TESS+ India

Conjecturing and generalising in mathematics: introducing algebra

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TESS-India OERs have been collaboratively written by Indian and international authors to address Indian curriculum and contexts and are available for online and print use (http://www.tess-india.edu.in/). The OERs are available in several versions, appropriate for each participating Indian state and users are invited to adapt and localise the OERs further to meet local needs and contexts.

TESS-India is led by The Open University UK and funded by UK aid from the UK government.

Video resources

Some of the activities in this unit are accompanied by the following icon: . This indicates that you will find it helpful to view the TESS-India video resources for the specified pedagogic theme.

The TESS-India video resources illustrate key pedagogic techniques in a range of classroom contexts in India. We hope they will inspire you to experiment with similar practices. They are intended to complement and enhance your experience of working through the text-based units, but are not integral to them should you be unable to access them.

TESS-India video resources may be viewed online or downloaded from the TESS-India website, http://www.tess-india.edu.in/). Alternatively, you may have access to these videos on a CD or memory card.

What this unit is about

Algebra is the topic where many students start to say that mathematics is difficult. This may be for many reasons; young students like things to be straightforward and concrete, whereas algebra is about abstract symbols standing for variables and constants. However, many difficulties are caused because the differences between the way that students work with numbers and the way that they work in algebra are not addressed, and so students start off feeling confused.

In this unit you will think about how to introduce algebra and what differences need to be explored to help students think algebraically without feeling lost and confused. The activities ask the students to work with algebraic ideas whilst safely using numbers. They will build on their thinking towards conjecturing and generalising, two important algebraic ideas.

Two of the activities use cards in different ways to encourage the students to explore algebraic ideas and to extend those ideas using their own thinking. Working out whether something they are presented with is always true, sometimes true, or false, is a further theme of the activities.

What you can learn in this unit

- How to help your students understand the differences between arithmetic and algebra.
- Some suggestions on using conjecturing and generalising to enable students to think algebraically.
- Some methods that help students to decide for themselves if statements are right or wrong, and to explore mathematics together.

This unit links to the teaching requirements of the NCF (2005) and NCFTE (2009) outlined in Resource 1.

1 The equals sign in algebra

In arithmetic, the equals sign is often seen as a command to take an action and find an answer. Therefore, when a student sees an equals sign in an equation, they may want to carry out the operation that precedes it. To many students, the equals sign means 'and the answer is', which is not helpful when doing algebra.

The equals sign is always a representation of a relationship between two expressions. The equals sign represents quantitative sameness – in other words, the expression on the left-hand side of the equals sign represents the same quantity as the expression on the right-hand side. The equals sign can be read as 'is the same as', or 'is equivalent to', or 'is of equal value'. Understanding this will help your students when they work with equations.

Figure 1 The equals sign indicates balance.

The concept of 'the equal sign indicates balance' can be used to reinforce the idea of equality – that both sides of the number sentence need to be the same, and the equation needs to balance. A set of simple balance scales with different coloured blocks (or other small items all of the same weight) in each pan could be used. Alternatively, string bags containing small packets or cans suspended from a stick or a coat hanger, would represent this concept visually.

Pause for thought

Think about some uses of the equals sign that your students may see around them that may lead to a misinterpretation or misunderstanding. For example, the equals sign can sometimes be used outside mathematical equations, such as 'MATHS = FUN' or 'Ravi = 9'.

2 Thinking algebraically

Thinking algebraically and using algebra in school involves recognising and analysing patterns and relationships, using symbols and developing generalisations. The 'language of arithmetic' focuses on finding answers, while the 'language of algebra' focuses on relationships. For example, 'a + 0 = a' is a symbolic representation for the generalisation that when zero is added to any number, it stays the same.

Algebra focuses on expressing a generalised relationship, whereas most mathematical lessons focus on finding the answer. So the first thing that needs to be understood is that algebra is different.

The activities in this unit will work on developing ideas about algebraic thinking:

- Activity 1 is about encouraging your students to play with numbers and making expressions that encourage them to think of the equals sign as meaning 'is the same as' rather than 'find the answer'.
- Activity 2 begins to extend the students' algebraic thinking, asking them to explore whether a statement is true or false and to make a conjecture about whether it is always true, sometimes true or always false.
- Activity 3 moves on to generalisation, encouraging students to consider whether their conjecture (or theory) works for all numbers. This means that they will be beginning to generalise about number properties in an algebraic way.

Before attempting to use the activities in this unit with your students, it would be a good idea to complete all, or at least part, of the activities yourself. It would be even better if you could try them out with a colleague as that will help you when you reflect on the experience. Trying them for yourself will mean you get insights into a learner's experiences which can, in turn, influence your teaching and your experiences as a teacher.

Activity 1: The game of equality

Preparation

This game is played by sets of two teams competing with each other. Decide how many sets of two teams you want to divide your class into.

For each set of two teams you will require the following:

- two number cards for each of the numbers 1 to 9
- operation cards for addition (+), subtraction (−), multiplication (×) and division (÷)– several of each

- a card for the sign of equality (=).

A number card can be made by writing the number on a large piece of paper. You can use a sketch or marker pen to write on the paper so that the ink is dark enough for all the students to see.

You will need some space for the students to move around. If the desks and benches cannot be moved sufficiently in your classroom then consider going outside. Regroup the class to create:

- two teams (A and B) with number cards.
- an Operation Team of four members who will have to hold the 'operation' cards
- one student with the 'equals' card (who will be called Professor Equals).

How the game is played

Team A makes a mathematical expression using any two of its members and the operation of addition or subtraction. For example:

$$9 + 8$$

$$7 - 4$$

Professor Equals then comes and stands at either end of the Team A expression.

Team B then makes another expression using any number of its members and any one of the remaining operations that has the 'same value' as the expression made by Team A. Members of Team B stand on the other side of Professor Equals.

For example, for the two expressions made by Team A above, Team B can make:

'9 + 8 = 19 – 2' or '9 + 8 = 21 – 4', etc.

'7 – 4 = 6 $\boxed{2}$' or '7 – 4 = 9 – 6', etc.

If Team B is successful in making an expression that equals the expression made by Team A, it earns as many points as the largest number it used in making its expression.

If Team B fails to make the expression that equals the expression made by Team A, then Team A gets as many points as the largest number it used in making its expression.

For the next move, Team B goes first. The two teams are allowed the same number of moves.

Case Study 1: Mrs Aparajeeta reflects on using the game of equality

This is the account of a teacher who tried Activity 1 with her elementary students.

This activity required a lot of management and did take some time before we could actually start. I got hold of some quite strong card and made the cards using this. When the activity was over I collected them all together and put them away in a safe place, determined to use them again.

One of my colleagues, Meena, saw me putting them away and asked about them. When I explained the activity [Figure 2], she said that she would like to use them as well, so they have been used twice now which makes the time spent more worthwhile. Next time I use cards – and I am sure that I will as the students learned so much – Meena and I will work together to make the cards.

Before the lesson we had to move the desks to make sure we had room to move. But it was well worth the

time. In fact, all the students really enjoyed the activity and I think learned a lot about the equals sign.

To start with I explained the game using two teams of ten students but I had already decided that I would need to make four teams instead of two to actually play the game as there were too many students in my class and many would not have been involved otherwise. I also had four students appointed as evaluators to evaluate the expressions made by each group of students to see whether they were correct or not, and had two students to keep score. Team A played Team B, and Team C played Team D, at different ends of the classroom, then they swapped.

I noticed that Team B had some deep-thinking students. They decided to always use the biggest numbers that they could in order to get more marks and to put the other team out. This of course meant that they had to do some quite difficult arithmetic with their big numbers. It was their choice to challenge themselves but I was pleased to see the sums they set themselves and the trouble they took to make sure they were accurate and scored the points.

We decided to have a knock-out competition at the end of the lesson. Team A played Team B and the winner played Team C, and so on. I felt this worked very well as we stopped at each try and everyone evaluated the answers given, why they were correct or why they were wrong. This led to a great deal of discussion and again I was surprised at how much each student in the class challenged themselves to do the arithmetic in their heads quickly.

Reflecting on your teaching practice

When you do such an exercise with your class, reflect afterwards on what went well and what went less well. Consider the questions that led to the students being interested and able to progress, and those you needed to clarify. Such reflection always helps with finding a 'script' that helps you engage the students to find mathematics interesting and enjoyable. If they do not understand and cannot do something, they are less likely to become involved. Use this reflective exercise every time you undertake the activities, noting as Mrs Aparajeeta did, some quite small things that made a difference.

Pause for thought

Good questions to trigger such reflection are:

- How did your students respond to this activity? What responses from students were unexpected? Why?
- Did you feel you had to intervene at any point?
- What points did you feel you had to reinforce?
- Did you modify the task in any way? If so, what was your reasoning for this?

3 Conjecturing and generalising

Making conjectures (theories), and then reasoning out whether they are true, sometimes true, or false, is part of developing the ideas of generalising that algebraic thinking depends on.

The 'additive identity' – that is, the idea that adding or subtracting zero leaves the original number intact – is relatively easily grasped. However, because of its later application in solving algebraic equations, this identity is well worth exploring.

It is important for students to be able to articulate their understanding of such identities and this can be done by asking the students to develop conjectures. The class can develop statements or conjectures about what happens when zero is added to, or subtracted from, a number.

Students often try out a range of different numbers in order to test their ideas. It is important to encourage students to consider whether their conjecture (or theory) works for all numbers. In this way, your students will begin to generalise about number properties in an algebraic way.

The rules or conjectures developed by a class can be displayed and/or the student who came up with the concept could have their name attached to it, for example 'Prem's Rule'.

Here are some examples of conjectures developed by students about addition:

> Prem's rule: 'When you add zero with a number it doesn't change the number that you started with.' ($a + 0 = a$)

> Anisha's rule: 'When you subtract zero from a number it doesn't change the number you started with.' ($a - 0 = a$)

> Jyotsna's rule: 'If you take the number you started with away from the same number you get 0.' ($a - a = 0$)

> Vishal's rule: 'It doesn't matter if the numbers are swapped around on each side of the number sentence. If the numbers are the same, the number sentence will still balance.' ($a + b = b + a$)

> Simi's rule: 'When you add two numbers, you can change the order of the numbers you add, and you will still get the same number.' ($a + b = b + a$)

Exploring and conjecturing

The following activity shows how you can help students start to think algebraically by exploring arithmetic statements and making conjectures about whether they are always true, sometimes true or never true. It can sometimes come as a surprise to students that they are allowed to say 'this is not true'. It is really important that they should not just accept everything they see with numbers in it, but should be willing to think 'is this always true or can I refute it?'

Activity 2: Conjectures

Preparation

Write several arithmetic statements on the blackboard. Some examples you might use are:

- $(3 + 5) + 8 = 3 + (5 + 8)$
- $(3 + 5) \times 8 = 3 + (5 \times 8)$
- $(3 - 5) - 8 = 3 - (5 - 8)$
- $(3 \times 5) + 8 = 3 \times (5 + 8)$

Note that some of the statements should be true and some not.

There are more examples of statements in Resource 2.

The activity

Ask your students to do the following:

- Check the validity of each statement.
- For all the correct statements, change one, two or all three numbers to write several similar statements. Are all of these true? If yes, do you feel that these statements would be true for all possible choices of numbers? Write down your thoughts as a conjecture.
- For all the incorrect statements, change one, two or all three numbers to write several similar statements. Are all of these incorrect or can you find a correct statement? Do you feel that these statements would be incorrect for all possible choices of numbers? Write down your thoughts as a conjecture.

This activity provides students with valuable opportunities to learn through talk. You may want to look at the key resource 'Talk for learning' (http://tinyurl.com/kr-talkforlearning) to help your planning for this aspect of the activity.

Video: Talk for learning

http://tinyurl.com/video-talkforlearning

Case Study 2: Mrs Kapur reflects on using Activity 2

I divided the students into groups of five and then gave them ten minutes to discuss the validity of the statements I had written on the blackboard.

There was a lot of discussion among the groups [Figure 3]. This made me extremely happy because, when I listened into their conversations they were all thinking of the reasons why the statements were true or if they could think of numbers that would make them untrue, or the other way round.

There were a few who were not contributing to the thinking in their group, so I told the group to make sure to involve them in the discussion too. One of them had missed some time at school and needed help feeling part of the group again. His arithmetic was particularly good so they soon appreciated his contributions. I told the class that everyone had to do their share of the thinking and that sharing ideas would help everyone. Also I said that I would pick the student who would make the presentation so everyone should be able to report on what was said. So then they all got involved in the exercise.

I asked different students to give the answers from their group's discussion and say if they thought the statement was always true, sometimes true, or never true. I asked them what numbers they had tried and also to try to explain why they had chosen those particular values. Then I also got other groups to contribute the numbers they had chosen, so that we then had a substantial number of examples.

This took a considerable time, especially with an incorrect statement as several groups were sure they could find a way to make it true. This meant we did not get through all of the statements I had prepared and so I asked them to do the rest as a home assignment, writing their own individual conjectures.

We discussed what they had found out the following day with a lot of contributions from most of them. I noticed that some students at the back of the class were very quiet. When I went to talk to them, some of them said they did not understand what was meant to be done, so I asked them to give the reasons why the conjectures made by others were correct or wrong.

Pause for thought

What do you think about the way Mrs Kapur intervened with the quiet students at the back of the class? What might the possible reasons for them not understanding what they needed to do?

Now think about how your students responded to the activity and reflect the following questions:

- What responses from students were unexpected? Why?
- What questions did you use to probe your students' understanding?
- Did you modify the task in any way? If so, what was your reasoning for this?

4 Moving on to more formal generalisations

Moving from conjecturing about statements to generalising using symbols might seem to be a big step, but if your students have been working with games like the ones in Activities 1 and 2 they may have already started to use symbols.

For example, they may have said things like 'If you take 2 from any number and then add five to it, the answer is always going to be three more'. Using x or n as a convenient way to show any number may well seem entirely natural in this context.

The next activity starts to encourage more formal generalisations.

Activity 3: Generalising

Preparation

Create flash cards of two types:

- S-cards – These have specific arithmetic statements that may or may not be true.
- G-cards – These have generalised statements (conjectures) which correspond to the statements on the S-cards.

Resource 3 provides examples of S-cards and G-cards. You may alter these to suit the level of your class.

Divide the class into groups. Groups of six to ten students work well for this activity. Shuffle all the S-cards and all the G-cards separately. You may want to look at the key resource 'Using groupwork' (http://tinyurl.com/kr-usinggroupwork) when thinking about organising your class into groups. Divide each group into two halves. Distribute S-cards to one half of the group and G-cards to the other half.

The activity

Part 1

Ask your students to create pairs of S- and G-cards and then explore if the conjecture made is always true, sometimes true or false. Another idea is to ask the students to work in groups of five or six, hand them six assorted S- and G-cards. If they have a specialised (S) card then they have to generalise it (make a G card). If they have a G-card they have to create an S-card for it and then discuss if it is always true, sometimes true or never true.

Resource 4 provides some examples for the content of each type of card.

Part 2

Ask the students to make up their own S-cards and G-cards.

Video: Using groupwork

http://tinyurl.com/video-usinggroupwork

Case Study 3: Mrs Agarwal reflects on using Activity 3

I used the suggestions [in Resource 3] to make the S- and G-cards. I liked the activity because I thought it would encourage the students to compare the expression and then see how each expression could be referred to in mathematical language.

I made the groups such that each group had one student who already seemed to have a good understanding of algebra. I then asked them to see that all the students in their group participated in the discussion of whether their conjectured generalisations were completely false, sometimes true (and if so, when), or always true. The groups were also warned that the explanation for their group would have to be given by any student, so they all had to have a consensus.

This worked very well. The level of discussion that I overheard in the groups was quite exceptional for the class as they each tried to match up the specific with the generality and work out whether it was always true or not. I asked each group to present one pair of cards and give their reasons for what they had decided. This took some time but only because they were discussing so much with each other! We had to leave the second part of the activity until the next day.

Making their own S- and G-cards was also a good exercise, as I made them do these individually, and then they talked about what each had made in their groups. So they gained a lot of input from their classmates about whether they were right about their statements or whether they had some misunderstandings.

Pause for thought

- What questions did you use to probe your students' understanding?
- Did you feel you had to intervene at any point?
- What points did you feel you had to reinforce?
- How did you organise your students into teams?
- Will you use these teams again?
- What assessment of your students' understanding could you make for this activity?
- Are there students who need more support?

5 Summary

This unit has focused on algebraic thinking and helping your students to consider the similarities and differences between algebraic thinking and arithmetic.

In studying this unit you will have identified the importance of clarifying the equals sign as meaning 'is the same as' instead of 'and the answer is'. You will also have considered how to enable your students to develop their reasoning about whether statements are true or false, and whether they are true are they true for all numbers?

You have considered how to help students understand that they are able to work as mathematicians, exploring and producing reasons for whether statements work. They have had to be the ones to judge whether something is true or false. In this way you will have helped your students to grow in confidence as mathematicians using algebraic thinking rather than becoming confused or puzzled when using symbols or generalising.

You will also have seen how reflecting on your teaching is important in becoming better at supporting your students' learning.

Pause for thought

Identify three techniques or strategies you have learned in this unit that you might use in your classroom, and two ideas that you want to explore further.

Resources

Resource 1: NCF/NCFTE teaching requirements

This unit links to the following teaching requirements of the NCF (2005) and NCFTE (2009) and will help you to meet those requirements:

- View students as active participants in their own learning and not as mere recipients of knowledge; how to encourage their capacity to construct knowledge; how to shift learning away from rote methods.
- Let students see mathematics as something to talk about, to communicate through, to discuss among themselves, to work together on.
- Let students learn important mathematics and see mathematics is more than formulas and mechanical procedures.

Resource 2: Examples of statements for use in Activity 2

Write several arithmetic statements like these on the blackboard.

Note that some of the statements you use should be true and some not.

$$(3 + 5) + 8 = 3 + (5 + 8)$$

$$(3 + 5) \times 8 = 3 + (5 \times 8)$$

$$(3 - 5) - 8 = 3 - (5 - 8)$$

$$(3 \times 5) + 8 = 3 \times (5 + 8)$$

$$3 - (5 + 8) = (3 - 5) + 8$$

$$(8 - 5) \times 3 = (3 - 5) \times 8$$

$$(8 + 5) \times 3 = 8 \times 3 + 8 \times 5$$

$$3 \times 5 + 3 \times 8 = (3 + 5) \times 8$$

$3 \times 5 - 8 = 8 - 3 \times 5$

$3 \times (5 - 8) = 3 \times 5 - 3 \times 8$

$(5 - 3) \times 8 = 8 \times (3 - 5)$

$3 \times (8 - 5) = 3 \times 8 - 3 \times 5$

Resource 3: Examples of S-cards and G-cards

Table R3.1 Examples of S-cards and G-cards.

S-card (specialising)	G-card (generalising)	Generalisation is always true (A), sometimes true (S) or false (F)
$(3 \times 2) \times 4 = 3 \times (2 \times 4)$	The product of three numbers remains the same whichever two numbers are multiplied first	A
$12 \div 3 = (12 \div 4) + 1$	$ab/b = ab/a + (a - b)$	A
$12 + 20 = 4 \times 8$	$ab + bc = b(a + c)$	A
$2 \times 4 + 3 \times 4 = 4 \times 5$	$a(a + 2) + (a + 1)(a + 2) = (a + 2)(a + 3)$	S
$2 \times 1^2 = (2 \times 1)^2$	Two times a squared number is equal to two times that number squared	S
$4 + 16 - 8 = 8 + 8 - 4$	$4 + 4(a - 2) = 2a + 2(a - 2)$	S
$4 + 4 \times 1 = 6 + 1 + 1$	$4 + 4(a - 2) = 3(a - 1) + (a - 2) + 1$	A
$3 + 2 + 1 = 3 \times 2 \times 1$	The sum of three consecutive numbers is the same as the product of those numbers	S
$4 + (6 \div 2) = 4 + 3$	$a + bc/c = a + b$	A
$461 + 200 = 200 + 461$	If you add two numbers together you can change the order and you still get the same answer	A
$7 \times 4 = 9 \times 7 - 5 \times 7$	$c(a - b) = ac - bc$	A

Resource 4: Use of S-cards and G-cards

Table R4.1 *Use of S-cards and G-cards.*

S-card (specialising)	G-card (generalising)	Generalisation is always true (A), sometimes true (S) or false (F)
(3 × 2) − 1 = (3 + 2)	The predecessor of the product of two numbers is the sum of the two numbers	
(3 × 2) × 4 = 3 × (2 × 4)	The product of three numbers remains the same if the product of any two of the numbers is multiplied by the third number	
12 ÷ 3 = (12 ÷ 4) + 1	ab/b = ab/a + (a − b)	

Additional resources

- A newly developed maths portal by the Karnataka government: http://karnatakaeducation.org.in/KOER/en/index.php/Portal:Mathematics
- National Centre for Excellence in the Teaching of Mathematics: https://www.ncetm.org.uk/
- National STEM Centre: http://www.nationalstemcentre.org.uk/
- National Numeracy: http://www.nationalnumeracy.org.uk/home/index.html
- BBC Bitesize: http://www.bbc.co.uk/bitesize/
- Khan Academy's math section: https://www.khanacademy.org/math
- NRICH: http://nrich.maths.org/frontpage
- Art of Problem Solving's resources page: http://www.artofproblemsolving.com/Resources/index.php
- Teachnology: http://www.teach-nology.com/worksheets/math/
- Math Playground's logic games: http://www.mathplayground.com/logicgames.html
- Maths is Fun: http://www.mathsisfun.com/
- Coolmath4kids.com: http://www.coolmath4kids.com/
- National Council of Educational Research and Training's textbooks for teaching mathematics and for teacher training of mathematics: http://www.ncert.nic.in/ncerts/textbook/textbook.htm
- AMT-01 *Aspects of Teaching Primary School Mathematics*, Block 1 ('Aspects of Teaching Mathematics'), Block 2 ('Numbers (I)'), Block 3 ('Numbers (II)'): http://www.ignou4ublog.com/2013/06/ignou-amt-01-study-materialbooks.html
- LMT-01 *Learning Mathematics*, Block 1 ('Approaches to Learning') Block 2 ('Encouraging Learning in the Classroom'), Block 4 ('On Spatial Learning'), Block 6 ('Thinking Mathematically'): http://www.ignou4ublog.com/2013/06/ignou-lmt-01-study-materialbooks.html
- *Manual of Mathematics Teaching Aids for Primary Schools*, published by NCERT: http://www.arvindguptatoys.com/arvindgupta/pks-primarymanual.pdf
- *Learning Curve* and *At Right Angles*, periodicals about mathematics and its teaching: http://azimpremjifoundation.org/Foundation_Publications
- Textbooks developed by the Eklavya Foundation with activity-based teaching mathematics at the primary level: http://www.eklavya.in/pdfs/Catalouge/Eklavya_Catalogue_2012.pdf

- Central Board of Secondary Education's books and support material (also including *List of Hands-on Activities in Mathematics for Classes III to VIII*) – select 'CBSE publications', then 'Books and support material': http://cbse.nic.in/welcome.htm

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Acknowledgements