TESS+
India

Learning from misconceptions: algebraic expressions

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TESS-India (Teacher Education through School-based Support) aims to improve the classroom practices of elementary and secondary teachers in India through the provision of Open Educational Resources (OERs) to support teachers in developing student-centred, participatory approaches. The TESS-India OERs provide teachers with a companion to the school textbook. They offer activities for teachers to try out in their classrooms with their students, together with case studies showing how other teachers have taught the topic and linked resources to support teachers in developing their lesson plans and subject knowledge.

TESS-India OERs have been collaboratively written by Indian and international authors to address Indian curriculum and contexts and are available for online and print use (http://www.tess-india.edu.in/). The OERs are available in several versions, appropriate for each participating Indian state and users are invited to adapt and localise the OERs further to meet local needs and contexts.

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Video resources

Some of the activities in this unit are accompanied by the following icon: . This indicates that you will find it helpful to view the TESS-India video resources for the specified pedagogic theme.

The TESS-India video resources illustrate key pedagogic techniques in a range of classroom contexts in India. We hope they will inspire you to experiment with similar practices. They are intended to complement and enhance your experience of working through the text-based units, but are not integral to them should you be unable to access them.

TESS-India video resources may be viewed online or downloaded from the TESS-India website, http://www.tess-india.edu.in/). Alternatively, you may have access to these videos on a CD or memory card.

What this unit is about

Algebraic expressions are mathematical sentences such as 3x + 4. They do not have an equals sign (=), which makes them different from algebraic equations. Algebraic expressions play an important role in the mathematics curriculum and in mathematics in general. In order to progress and do well in mathematics, students need to be able to read and write expressions, and to be skilled in computations and manipulations of algebraic expressions.

For many students the issue with learning about algebraic expressions is that the work is purely a question of memorising and following algorithms. The power and beauty of algebra to express generality, describe relationships between variables and constants, and explore possibilities in a playful and creative way, is often lost. Algebra and its expressions are considered as the language of mathematics, and are used to describe relationships between people, thoughts, elements and structures. Students often do not experience this in their learning at school and thus cannot see the purpose of learning about algebraic expressions and how they relate to real life, apart from passing mathematics examinations.

This unit will explore some different approaches to teaching algebraic expressions by using and developing contexts to help students see the purpose in algebraic expressions. It will first explore the role of variables and constants in a real-life context; it will then look at the power of substitution and how this can stimulate thinking creatively and learning from misconceptions.

Pause for thought

Think of your own classroom. What do you think are the issues for your students when learning about algebraic expressions? What do you think they like about it? What do they dislike? What do you think they would like to be different?

Then think back to when you were learning about algebraic expressions in school – how did you feel? What about it did you like? What about it did you dislike? What would you have liked to be different?

What you can learn in this unit

- How to help students to identify relations between variables and constants.
- Some ideas on using and developing contexts to help students see the purpose in algebraic expressions.
- Some ideas on eliciting misconceptions and using them as a learning tool.

The learning in this unit links to the NCF (2005) and NCFTE (2009) teaching requirements in Resource 1.

1 Variables and constants in a real-life context

Nehru Place in Delhi, Asia's largest market for computers and peripherals, can always become crowded. During business hours there is an extremely dynamic atmosphere. Everything from a hawker to the car park or the number of staff required in a shop is affected by how fast the environment changes from morning to evening (Figure 1). This change in an environment is called dynamics.

Figure 1 Dynamics in real life: Nehru Place, Delhi, when it is quiet (left) and busy (right).

Professional mathematicians develop models to predict and describe these dynamics. In doing so they make it possible for urban planners, local policy makers and law enforcers to foresee what might be needed at different times in terms of labour, provisions, support structures, and so on.

This mathematical modelling relies on deciding what the variables (the numerical quantities that will vary) and the constants are (the quantities that will stay the same) in this setting. Activity 1 introduces a way to teach this with your students using an example from city life. (If your students are unfamiliar with Nehru Place or a similar environment, you could amend this example for a context they know.) The next step is to decide which variables are connected and in what way, and Activity 2 gives you an idea for how to do this with your students.

In Activities 1 and 2, you and your students will think about how to make a simplified version of such a model; note that there is no single right or wrong answer. These tasks work particularly well for students working in pairs or small groups, because this allows more ideas to be generated and students can offer mutual support when stuck.

Before attempting to use the activities in this unit with your students, it would be a good idea to complete all (or at least part) of the activities yourself. It would be even better if you could try them out with a colleague, as that will help you when you reflect on the experience. Trying the activities yourself will mean that you get insights into learners' experiences that can in turn influence your teaching and your experiences as a teacher. When you are ready, use the activities with your students. After the lesson, think about the way that the activity went and the learning that happened. This will help you to develop a more learner-focused teaching environment.

Activity 1: Identifying constants and variables

Tell your students the following:

- Imagine you are a professional mathematician and you are working on developing a mathematical model to describe the dynamics of Nehru Place in Delhi. You first have to identify all the variables (quantities that vary) and constants (quantities that stay the same) playing a role in Nehru Place.
- Make a list of all the 'players' or 'elements' in this setting. Some examples could be the car park, the hawkers or the number of shops on the first floor.

When your students have generated some ideas write the list below on the board:

The number of:

- police men and women who work at the police department in charge of security at the complex
- car parks
- people employed by the municipal corporation that is in charge of civic maintenance of the complex
- parking lot attendants
- hawkers
- escalators
- shop owners whose shop is on the first floor
- restaurant owners on the ground floor
- electricity supply companies
- visitors wanting to purchase a laptop.

Then tell your students:

On this list are some more examples of 'players' or 'elements' in this context. Between this list and your own examples, decide which are variables (with quantities that vary) and which are constants (with quantities that stay the same). Will any of these be both? If so, what would this depend on?

Activity 1 asked the students to identify variables and constants in Nehru Place. To develop a mathematical model, the students now need to think how these constants and variables influence and relate to each other.

The activity asks the students to make a mind map. A mind map is typically a series of words or phrases to represent the concept (as a node), and a line (or link) joining it to another concept, expressing a relationship of the two. A concept map is similar to a mind map, except mind maps have a centre whereas concept maps can be linear. The mind map is a good tool and provides an effective strategy to help the students explore and review their own understanding; it can also be used as an assessment tool to find out what the students know and what their misconceptions are. There are no right or wrong answers in Activity 2.

Activity 2: Developing algebraic expressions

With your students, imagine again that you are professional mathematicians working on developing a mathematical model to describe the dynamics of Nehru Place in Delhi. You have already identified the variables (with quantities that vary) and constants (with quantities that stay the same) that play a role in Nehru Place.

The next step is to identify how the variables relate to each other and to the constants. To keep it manageable, each group of students should decide which four variables they will focus on. Now tell your students the following:

- Make a mind map of these variables and write on the lines connecting the ideas how you think they might relate. Add some constants to the mind map if you think they play a role in the relationship. Remember there are no right or wrong answers for this! For example, you could think that the number of police officers should vary depending on how many visitors (buyers) there are at any given time, or on the number of shops or cars.
- Now decide which quantifiers you would use in the relationships your group described above. Write these as a mathematical expression. For example, you could state that you would need one police

officer for a combination of every ten shops, 100 visitors or 50 cars; in which case you could write a model of the number of police officers like this: s/10 + v/100 + c/50. Remember there is no right or wrong answer!

When the students have generated some mathematical expressions, move them into thinking out possible outcomes for their modelling. Tell them to do the following:

- Predict the range of values for each variable. In cases where you are having difficulty predicting a range, identify the reasons for the difficulty. For example, the number of escalators cannot be less than one, because you cannot have half of an escalator. You also cannot have an unlimited number of escalators, because they take up space. Deciding on the maximum number of escalators is harder to do because it will depend on several factors.
- Decide which of the variables you think can be controlled easily? Controlling a variable could mean either that its range can be restricted or that its value can be fixed without affecting the situation very much.
- At the end of the activity, ask the whole class to discuss this point: in reality, the quantifiers used in modelling will be based on data. If you had to organise this, how could you collect the information?

Case Study 1: Mrs Aparajeeta reflects on using Activities 1

This is the account of a teacher who tried Activities 1 and 2 with her secondary students.

I wanted to do these activities with my students because I thought it was a lovely example of seeing and recognising mathematics in real life. We first thought of a few examples with the whole class. Straight away I asked them to sort these into variables and constants. This early discussion made them aware that this was not always easy to determine. For example, the number of car parks might be considered constants; however, if you looked at the situation over a longer time period – for example, two years – then it could become a variable because, in theory, more car parks could be built in that time if there was the space and the money.

They worked in pairs on finding more examples and thinking of reasons why and when the example was a variable or a constant. Their examples and classifications were all recorded on the blackboard. These were then used to work on Activity 2: thinking about how they relate to each other and how to record this mathematically by writing it as expressions and deciding on coefficients. A student said that she had never considered coefficients to indicate a proportion and that she now suddenly understood why there are these rules when working with expressions.

At first the students felt uncomfortable with the idea that there could be no right or wrong answers. However, after sharing some ideas about the possible expressions involving the same variables and constants, they could see why this was so and became more creative with their answers.

Because I really wanted these activities to make the students see and recognise mathematics in real life, as homework I asked them to do the same for a different situation they would encounter that night. For example, to identify variables and constants and how they related while waiting at the bus stop, having dinner with their families or doing their homework.

Reflecting on your teaching practice

When you do such an exercise with your class, reflect afterwards on what went well and what went less well. Consider the questions that led to the students being interested and being able to get on, and those where you needed to clarify. Such reflection always helps with finding a 'script' that helps you engage the students to find mathematics interesting and enjoyable. If they do not understand and cannot do something, they are less likely to become involved. Use this reflective exercise every time you undertake the activities, noting, as Mrs Aparajeeta did, some quite small things that made a difference.

Pause for thought

Good questions to trigger reflection are:

- How did it go with your class? Did all the students participate?
- What responses from students were unexpected? Why?
- What questions did you use to probe your students' understanding?
- Did you feel you had to intervene at any point?
- What points did you feel you had to reinforce?
- Did you modify the task in any way? If so, what was your reasoning for this?

2 Using substitution to think about possibilities

Substitution is a powerful and necessary thinking tool. In real life, substitution is used constantly: for example, deciding which different herbs and spices to use in today's meal to make it different from yesterday's; which mode of transport to use (will you walk, or take a rickshaw or bus?), or which sari to wear. Substitution involves thinking about possibilities and alternatives whilst also considering limitations. For example, substituting a silk sari for a calculator will not be very valid when the aim is to change your dress, but could be valid when considering what you are going to spend your money on. Substitution thus allows for variation – the spice of life.

The thinking process of substitution in mathematics is similar to that in real life. It involves considering examples, alternatives and possibilities, while at the same time being aware of limitations and restrictions.

Pause for thought

- In what way do you think is mathematical substitution is different or similar to substitution in real life?
- Why would you spend time in class working on substitution?

Learning opportunities when working on substitution

For their studies in mathematics, students have to learn the skill of substitution – many exercises and problems in exercise books and examination require students to demonstrate this skill.

Giving students problems to solve that involve substitution also offers other opportunities for learning mathematics that are perhaps not so obvious but which are equally important and worthwhile. Some of these, which will be explored further, are:

- Substitution as a tool to make mathematical expressions come alive by making a story of what an algebraic expression can mean. Activity 3 is an example of this.

- Substitution as a tool to generalise a specific numerical example. Activity 4 is an example of this.
- Substitution to explore mathematical relationships, properties and limits. Activity 4 has some examples of this.

Using substitution to develop imagination

Algebra and its expressions are often considered to be the language of mathematics because they describe relationships. When doing page after page of practice from the textbook on manipulating algebraic expressions, students might be forgiven for not recognising this as eloquent language!

The next activity aims to introduce some playfulness in reading algebraic expressions and asks students to come up with their own stories of what the expression might mean and what situation it models. This means that the students are asked to give meaning to the symbolised mathematics by substituting the symbols with a narrative or story. For example:

- The expression $\frac{x}{n}$ + 3n could be about a group of people (x) who in equal numbers boarded a number of buses (n). There were some people who arrived late and each bus had to accommodate another three people.
- The expression 1.3(2100m) could have come from the expression 1.3 [3(700)m] and be about 700 space aliens who landed in Udaipur. This was their first time in India. They saw that people seemed to like a drink called chai which cost Rs. 1 per cup. They tried it, and loved it so much that they had three cups each. When they wanted to pay for the tea they were so pleased with the quality of the chai and the service they had received that they paid a tip of 30 per cent.

Activity 3: What could it mean?

Tell your students the following.

Below are some algebraic expressions adapted from textbooks:

- 30u + 44v
- 3x +1
- 3√4
- $\frac{18(3x + 1)}{6}$
- $\frac{1}{a} + \frac{1}{b}$

Use your imagination and make up a story for what each expression could be about. What situation could the expression be modelling? What else could it be about?

Video: Storytelling, songs, role play and drama

http://tinyurl.com/video-ssrpd

You may also want to have a look at the key resource 'Storytelling, songs, role play and drama' (http://tinyurl.com/kr-ssrpd).

Case Study 2: Mrs Kapur reflects on using Activity 3

Thinking of an actual story to fit an expression was not something I myself found easy. But I found that the examples given about the people on the bus and the aliens drinking copious amounts of chai tea really helped me and made me think more freely and creatively of other examples. I decided that because this had worked well for me, I would give it a go and use the same approach with my students. So I started the activity by writing the expressions for the bus and alien stories on the blackboard, giving them a few moments (not too long) to think of a story and then sharing it with the class.

I then wrote the expressions of Activity 3 on the blackboard. Rahul immediately thought that the expression 3x + 1 could tell the story of how much it cost to make three cups of tea: x rupees for the water and tea, and a fixed cost of 1 rupee for something. Some students agreed and others did not, but an interesting discussion followed. It made me realise that working from a somewhat imperfect example is actually more effective in generating discussion than having a perfectly correct answers.

Pause for thought

- What questions did you use to probe your students' understanding?
- Did you modify the task in any way like Mrs Kapur did? If so, what was your reasoning for this?

3 Using generalisation to find misconceptions

When you see an expression like 2(3 − 8), you might spot immediately that this could be rewritten as (2 × 3) − (2 × 8), because it is an example of the law of distributivity where a(b − c) = a × b − a × c.

Spotting such patterns and generalisations is helpful in mathematics because it makes it easier to solve problems. The downside is that sometimes you might think something is a pattern and so you generalise it, when actually it is a special case and will only be true in some circumstances. The next activity gives some examples of writing generalised algebra from numerical expressions.

Activity 4: Substituting the specific with the generalised form

Tell your students the following:

- Below are expressions and equations that are arithmetic examples of a generalised form that algebraic expressions and equations can take. Some of these are equations, not just expressions – make sure you know the difference.
  1. $2\,(3 - 8)$
  2. $12 + (13 + 81) = (12 + 13) + 81$
  3. $2 + 2 = 2 \times 2$
  4. $1/(\frac{1}{4}) = 4$
  5. $(-7) = 7$
  6. $42 + 0 = 42$
  7. $23 \times 1 = 23$
  8. 5 per cent of 120
  9. $(12 + 51)/(12 \times 51)$

10. $2 + 3 = 3 + 2$
11. $\frac{1}{2} + \frac{3}{4} + \frac{5}{6}$

- Write down general algebraic expressions or equations for these. There is more than one solution for some of them, so be inventive!
- Will these always hold true? Can you say that they will be valid for any number?

Video: Monitoring and giving feedback

http://tinyurl.com/video-monitoringandfeedback

You may also want to have a look at the key resource 'Monitoring and giving feedback' (http://tinyurl.com/kr-monitoringandfeedback).

Case Study 3: Mrs Agarwal reflects on using Activity 4

Mohit spotted straight away that the first expression, $2(3 - 8)$, was an example of the distributive property over subtraction, and according to him it would hold for any value of number. He gave the general form as $a(b - c) = a \times b - a \times c$.

The example of $2 + 2 = 2 \times 2$ led to interesting discussions and made some misconceptions come to the fore about indices and what 'to the power of' means when Rima said that '$= 2^2$' could be added to that equation. Questions 6 and 7 made the students think about identities and some were surprised that these were not the same for all operations – they actually knew this when working with arithmetic examples, but seemed to have forgotten when using a generalised form!

Although I was happy to try out this task, I had not expected that it would offer me so many insights into students' misconceptions. I am still thinking about why this activity lets this happen. Was it the discussion, or the 'simple' examples? Why does it not happen so much when doing exercises from textbooks? When I am thinking about ways that I assess my students in class, I must remember that using tasks that ask the students to apply what they know in an unusual but not necessarily difficult context lets me evaluate what they know very clearly.

Pause for thought

- What responses from students were unexpected? Why?
- What questions did you use to probe your students' understanding?
- What do you think is the reason that this task allows misconceptions to emerge as they did in Mrs Agarwal's class?

4 Summary

This unit has asked you to think about writing and working with algebraic expressions. The activities have introduced ideas that are used extensively in the real world to model complex situations and allow decisions to be made. Asking the students to come up with mathematical expressions for themselves for a situation of their choosing helps to get away from mundane practice in using expressions and to help them

see that these things are important. It is much easier to use the mathematical language of expressions to communicate some ideas, but this is something that students are rarely exposed to.

The final activities asked the students to discuss ideas about expressions in a relaxed way so that misconceptions can be uncovered and dealt with. The focus on generalising – that is, being sure that the equivalence in expressions is always true – helped them to connect naturally to many other ideas in algebra.

Pause for thought

Identify three ideas that you have used in this unit that would work when teaching other topics. Make a note of two topics you have to teach soon where those ideas can be used with some small adjustments.

Resources

Resource 1: NCF/NCFTE teaching requirements

This unit links to the following teaching requirements of the NCF (2005) and NCFTE (2009), and will help you to meet those requirements:

- View students as active participants in their own learning and not as mere recipients of knowledge; to encourage their capacity to construct knowledge; to ensure that learning shifts away from rote methods.
- View learning as a search for meaning out of personal experiences and knowledge generation as a continuously evolving process of reflective learning.
- Help students to see mathematics as something to talk about, to communicate through, to discuss among themselves, to work together on.
- Connect school knowledge with community knowledge and life outside the school.

Additional resources

- A newly developed maths portal by the Karnataka government: http://karnatakaeducation.org.in/KOER/en/index.php/Portal:Mathematics
- Class X maths study material: http://www.zietmysore.org/stud_mats/X/maths.pdf
- National Centre for Excellence in the Teaching of Mathematics: https://www.ncetm.org.uk/
- National STEM Centre: http://www.nationalstemcentre.org.uk/
- OpenLearn: http://www.open.edu/openlearn/
- BBC Bitesize: http://www.bbc.co.uk/bitesize/
- Khan Academy's math section: https://www.khanacademy.org/math
- NRICH: http://nrich.maths.org/frontpage
- Mathcelebration: http://www.mathcelebration.com/
- Art of Problem Solving's resources page: http://www.artofproblemsolving.com/Resources/index.php
- Teachnology: http://www.teach-nology.com/worksheets/math/
- Maths is Fun: http://www.mathsisfun.com/
- National Council of Educational Research and Training's textbooks for teaching mathematics and for teacher training of mathematics: http://www.ncert.nic.in/ncerts/textbook/textbook.htm

- AMT-01 *Aspects of Teaching Primary School Mathematics*, Block 3 ('Numbers (II)'): http://www.ignou4ublog.com/2013/06/ignou-amt-01-study-materialbooks.html
- LMT-01 *Learning Mathematics*, Block 1 ('Approaches to Learning') Block 2 ('Encouraging Learning in the Classroom'), Block 6 ('Thinking Mathematically'): http://www.ignou4ublog.com/2013/06/ignou-lmt-01-study-materialbooks.html
- *Learning Curve* and *At Right Angles*, periodicals about mathematics and its teaching: http://azimpremjifoundation.org/Foundation_Publications
- Central Board of Secondary Education's books and support material (also including the *Teachers Manual for Formative Assessment – Mathematics (Class IX)*) – select 'CBSE publications', then 'Books and support material': http://cbse.nic.in/welcome.htm

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