfall in Lilongwe is very high in November to January.

You are now looking at the **dispersion** of how the separate values are spread about the mean.

Here are two histograms showing the distribution of the heights of two groups of 100 men. The mean heights are almost the same but the histograms look very different.

What do you notice about the dispersion or spread?

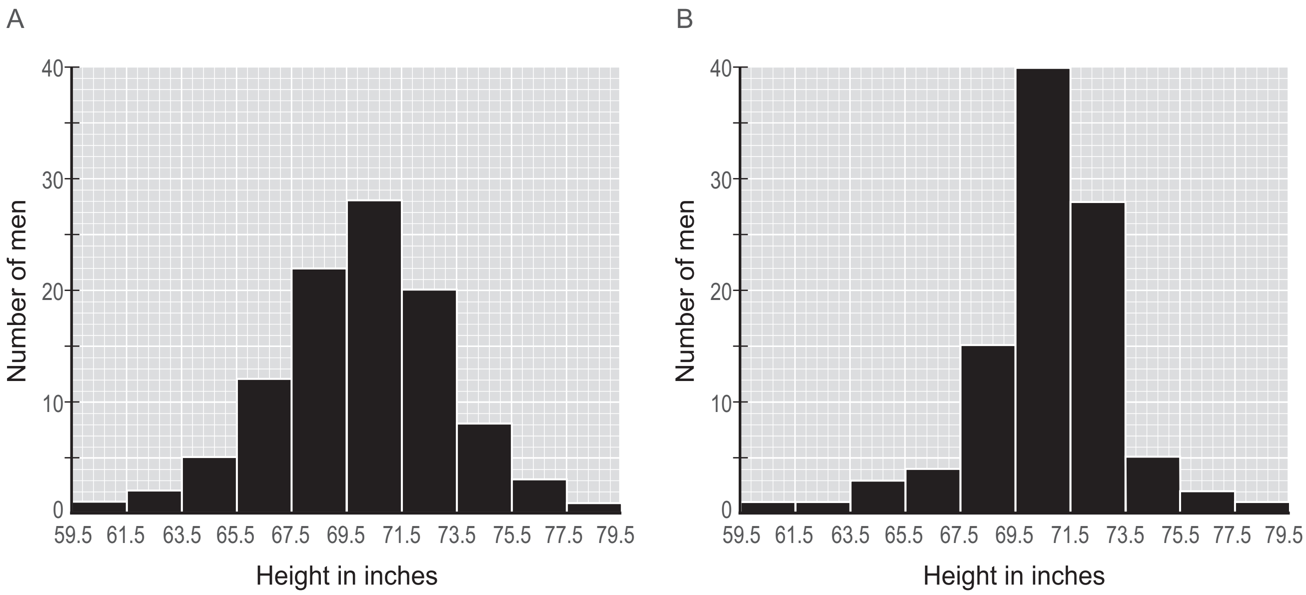


Figure 9: Histograms to show the height of two groups of 100 men

Describing them in words is useful but there is a method that we can use to show the dispersion mathematically. This is called finding the **variance** and then the **standard deviation.**

The standard deviation is a measure of how spread out your data is from the mean. It is given as a number. If the number is high, then the data is well spread out. If the number is low, then the data is clustered nearer to the mean; it is less spread out.

In our example of rainfall, the standard deviation for Lilongwe would be a much larger number than the standard deviation for Edinburgh.

Now look at the two graphs above. Which would have a larger standard deviation; A or B? ....

Example

Two batsmen from a cricket team made the following runs during the season. From the table you can see that Batsman A has a much larger dispersion than Batsman B. Batsman B is a much steadier batsman than Batsman A.

Batsman A	73	92	0	80	41	70	16	28
Batsman B	59	42	60	38	57	41	55	48

The mean for each is 50.

To find the standard deviation:

Complete each step and check your answers as you go.

Step 1

For both A and B separately, find the deviations from the mean for each of the scores.

Batsman A	73	92	0	80	41	70	16	28
Deviation								

Batsman B	59	42	60	38	57	41	55	48
Deviation								

Step 2

Square the deviations.

Batsman A	73	92	0	80	41	70	16	28
Deviation	23	42	-50	30	-9	20	-34	-22
$(\text{deviation})^2$								

Batsman B	59	42	60	38	57	41	55	48
Deviation	9	-8	10	-12	7	-9	5	-2
$(\text{deviation})^2$								

Step 3

Now calculate the variances.

Batsman A	73	92	0	80	41	70	16	28
Deviation	23	42	-50	30	-9	20	-34	-22
$(\text{deviation})^2$	529	1764	2500	900	81	400	1156	484

Batsman B	59	42	60	38	57	41	55	48
Deviation	9	-8	10	-12	7	-9	5	-2
$(\text{deviation})^2$	81	64	100	144	49	81	25	4

$$\text{Variance} = \frac{\text{sum of (deviation)}^2}{\text{number of values}}$$

For Batsman A:

$$\text{Variance} = \frac{529 + 1764 + 2500 + 900 + 81 + 400 + 1156 + 484}{8}$$

$$= \frac{7814}{8} \quad = 976.75$$

Now you find the variance for Batsman B.

Step 4

You should have found the variance for B to be 68.5.

Now square root the variance to give the standard deviation.

For Batsman A:

Standard deviation $= \sqrt{976.75}$

$= 31.252... = 31.3$ (to 1dp)

The standard deviation for Batsman B should be 8.3 (to 1dp). Check this answer.

The standard deviation for A is almost 4 times the standard deviation for B. This means that distribution A is definitely more spread out than distribution B.

In textbooks the following formulae are also given:

$$\textit{variance} = \frac{\Sigma f(x - \bar{x})^2}{\Sigma f}$$

and

$$\textit{standard deviation} = \sqrt{\frac{\Sigma f(x - \bar{x})^2}{\Sigma f}}$$

Can you see a connection between these and the formulae you have just used?

These are the formulae that you need to use when given frequencies.

Activity 4

1. MSCE Paper 2 2008 Q9b

   The table below shows the deviation from a mean of marks and the frequencies of the marks students scored in a test. Using the information from the table, calculate:

   (a) total number of students

   (b) the mean

   (c) the standard deviation to 3 significant figures.

Mark	f	d	d^2	fd^2
20	1	-11	121	121
23	1	-8	64	64
26	2	-5	25	50
27	1	-4	16	16
30	3	-1	1	3
34	2	3	9	18
35	3	4	16	48
40	2	9	81	162

2. MSCE Paper 2 2009 Q5b

The ages of five students at Mayeso Secondary School are 14, 16, 19, 20 and 21. Calculate the variance, showing all your working.

Answers:

1

Mark	f	d	fd	d^2	fd^2
20	1	-11	-11	121	121
23	1	-8	-8	64	64
26	2	-5	-10	25	50
27	1	-4	-4	16	16
30	3	-1	-3	1	3
34	2	3	6	9	18
35	3	4	12	16	48
40	2	9	18	81	162
	$\Sigma f = 15$		$\Sigma fd = 0$		$\Sigma fd^2 = 482$

(a) Total number of students = $\Sigma f = 15$

(b) The estimated/assumed mean is 31 because the table shows that the deviation from the mean for 30 marks is -1.

Mean deviation $= \dfrac{\Sigma fd}{\Sigma f} = \dfrac{0}{15} = 0$

So mean $= 31 + 0 = 31$

(c) Variance $= \dfrac{\Sigma fd^2}{\Sigma f}$

$= \dfrac{482}{15} = 32.13$ (to 2dp)

Standard deviation $= \sqrt{32.13} = 5.67$ (to 2dp)

2.

x	d	d^2
14	-4	16
16	-2	4
19	1	1
20	2	4
21	3	9
$\Sigma x = 90$		$\Sigma d^2 = 34$

Mean $= \dfrac{90}{5} = 18$

Variance $= \dfrac{\Sigma d^2}{n} = \dfrac{34}{5} = 6.8$

Practice questions

1. Find the mean, variance and standard deviation of the following marks obtained by 7 students: 1, 5, 8, 7, 13, 11 and 4.

2. Once a week there is a market in the local village. The fruit stall sells fruit to 30 people.

   The table shows the different types of fruit sold. Each person only buys one type of fruit.

Type of fruit	Number of people
Bananas	7
Mangoes	12
Oranges	8
Lemons	3

   Draw a pie chart to represent the information.

3. Find the mean of a distribution where the assumed mean is 4.6, the sum of the deviations from the mean is -2.4 and there are 60 measurements in the distribution.

4. Asale was celebrating her birthday. She had a party to which her friends and family were all invited. The table shows the different age groups of the 50 people who attended the party.

   Find an estimate for the mean age of the people attending.

Age	Number of people
$0 \le$ age < 10	8
$10 \le$ age < 20	15
$20 \le$ age < 40	13
$40 \le$ age < 60	7
$60 \le$ age < 80	6
$80 \le$ age < 100	1

5. 20 students ran the 100 metres. Their times are shown in the table below.

   By using the mid-point of each class interval, copy and complete the table to find:

   (a) the estimated mean

   (b) the standard deviation of the data.

Time (in seconds)	f	Mid-point (x)	fx	d	d^2	fd^2
$10 < t \le 11$	3	10.5				
$11 < t \le 12$	7	11.5				
$12 < t \le 13$	5	12.5				
$13 < t \le 14$	3	13.5				
$14 < t \le 15$	2	14.5				

How am I doing?

	☺ **Easy** (Tick this box if you feel confident that you **understand this section well**)	**Fine** (Tick this box if you still **need a little work** on this section)	☹ **Difficult** (Tick this box if you still **need a lot of work** on this section)
Grouping data and illustrating it			
Drawing a frequency polygon			
Calculating the mean of ungrouped data			
Finding the standard deviation and variance of data			

Notes on what to do next:

Signed (by Scholar): .... Date:..................

Signed (by Tutor): .... Date:..................

Answers to practice questions

1. There are 7 students.

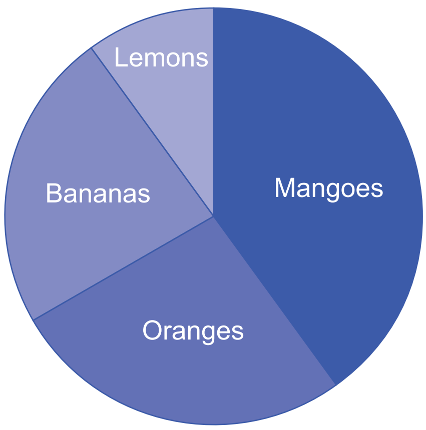
Marks (x)	d	d^2
1	-6	36
5	-2	4
8	1	1
7	0	0
13	6	36
11	4	16
4	-3	9
$\Sigma x = 49$		$\Sigma d^2 = 102$

Mean $= \frac{49}{7} = 7$ marks

Variance $= \frac{\Sigma d^2}{n} = \frac{102}{7} = 14.57$ (to 2dp)

Standard deviation $= \sqrt{14.57} = 3.82$ (to 2dp)

2.



Bananas = 84°

Mangoes = 144°

Oranges = 96°

Lemons = 36°

3. Mean $=$ assumed or estimated mean $+ \frac{\Sigma d}{n}$

$= 4.6 - \frac{2.4}{60}$

$= 4.6 - 0.04$

$= 4.56$

4.

Age	Number of people (f)	Mid-point (x)	fx
$0 \le$ age < 10	8	5	40
$10 \le$ age < 20	15	15	225
$20 \le$ age < 40	13	30	390
$40 \le$ age < 60	7	50	350
$60 \le$ age < 80	6	70	420
$80 \le$ age < 100	1	90	90

Estimated mean $= \frac{1515}{50} = 30.3$ years old

5.

Time (in seconds)	f	Mid-point (x)	fx	d	d^2	fd^2
$10 < t \leq 11$	3	10.5	31.5	-1.7	2.89	8.67
$11 < t \leq 12$	7	11.5	80.5	-0.7	0.49	3.43
$12 < t \leq 13$	5	12.5	62.5	0.3	0.09	0.45
$13 < t \leq 14$	3	13.5	40.5	1.3	1.69	5.07
$14 < t \leq 15$	2	14.5	29	2.3	5.29	10.58

Estimated mean $= \dfrac{244}{20} = 12.2$

Variance $= \dfrac{28.2}{20} = 1.41$

Standard deviation $= \sqrt{1.41} = 1.19$ (to 2dp)

MSCE M6:
Angles and circles

What you are studying and why

Subject: Mathematics Unit M6

This is the sixth mathematics unit for your MSCE study.

At the end of this unit you should be able to:

1. revise angle properties of parallel lines and triangles
2. revise the construction of special angles, accurately using only a ruler and a pair of compasses
3. state the chord properties of a circle
4. state the angle properties of a circle
5. construct a tangent to a circle and construct tangents from an external point
6. solve problems using the chord, angle and tangent properties of a circle.

Introduction

The geometry which is required for the MSCE has two main topics associated with angles:

- construction of angles and tangents using only a ruler and a pair of compasses
- questions on the chord, angle and tangent properties of circles.

You must also remember some of the earlier work on angles. We will quickly revise:

- angles associated with triangles and parallel lines
- the construction of special angles.

The rest of the time will be spent on the chord, angle and tangent properties associated with circles.

Angles associated with triangles and parallel lines

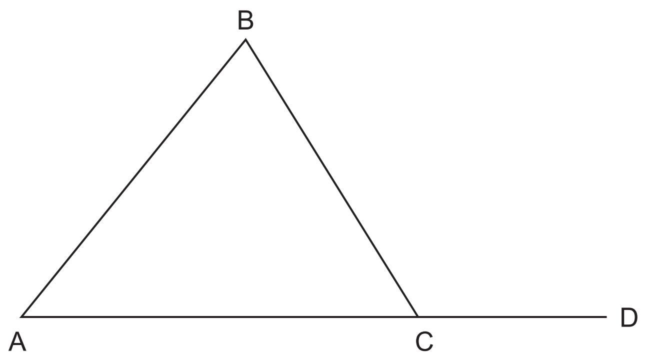


Figure 1

Using Figure 1, complete the following sentences:

The angles in a triangle add up to ....

Angles on a straight line add up to ....

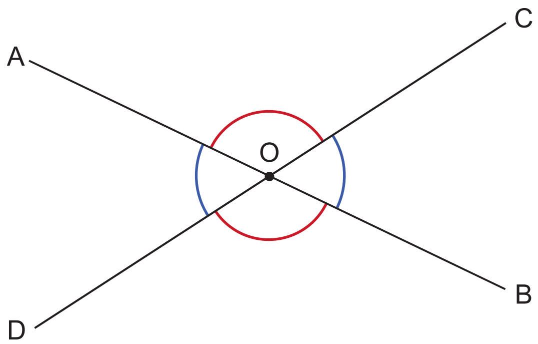
The **exterior angle** in a triangle is equal to ....

It might help to show this on the diagram in some way.

If you are unsure of your answers to these or any other questions in this unit, refer to the textbooks in your tutor group book box.

You might also need to know that when two straight lines cross, **vertically opposite angles** are formed.

Vertically opposite angles are equal:



Angle AOC = Angle DOB
Angle AOD = Angle COB

Figure 2

Sometimes straight lines never meet. They always remain the same distance apart. These are called **parallel lines**. How do we show that lines are parallel on a diagram?

Parallel lines have special angles associated with them. Write down the names of these three types of angles.

- ....
- ....
- ....

Alternate angles

Alternate angles on parallel lines are equal.

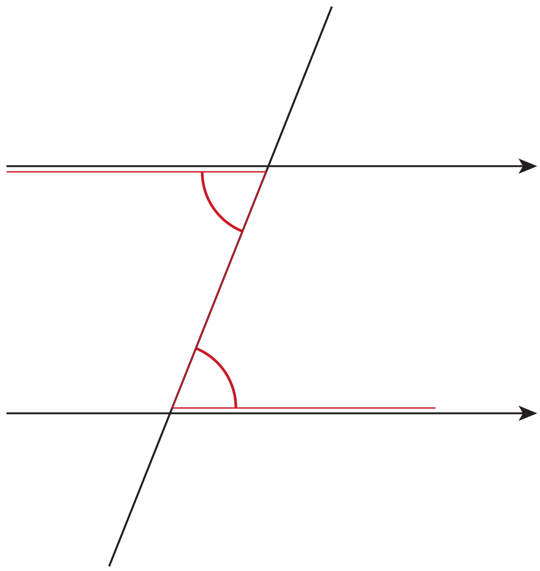


Figure 3

These are sometimes called ‘z’ angles but you must not call them this in the examination. You must always use their correct mathematical name; alternate angles.

Corresponding angles

Corresponding angles on parallel lines are equal. These are sometimes called ‘f angles but you must not call them this in the examination.

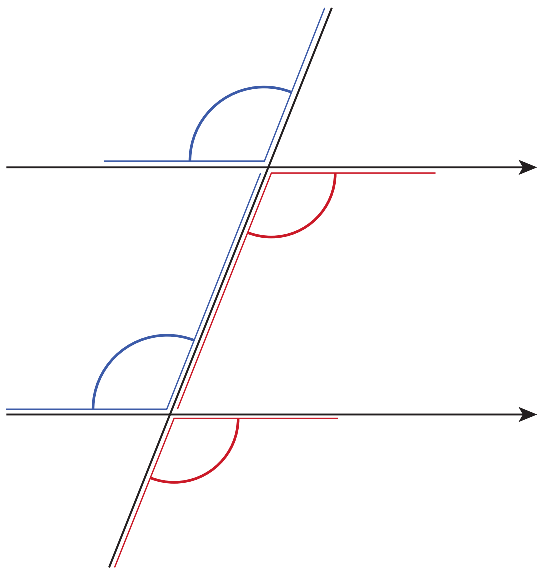


Figure 4

You can clearly see that the red arcs show corresponding angles. Why are the two blue angles equal?

....

Interior or allied angles

These are not equal. They are **supplementary.** This means that they add up to 180°.

Allied angles are angles inside the parallel lines.

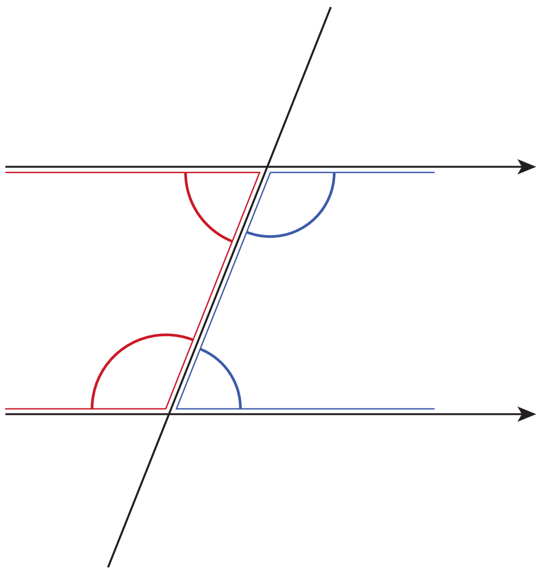


Figure 5

The construction of special angles

It is common for the MSCE to include a question on constructions in the examination paper. You may be asked to draw a triangle using certain lengths and angles. These angles are often one of the special angles of 90°, 45°, 30°, 60° or 120°.

You cannot use a protractor to draw these angles in the examination. You are only allowed to use a ruler and a pair of compasses to draw them. Make sure that you use a sharp pencil.

Drawing right angles (90° angles)

A right angle is formed when you draw a line perpendicular to a given line.

AB is perpendicular to CD.

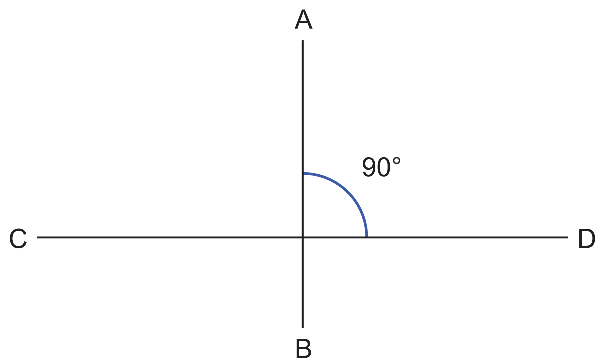


Figure 6

There is another way of showing that lines are at 90° to each other. Draw this on the diagram. If you cannot remember this, look at the perpendicular bisector shown in Figure 8.

Follow the instructions below to draw a **perpendicular** line to the line RS at O. (You may choose to copy the diagram below onto the middle of a large piece of paper to give yourself more space.)

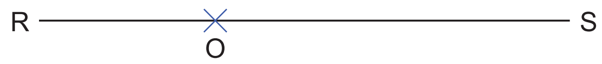


Figure 7

To draw a perpendicular line to a straight line:

- Place your compasses to a radius not greater than OR. Place your compasses at centre O, and draw arcs (curved lines) to cut the line RS in two places; one on each side of O. Label these A and B.
- Open your compasses to a slightly larger radius. With centre A, draw an arc above the line RS.
- With the same radius, centre B, draw another arc above the line RS.
- These two arcs should cross. Join this new point to O.

You have now constructed a perpendicular line through O.

You may also be asked to draw the **perpendicular bisector** of a line. Complete this sentence:

A perpendicular bisector is a line that cuts a given line in half and is at ......° to the given line.

- Open your compasses again, this time drawing an arc above and below the line with centre A.
- Then, repeat this for centre B. Remember to draw arcs above and below the line.
- If the arcs drawn from A and B do not cross, then you need to open your compasses to a larger radius and try again. The radius needs to be larger than half the length of the line.
- Join the two crosses to give the perpendicular bisector of the line.

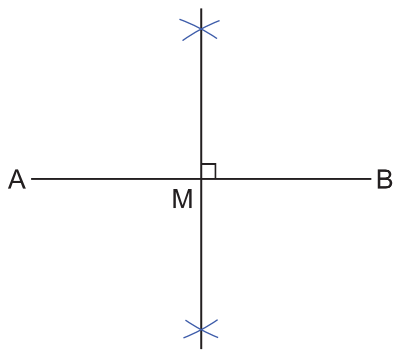


Figure 8

Try this out for yourself on a blank piece of paper. Remember to draw the line in the middle of the page as you will need plenty of space above and below the given line.

Drawing angles of 45°

To produce an angle of 45° you can bisect a right angle. First, you need to know how to bisect an angle. Remember, the word ‘bisect’ means to cut something into two equal parts.

To bisect an angle:

- With any radius and centre A, draw an arc to cut the arms of the angle at B and C.

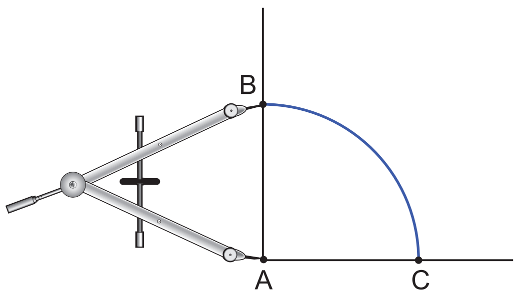


Figure 9

- With the same radius, draw an arc inside the arms of the original angle from centre B.
- Repeat this with centre C.
- The two arcs should cross each other at a point D. If the arcs do not cross then repeat the process with a larger radius.

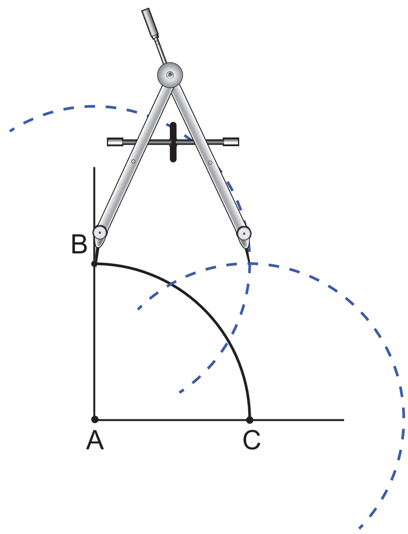


Figure 10

- Join AD. This line now bisects the angle BAC. So angle BAD = angle DAC.

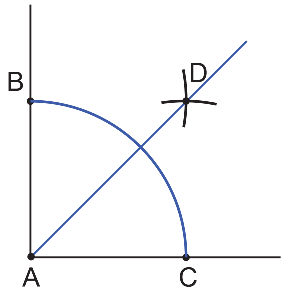


Figure 11

To produce an angle of 45°, you simply construct a right angle (as we did earlier) and then bisect it to get one of 45°.

Angles of 30° and 60°

To construct 30° and 60° angles, you must first construct an equilateral triangle, which has angles of 60°, and then bisect one of the angles to make 30°.

To construct an equilateral triangle:

- Draw any line AB.
- Using a radius of AB, draw an arc above the line.
- From centre B, with the same radius, draw another arc above the line.
- The two arcs will cross at C.

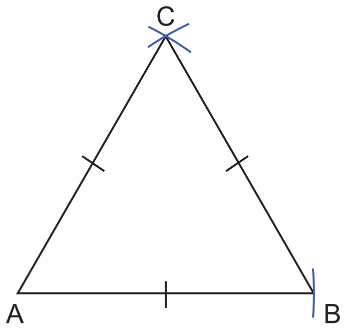


Figure 12

On a clean sheet of paper, following the instructions above, draw an equilateral triangle ABC of side 10 cm.

Bisect the angle CAB to form two 30° angles at the vertex A.

Should you be asked to form an angle of 120°, then you need to construct another equilateral triangle as shown in Figure 13 on the next page.

Keep the same radius as you used for constructing the first triangle, but this time work with centres B and C.

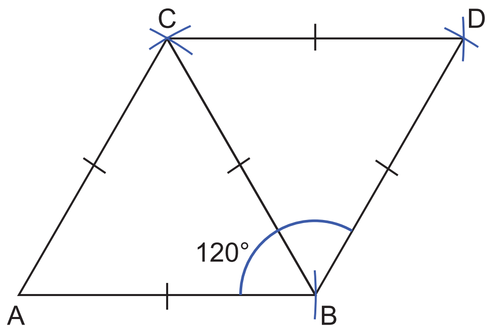


Figure 13

The MSCE examination may also ask you to draw a circle which passes through all three vertices of a triangle. This circle is on the outside of the triangle and is called a circumcircle or an escribed circle.

A circle that passes through the mid-points of each of the sides of a triangle is formed inside the triangle. It is called an inscribed circle.

Draw sketches below showing a triangle with an inscribed circle and the same triangle with its circumcircle.

To draw a circumcircle, find where the perpendicular bisectors of each side meet. This is called the circumcentre of the triangle. Use it to draw your circle.

To draw an inscribed circle, find where the angle bisectors of each side meet. This is called the incentre of the triangle. Use it to draw your circle.

Example

Q6b Paper II 2009

1. Using a ruler and compasses only, construct in the same diagram:
   (a) a triangle ABC in which AB = 10 cm, AC = 7 cm and angle BAC = 60°
   (b) two points M and N which lie on the circumference of circle ABC and which are equidistant from B and C.
2. Measure and state the length of MN.

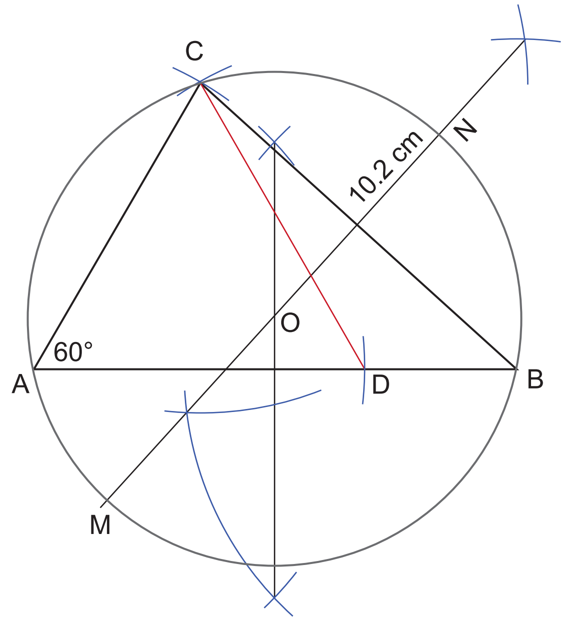


Figure 14

If you have time you might like to work this through for yourself. Use the steps given in the answer to help you. You should end up with a diagram similar to Figure 14.

Answers:

1. (a)

- Using a ruler, draw line AB.
- From centre A, with a 7 cm radius, draw an arc above the line and an arc to cut AB at the point D.
- From centre D, with the same radius, draw an arc to cut the first arc at C.
- Join A to C and C to D. Triangle ACD is an equilateral triangle so angle BAC is 60°.
- Now join B to C. This is the required triangle.

(b) You are being asked to use the circumcircle.

- Construct the perpendicular bisector of BC.
- Construct the perpendicular bisectors of AB and AC.
- This diagram only shows two perpendicular bisectors. This is so you can see what is going on more easily. You should always draw three. The third is a check of the other two. They should all cross in the same place, at O.
- With centre O, draw in the circumcircle.
- The points M and N are where the perpendicular bisector of BC crosses this circle.

2. MN = 10.2 cm. Notice that MN is the diameter of the circumcircle.

Constructing tangents

Some of the construction questions set may involve tangents.

A tangent to a circle is a straight line that .... the circle at a single point.

Draw a sketch of a tangent in the space below.

To construct a tangent to a point on a circle:

- Draw a straight line from the centre of the circle O to the given point P on the circle. Extend this line through P to Q.
- Construct a perpendicular to the line OQ at P.

The perpendicular line APB is the tangent to the circle O at the point P.

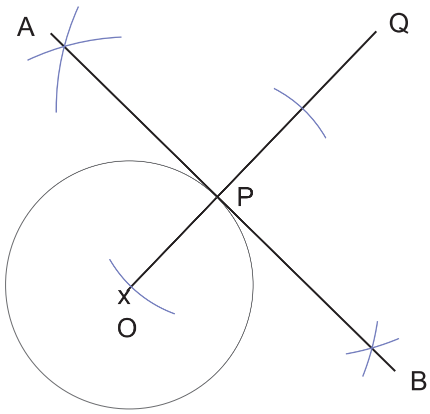


Figure 15

To construct tangents to a circle from a given external point:

- Draw a straight line from the centre of the circle O to the external point P.
- Construct the perpendicular bisector of the line OP. This perpendicular, AB, intersects the line OP at M. M is the mid-point of OP.

- Open your compasses to the length of the line OM and with centre M, draw two arcs; one above and one below OP, cutting the circumference of the circle at two points C and D.
- Join P to C and P to D. These are the two required tangents.

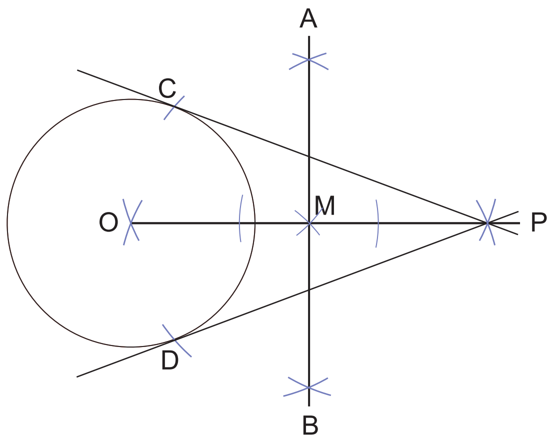


Figure 16

Activity 1

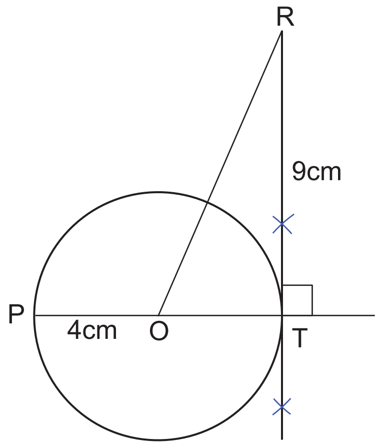
Question 11a Paper II 2008

1. Using a ruler and a pair of compasses only, construct in the same diagram:
   (a) a circle with centre O and a radius of 4 cm
   (b) a diameter PT and a tangent to the circle at T
   (c) a point R on the tangent such that TR = 9 cm.
2. Using a ruler and a pair of compasses only:
   (a) draw a circle with centre O and a radius of 3 cm
   (b) mark a point P on the circumference of the circle and join OP
   (c) draw a right angle OPM using only compasses and a ruler
   (d) from centre O, draw the arc of a circle with a radius of 8 cm to cut PM at T
   (e) join O and T; then measure and state the length of OT.

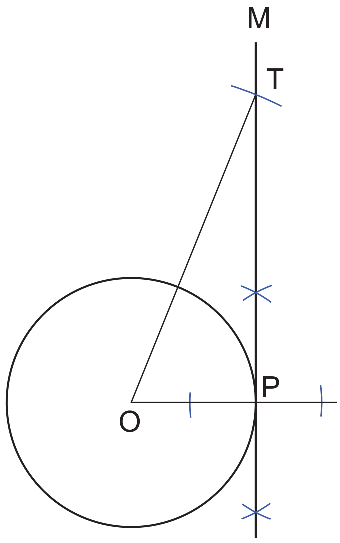
You might find it useful to sketch a rough diagram before you begin to construct it accurately.

Answers:

1.



2.



Angles associated with circles

Before you look at the circle theorems themselves, you may be asked to work with chords as well as with tangents. Use your tutor group book box to complete the following statements:

A chord is a straight line joining two points on the .... of a circle.

If it also passes through the centre, it is called a ....

Below are some facts that you should know about chords. Draw a sketch to show each of these theorems on a separate sheet of paper. Their converses are written underneath. What do we mean by a converse?

....

The line from the centre of a circle bisecting a chord is perpendicular to it.

Conversely, the line from the centre perpendicular to a chord bisects it.

Chords which are equidistant from the centre of a circle are equal.

Conversely, chords which are equal are equidistance from the centre of the circle.

Below is an example using these theorems.

Example

Q7a Paper II 2009

This figure shows a circle ACB with centre O.

OM = 3x cm

MC = 2x cm and angle OMB = 90°

Find AB in terms of x.

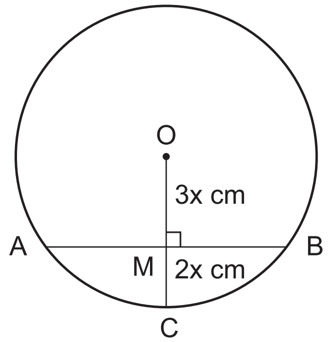


Figure 17

Answer:

AB is a chord. OC is a line from the centre perpendicular to this chord. So OC bisects AB. Therefore: AM = MB

Join OA. In the right angled triangle OAM, AO = 3x + 2x = 5x (radius of the circle).

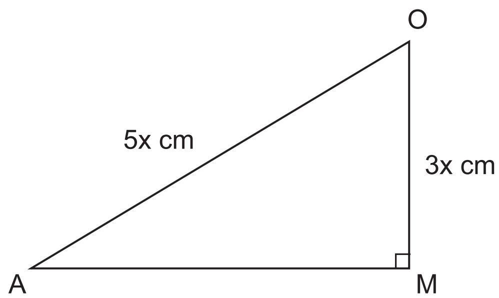


Figure 18

By Pythagoras' theorem:

$AO^2 = AM^2 + OM^2$

$(5x)^2 = AM^2 + (3x)^2$

So: $AM^2 = (5x)^2 - (3x)^2$

$= 25x^2 - 9x^2$

$= 16x^2$

$AM = \sqrt{16x^2}$

$= 4x$ cm

$AB = 2AM$

So: $AB = 8x$ cm

You may also have recognised this as a 3 : 4 : 5 triangle.

So AM = 4x cm

The circle theorems

The angle subtended at the centre of a circle is twice any angle subtended at the circumference.

You might like to replace the word 'subtended' by 'made'.

The angle made at the centre of the circle is twice any angle made at the circumference.

This is easier to remember but these angles must be made from the same two points, A and B.

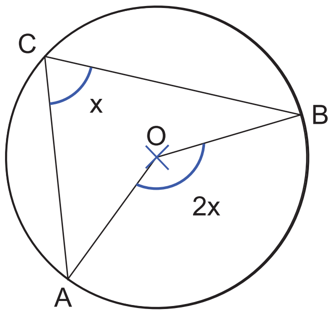


Figure 19

Put a finger on each of A and B, follow your fingers along the lines AO and BO until they meet at O. This is the angle made at the centre.

Put your fingers on A and B again. This time follow your fingers along the lines AC and BC until they meet at C. This is the angle made at the circumference.

Angle AOB = Angle ACB

Could you draw another angle on the diagram that is also ½ of AOB?

....

Check your answer with a fellow scholar. Have you all drawn this in the same place? ....

What does this tell you?

....

....

It tells you that any angles made at the circumference from the same two points are equal.

Angles in the same segment are equal.

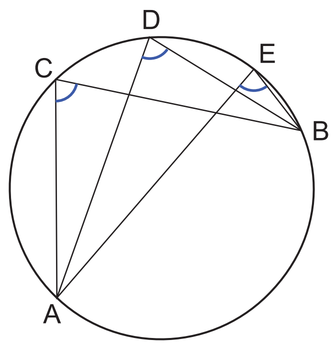


Figure 20

They are in the same segment because A and B are the ends of a chord.

Angle ACB = Angle ADB = Angle AEB

The chord cuts the circle into two segments; the major segment and the minor segment.

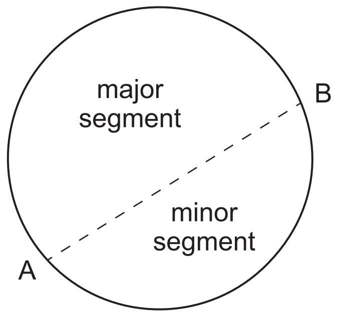


Figure 21

A diameter passes through the centre of a circle. This diameter cuts the circle into two parts of equal sizes. These are called semicircles. You will remember that an angle made at the centre, from two points on the circumference, is twice that made at the circumference. Here the angle at the centre is 180°, so the angle made at the circumference (at C) is half of this, which is 90°.

The angle made in a semicircle by the end points of a diameter is always a right angle.

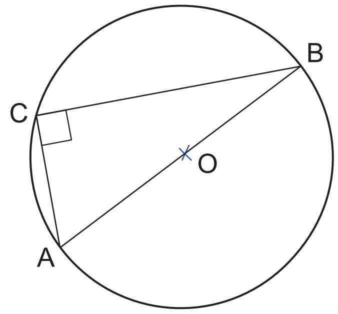


Figure 22

The remaining two theorems deal with **cyclic quadrilaterals**. A cyclic quadrilateral is a quadrilateral in which all four .... lie on the circumference of a circle. On the circle below, draw and label any cyclic quadrilateral.

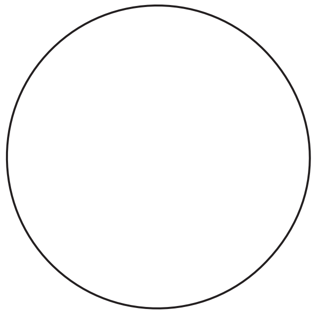


Figure 23

The opposite angles of a cyclic quadrilateral are supplementary.

This means that they add up to .... degrees.

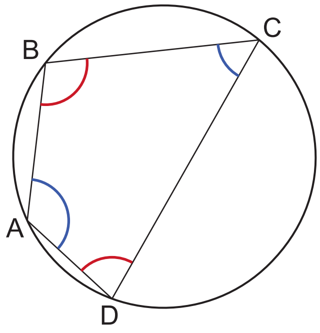


Figure 24

Angle DAB + Angle .... = 180°

Angle .... + Angle ADC = 180°

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

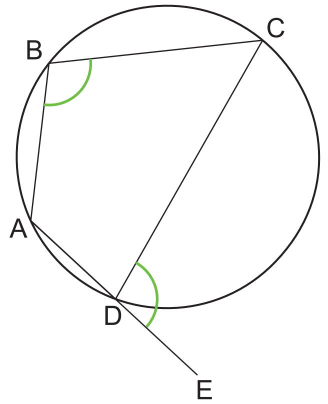


Figure 25

In this diagram the line AD has been extended to give an exterior angle at D.

In the diagrams below, different lines have been extended. On each of these diagrams the exterior angles have been shown. Mark the interior opposite angle on each.

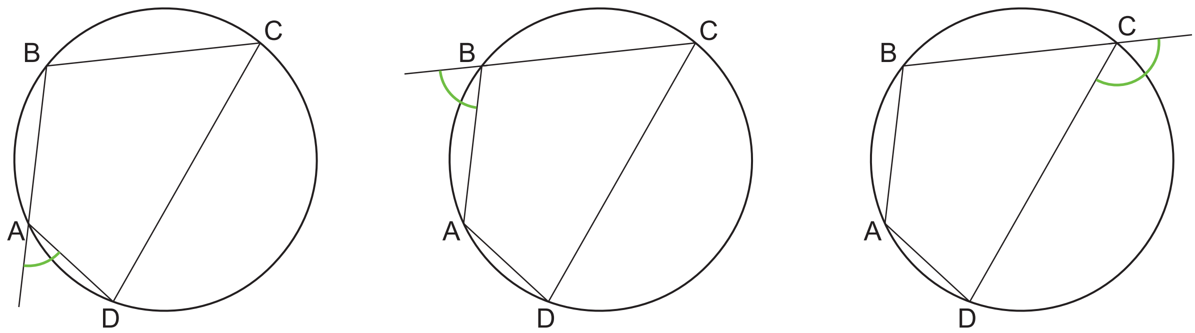


Figure 26

Example

Q6b Paper II 2008

This figure shows a circle ABC with centre O. AOB and ABC are triangles in which AB = BC. If angle AOB = 80°, calculate Angle OAC.

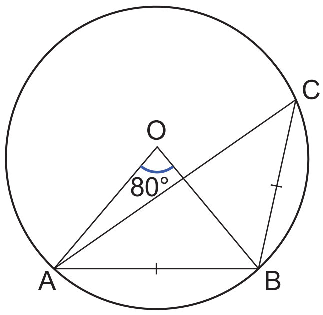


Figure 27

Answer:

When finding missing angles, it is a good idea to mark the angles given in the question in pen.

You can then work in pencil on the diagram. Once you have worked out the angle then carefully write it down in stages giving your reasoning.

Triangle AOB is isosceles. (AO = BO radii)

So:

Angle OAB = Angle OBA = (180° - 80°) ÷ 2
= 100° ÷ 2
= 50°

Angle AOB = 2 × Angle ACB (Angle at the centre is twice that at the circumference.)

Or:

Angle ACB = ½ × Angle AOB (Angle at the circumference is half that at the centre.)
= ½ × 80°
= 40°

Triangle ACB is isosceles. (Notice the little lines showing equivalence.)

So:

Angle CAB = 40°

Angle OAC = Angle OAB - Angle CAB
= 50° - 40°
= 10°

Now check that you understand by working through this step by step. You can write each angle found on the original diagram to help you.

Notice how all the reasoning was shown at every stage of the working. It is very important that you do this as you will get extra marks for giving these reasons.

Here are some questions for you to try yourself. Only after you have tried it yourself should you check the answers at the end of the unit.

Activity 2

In this activity, make sure that you always show the reasons for your answers.

1. AB is a diameter. Angle CAB = 52°.
   Calculate Angles ACB and CBO.

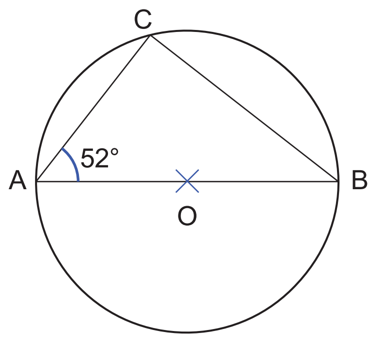


Figure 28

2. Angle DAC = 25° and Angle CEB = 60°.
   Calculate Angles CBD and ACB.

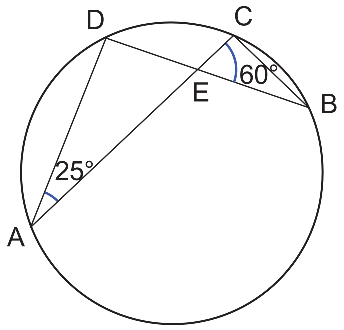


Figure 29

3. Angle DAC = 30° and Angle DEB = 100°.
   Calculate angles DCA and CDA.

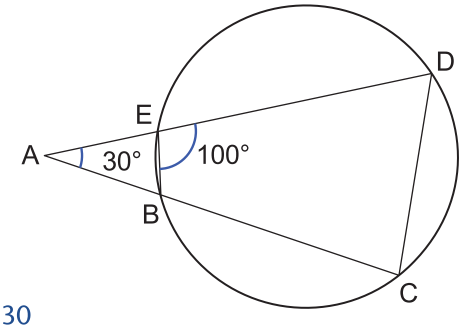


Figure 30

Answers:

1.	Angle ACB = 90°		(angle in a semicircle)
	Angle CBO	= 180° - (52° + 90°) = 180° - 142° = 38°	(angles in a triangle add up to 180°)
2.	Angle CBD = 25°		(angles in same segment)
	Angle CEB = Angle DEA		(vertically opposite angles)
	Angle ACB	= 180° - (60° + 25°) = 180° - 85° = 95°	(angles in a triangle add up to 180°)
3.	Angle DCA	= 180° - 100° = 80°	(opposite angles of a cyclic quadrilateral are supplementary)
	Angle CDA	= 180° - (80°+ 30°) = 180° - 110° = 70°	(angles in a triangle add up to 180°)

You have covered the main circle theorems but there are some more, the tangent theorems, which involve the alternate segment theorem. These theorems are listed below. The converses of these are also true.

Tangent theorems

There are three theorems concerning tangents to a circle.

1. The angle between a tangent and the radius, at its point of contact, is a right angle.

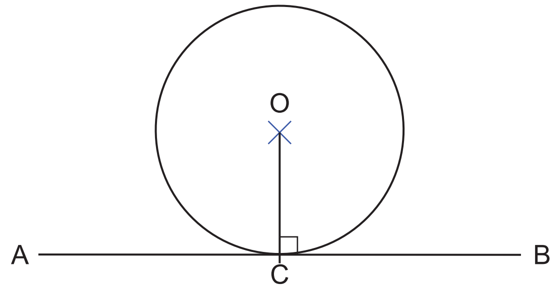


Figure 31

Angle OCA = Angle OCB = 90°.

2. The lengths of the tangents, from any two points on the circumference to the external point where they meet, are equal.

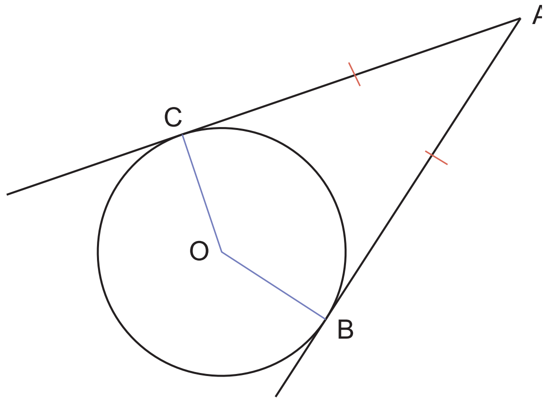


Figure 32

AC = AB.

3. The angle between a tangent and its chord is equal to the angle in the alternate segment.

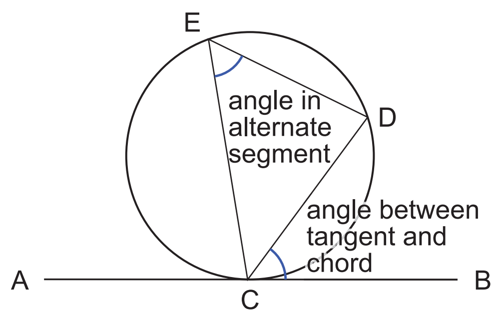


Figure 33

Angle BCD = Angle DEC.

Also, Angle ACE = Angle CDE.

There are often questions in the MSCE on tangents to circles. Here is an example.

Example

Q5 Paper I 2008

In this figure, QN is a tangent to the circle LMN at N. If angle QNM = 55° and angle LNM = 24°, calculate angle NML.

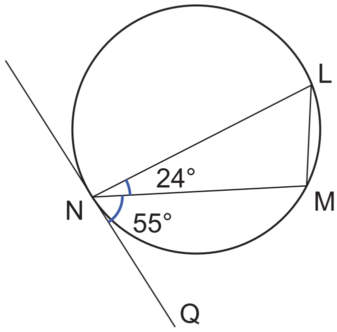


Figure 34

Answer:

Angle QNM = Angle NLM = 55° (angles in the alternate segment)

So, in triangle NML:

Angle NML	= 180° - (24° + 55°)	(angles in a triangle add up to 180°)
	= 180° - 79°	
	= 101°	

Activity 3

1. Q4 Paper I 2009

   This figure shows a circle ABC with centre O. OCP is a straight line and AP is a tangent to the circle at A. If angle ABC = 35°, calculate the value of angle APO.

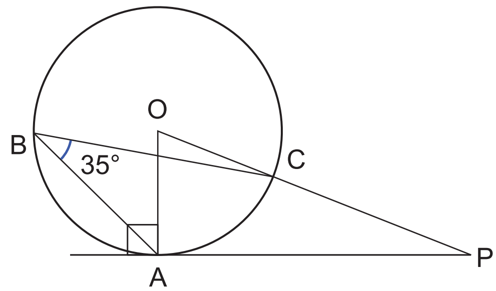


Figure 35

2. AB and BC are two equal tangents from the external point B. Angle ABC = 40°.

   Find angle ADC.

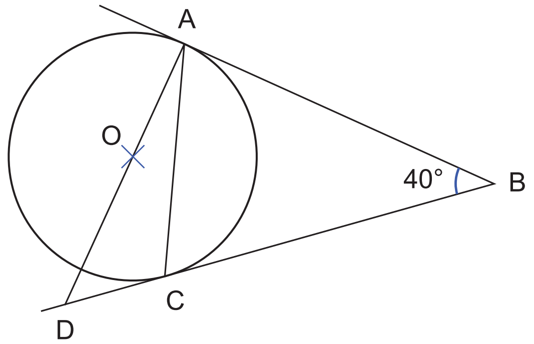


Figure 36

Answers:

1. Angle AOP = 70° (angle at centre twice that at circumference)
   Angle OAP = 90° (tangent perpendicular to radius)

   In triangle OAP:

   Angle APO = 180° - (70° + 90°) (angles in a triangle add up to 180°)
   = 180° - 160°
   = 20°

2. AB = BC (tangents from an external point are equal)

   Triangle ABC is isosceles.

   So:

   Angle BAC = Angle ACB
   = (180° - 40°) ÷ 2
   = 140° ÷ 2
   = 70°

   Angle OAB = 90° (tangent perpendicular to radius)

   So:

   Angle OAC = Angle OAB - Angle BAC
   = 90° - 70°
   = 20°

   Angle ACD = 180° - 70° (angles on a straight line)
   = 110°

   In triangle ADC:
   Angle ADC = 180° - (110° + 20°) (angles in a triangle add up to 180°)
   = 50°

Practice questions

1. ABCD is a cyclic quadrilateral and KABL is a straight line so that Angle KAD = 70° and Angle LBC = 68°.

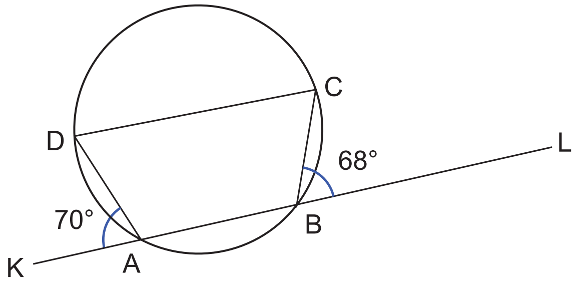


Figure 37

Calculate the following angles:

(a) angle ABC

(b) angle DAB

(c) angle ADC

(d) angle BCD.

2. ABCD is a cyclic quadrilateral with AB as a diameter. Angle CAB = 34° and AD = DC.

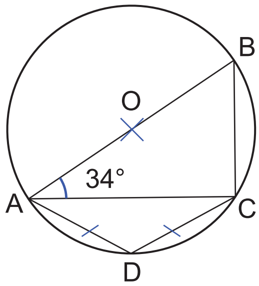


Figure 38

Calculate all the angles of the quadrilateral. These are angles CBA, ADC, BAD and DCB.

3. OP and OQ are radii of circle centre O such that angle PQO = 20°. TP is a tangent at P.

ORM is drawn such that angle POR = 50°. The tangent at P meets ORM at T. Show that TP = TR.

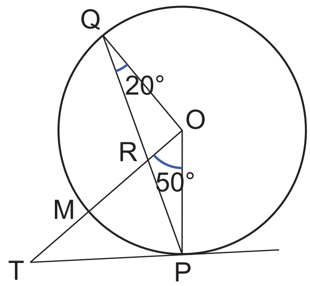


Figure 39

How am I doing?

	☺ **Easy** (Tick this box if you feel confident that you **understand this section well**)	**Fine** (Tick this box if you still **need a little work** on this section)	☹ **Difficult** (Tick this box if you still **need a lot of work** on this section)
Revising angle properties of parallel lines and triangles			
Constructing different angles using a ruler and a pair of compasses			
Stating the chord properties of a circle			
Stating the angle properties of a circle			
Constructing a tangent to a circle and tangents from an external point			
Solving problems using the chord, angle and tangent properties of a circle			

Notes on what to do next:

Signed (by Scholar): .... Date:..................

Signed (by Tutor): .... Date:..................

Answers to practice questions

1.	(a) Angle ABC = 180° - angle LBC		(angles on a straight line)
	= 180° - 68°		
	= 112°		
	(b) Angle DAB = 180° - angle KAD		
	= 180° - 70°		(KAB is a straight line)
	= 110°		
	(c) Angle ADC	= angle CBL	(Exterior angle of cyclic quad equals interior opposite angle)
	= 68°		
	(d) Angle BCD	= angle KAD	(Exterior angle of cyclic quad equals interior opposite angle)
	= 70°		
2.	Angle ACB = 90°		(angle in a semicircle)
	Angle CBA	= 180° - (90° + 34°)	(angles in a triangle add up to 180°)
		=180° - 124°	
		= 56°	
	Angle ADC	= 180° - 56°	(opposite angles of cyclic quad are supplementary)
		= 124°	
	Angle ACD	= angle DAC	(angles in an isosceles triangle)
		= (180° - 124°) ÷ 2	
		= 56 ÷ 2	
		= 28°	
	Angle BAD	= 28° + 34°	
		= 62°	
	Angle DCB	= 90° + 28°	
		= 118°	
3.	In triangle OPQ:		
	Angle OPQ	= angle OQP	(angles in an isosceles triangle)
		= 20°	
	Angle OPT	= 90°	(tangent perpendicular to radius)
	Angle TPR	= 90° - 20°	
		= 70°	
	In triangle OPT:		
	Angle OTP	= 180° - (50° + 90°)	(angles in a triangle add up to 180°)
		= 180° - 140°	
		= 40°	
	In triangle RPT:		
	Angle TRP	= 180° - (40° + 70°)	(angles in a triangle add up to 180°)
		= 180 - 110°	
		= 70°	

Since angle TRP = angle TPR = 70°:

Triangle TPR is isosceles and so TP = TR.

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