(b) From the position diagram, measure the linear displacements of points on the links or the angular displacements of the links and plot the values against crank angular displacement.

(c) Where the motion of a point on a link is on a curved path, resolve the linear displacement vectors into \(X\) and \(Y\) components and plot a displacement curve for each component.

**Answers to revision exercise**
2 and 3 are both true.

1 Displacement curves show the *change of position* of a link from an initial position during the motion of the mechanism.

### 3 Constraints on motion: degrees of freedom

In this Section 1 will be developing a straightforward set of ideas about an assembly of links. In particular, I will discuss the particular conditions that a kinematic chain must satisfy in order to make it into a useful mechanism. Just to remind you of the special properties of a mechanism, we might distinguish two principal features, i.e.:

1 A mechanism's links are modelled as rigid bodies and must be able to move. If they cannot, then the mechanism is a *structure*.

2 The relative positions of the links of the mechanism depend on (a) the link dimensions and (b) one or more independent quantities which 'label' the position of the mechanism. The minimum number of these independent quantities needed to locate all the links in an assembly is the *number of degrees of freedom*, \(F\), of the assembly. In the examples of Section 1 – the squeegee mop, the up-and-over garage door, the car suspension linkage – only one quantity was needed in each case to define or label the position of the mechanism. For each of the cases, the mechanism had one degree of freedom, \(F = 1\), and the motion of the mechanism is said to be *constrained*. Most mechanisms used in machinery have constrained motion; that is, a change in position of one link (input motion) produces a predictable change of position of all the other movable links. However, in mechanisms which have several input motions, the number of degrees of freedom will be more than one. In this Unit, I shall be dealing in detail only with constrained assemblies of rigid links which satisfy the condition \(F = 1\).

#### 3.1 Constraint in kinematic chains

In Unit 1 we saw that when two links are joined to form a sliding or a rotating pair, only one degree of freedom was allowed in each case in the form of a translation or a rotation of one link relative to the other. In both cases, one quantity was completely sufficient to specify the position of one link in relation to the other. However, any link which is free to move in planar motion has three degrees of freedom, i.e. three quantities are necessary to specify the position of the link – the position of one point on the link (two co-ordinates) plus the inclination of the link with respect to a baseline or axis (one co-ordinate). Thus we can say that by making a connection to the link, in the form of a sliding or rotating pair, we have constrained two of the degrees of freedom which the link would have in free planar motion.

As a general rule, whenever two links are connected as a sliding or a rotating pair, their relative motion is constrained by the loss of two degrees of freedom. In addition if one of the links is 'fixed', i.e. its position is fully...
specified, then the three degrees of freedom associated with this link are also ‘lost’.

In order to illustrate these simple rules I will build up some assemblies of links, adding one link at a time and noting the overall number of degrees of freedom. In Figure 39 I have made a start in this process. In (a) a single link with no connections needs three variables to fix its position in a plane. When another link is added, as in (b), it contributes a further three degrees of freedom, but the rotating pair connection between the links constrains two degrees of freedom, leaving four degrees of freedom or four variables required to specify the position of the mechanism. Rather like the hands of a clock, you can demonstrate the time by fixing the position of the pivot at the centre of the clock face (two quantities) and then give the inclinations of the minute hand (one quantity) and hour hand (one quantity) with respect to a reference line (12 o’clock). In (c) one link is fixed in position leaving just one degree of freedom for the pair of links. I can add another link, which will give the variety of configurations shown in (d) to (h).

<table>
<thead>
<tr>
<th>no of links</th>
<th>configuration</th>
<th>no of degrees of freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(a)</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>(b)</td>
<td>((2 \times 3) - 2 = 4)</td>
</tr>
<tr>
<td></td>
<td>(c)</td>
<td>((2 \times 3) - 2 - 3 = 1)</td>
</tr>
<tr>
<td>3</td>
<td>(d)</td>
<td>((3 \times 3) - (2 \times 2) = 5)</td>
</tr>
<tr>
<td></td>
<td>(e)</td>
<td>((3 \times 3) - (2 \times 2) - 3 = 2)</td>
</tr>
<tr>
<td></td>
<td>(f)</td>
<td>((3 \times 3) - (2 \times 2) = 5)</td>
</tr>
<tr>
<td></td>
<td>(g)</td>
<td>((3 \times 3) - (2 \times 3) = 3)</td>
</tr>
<tr>
<td></td>
<td>(h)</td>
<td>((3 \times 3) - (2 \times 3) - 3 = 0)</td>
</tr>
</tbody>
</table>

*Figure 39 Building an assembly of links*
Once again, to find the overall number of degrees of freedom each link contributes three degrees of freedom and each pair connection reduces the total by two degrees of freedom. A fixed link will further reduce the total by three degrees of freedom. In (f) a single connector joins two pairs of links.

Figure 39(h) is a triangular truss, the basis of all so-called pin-jointed structures. With one link fixed the resulting degree of freedom is zero. The assembly is immovable and completely determined by the lengths of the links; i.e. given the lengths of the links, there is no choice left as to the position these links can take once it is built. Being completely specified in this way, it is called a statically determinate structure.

**Example 2**

Figure 40 shows some four-link assemblies. What is the overall number of degrees of freedom \( F \) associated with each?

**Solution**

(a) and (b) should be familiar to you as versions of the four-bar linkage and slider-crank mechanisms you met in Unit 1. In each case

\[
F = (4 \times 3) - (4 \times 2) - 3 = 1
\]

In (c) we have a translation-to-translation transmitter with \( F = 1 \) again. All three devices satisfy our condition for a constrained mechanism – a change of position of one link produces a predictable change of position of the other movable links, or a single variable will 'label' the position of the mechanism. In (d) you may recognize a simple 'lazy tongs' device. As shown it has four degrees of freedom, but in practice one arm will be held steady (fixed) while the other is actuated to produce an increase or decrease in the distance between the connecting pins. Used in this way, the device becomes a mechanism \( (F = 1) \). For (e) and (f) we have the apparently strange situation where the number of degrees of freedom is negative

\[
F = (4 \times 3) - (5 \times 2) - 3 = -1
\]

This does not mean that they are even more immovable than a structure. What it does signify is that one of the links is unnecessary or redundant, in terms of fixing the assembly as a structure.

Such structures with redundant members (negative number of degrees of freedom) are called statically indeterminate structures. They are more difficult to analyse than simple determinate structures. It should be noted, however, that a redundant member may well be carrying large loads – it is only redundant in a kinematic sense.

**Question:** Which links from (e) and (f) could be removed from the configurations without altering the basic 'immobility' of the structures?

**Answer:** (e) 3 or 4, (f) 3

**SAQ 10**

Draw three five-link assemblies and decide the number of degrees of freedom associated with each.

We could continue building up assemblies of linkages with six, seven or more links. The number of variations will naturally increase, but by now you should have gathered the main point of the exercise, which is to establish whether a system of connected links is a constrained mechanism \( (F = 1) \), a structure \( (F \leq 0) \), or an unconstrained kinematic chain \( (F \geq 2) \).
3.2 Degrees of freedom of general kinematic chains

During the process of machine design, a proposed mechanism commonly has to be analysed to determine whether the various links have the required motions. It is fruitless attempting this if the kinematic chain turns out not to be a mechanism, so the rest of this Section aims to give you practice in establishing the degrees of freedom of kinematic chains.

The procedure for performing this simple operation can be expressed in algebraic form by the equation

\[ F = 3n - 2j - 3 \]

where \( F \) is the overall number of degrees of freedom for the kinematic chain, \( n \) is the number of links (including the fixed link), and \( j \) is the number of sliding or rotating pairs. If, like me, you have difficulty in remembering formulae of this kind, simply recall that each link contributes three degrees of freedom; each pair connection 'uses up' two degrees of freedom and the fixed link uses up three degrees of freedom.

**Example 3**

Figure 41 shows the line diagrams of some of the mechanisms which you have already come across in the course, together with some other devices used for special purposes. Check whether these examples conform to the conditions for a constrained mechanism.

\[ \begin{align*}
(a) & \quad \text{Diamond car jack} \\
(b) & \quad \text{Oil pumping rig} \\
(c) & \quad \text{Quick-return mechanism} \\
(d) & \quad \text{Watts rotative engine (1784)} \\
(e) & \quad \text{Aircraft undercarriage} \\
(f) & \quad \text{Dwell mechanism} \\
(g) & \quad \text{Variable stroke engine mechanism} \\
(h) & \quad \text{Double reciprocaton engine mechanism} \\
(j) & \quad \text{Three-shaft drive}
\end{align*} \]
Figure 42 shows the corresponding 'exploded' diagrams for each mechanism. The results are given in Table 2 for each case.

![Figure 42 'Exploded' diagrams](image)

(a) Diamond car jack
(b) Oil pumping rig
(c) Quick return mechanism
(d) Watts rotative engine
(e) Aircraft undercarriage
(f) Dwell mechanism
(g) Variable stroke mechanism
(h) Double reciprocation mechanism
(i) Three-shaft drive

Table 2 Review of mechanisms of Figure 41

<table>
<thead>
<tr>
<th>Reference</th>
<th>No. of links ( n )</th>
<th>No. of connecting pairs ( j )</th>
<th>No. of degrees of freedom ( F = 3n - 2j - 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>6</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>(b)</td>
<td>6</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>(c)</td>
<td>6</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>(d)</td>
<td>8</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>(e)</td>
<td>8</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>(f)</td>
<td>8</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>(g)</td>
<td>6</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>(h)</td>
<td>6</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>(i)</td>
<td>5</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

All the cases in Figure 41 conform to \( F = 1 \), except the last, the so-called three-shaft drive, which appears to give an answer of \( F = 0 \), a determinate structure. Is this the exception that proves the rule?

In fact this last result can be used to illustrate a final important point which the mechanism designer must bear in mind. The above procedure for finding the number of degrees of freedom does not consider the geometric specifications of dimensions of the links involved in a kinematic
chain. For example, the slider-crank mechanism is indeed constrained with $F = 1$, but in a reciprocating engine, if the crank radius is too large, the piston may run out of space to finish its stroke (coming up against the cylinder head). The mechanism would then be useless for performing complete cycles of operation. Similarly, by arranging the 'three-shaft drive' of Figure 41(j) so that the crank lengths $AD = CE = BF$ and the triangle $ABC$ is identical to the triangle $DEF$, the seeming structure will in fact perform complete revolutions.

Any small variation in these specifications, however, and the device will 'lock' into its true structure form.

Some care, therefore, needs to be exercised by the engineering designer to ensure that the ratios of link lengths in a mechanism are appropriate to the particular range of operation of the mechanism.

You can play some nice mathematical games with the 'formula' for degrees of freedom. For example, with $F = 1$, the condition for a mechanism with constrained motion becomes

$$3n - 2j - 3 = 1$$

or

$$n = \frac{2j + 4}{3}$$

Since the right-hand side of the equation will always be an even number, the number of links $n$ must also be an even number.

Does this mean that a kinematic chain with an odd number of links can never form a mechanism with constrained motion? According to the 'formula' this deduction is correct. However, as noted earlier the formula takes no account of the link dimensions. What is possible and what is not possible in the motion of kinematic chains depends entirely on the geometry of the linkages.

**SAQ 11**

Determine and tabulate the values of $n$, $j$ and $F$ for the kinematic chains of Figure 43.

\[\text{Figure 43 Kinematic chains (SAQ 11)}\]
3.3 The constrained motion of mechanisms

The 'formula' introduced above is a very useful technique for deciding whether a linkage assembly is a constrained mechanism or not. However, we all have a natural ability to 'see' how simple devices work, like the squeegee mop, the garage door, the toolbox drawer. We do this by following the action of the individual components round the chain of connections, '... if I pull this handle up, this rod will push on the hinge, which will squeeze the water out of the sponge ...'. It is only when we come to what look like fairly complicated devices with a large number of moving components and connections, that our ability to follow the motion is hampered. Now, if we can look at a complicated mechanism, not in terms of the individual moving components, but in terms of a collection of simple assemblies of components, then the 'internal workings' of many complex devices become easier to visualize. What are these simple assemblies? They are the basic four-link chains we met in Unit 1 – the four-bar linkage and the slider–crank linkage. All constrained linkage mechanisms are 'made-up' from one or more of these basic four-link chains.

The TV programme associated with these Units gives you the opportunity to 'see' the motion of several mechanisms. Here on the 'static' page of text, I can only indicate what you might look out for when you are watching a complicated mechanism in motion.

As an example of the approach, we shall look very briefly at some of the mechanisms used in the domestic sewing machine.

Figure 44 shows the sequence of events required to produce a single stitch joining two pieces of material.

The machine has to produce three main movements:
(a) The needle has to rise and fall, carrying the upper thread.
(b) The shuttle which contains the bobbin of under thread has to swing to and fro beneath the needle so as to catch the upper thread and make a stitch.
(c) The material has to be moved forward in between the stitches.

All these movements (and other subsidiary movements) are produced from a single input rotating shaft.

Figure 45(a) shows the needle drive and upper thread take-up mechanism for a sewing machine.

Figure 45(b) shows a line diagram for the device. The rotating shaft at A turns the crank (link 2) to produce the up-and-down motion of the needle attached to the sliding link 4, and a rocking motion of the thread hole T at the top of link 5. With six links and seven pairs, the device is a constrained mechanism, \( F = (6 \times 3) - (7 \times 2) - 3 = 1 \); but how does this work?

The key to visualizing the motion is in recognizing that there are two basic four-link chains contained in the mechanism as shown in Figure 45(c). The needle drive is actuated by a slider–crank form and the thread take-up is part of a crank and rocker four-bar linkage. The 'common' link between the two basic chains (apart from the fixed link of the frame) is the crank link 2 which is the 'input' to both chains.

![Figure 44](image_url)

![Figure 45](image_url)
Figures 46 (a) and (b) show part of the shuttle drive mechanism for the sewing machine.

A rotating shaft at A causes a to-and-fro motion of a shaft at B which is attached to the shuttle. Again the mechanism is made up of two basic four-link chains as shown in Figure 46(c) – a crank and rocker four-bar linkage and a form of slider–crank (quick-return inversion). The 'common' link (apart from the fixed link) is link 4 which is the 'output' of chain (i) and the 'input' to chain (ii).

The mechanism for moving the material along is a combination of a double-rocker four-bar linkage and a cam and follower. We shall not take the example any further. The main idea to grasp is that all plane-linkage mechanisms can be broken down into familiar four-link chains whose motion can be more easily visualized.
Figure 47 shows part of a foldaway loft-ladder mechanism. Draw the line diagram for the mechanism, determine the overall number of degrees of freedom for the device, and identify the constituent four-link chains.

3.4 Revision exercise

From the list of statements given below, decide which are true and which are false.

1. A mechanism is a kinematic chain with a single degree of freedom in its motion.
2. The number of degrees of freedom of an assembly of links is the minimum number of independent quantities needed to define completely the relative positions of all the links.
3. At least four links are required to form a mechanism.
4. All constrained mechanisms have an even number of links.
5. All plane-linkage mechanisms are composed of one or more four-link chains.

3.5 Summary

The number of degrees of freedom associated with an assembly of links can give valuable information as to the forms of motion (or lack of motion) which the assembly can take.

Table 3 summarizes this information.

<table>
<thead>
<tr>
<th>No. of degrees of freedom, $F$</th>
<th>Type of assembly</th>
</tr>
</thead>
<tbody>
<tr>
<td>less than 0</td>
<td>Statically indeterminate structure</td>
</tr>
<tr>
<td>0</td>
<td>Statically determinate structure</td>
</tr>
<tr>
<td>1</td>
<td>Mechanism – constrained kinematic chain</td>
</tr>
<tr>
<td>greater than 1</td>
<td>Unconstrained kinematic chain or mechanism with several inputs</td>
</tr>
</tbody>
</table>

The method of calculating the number of degrees of freedom of an assembly involves finding the number of links $n$ and the number of sliding and rotating pairs $j$. With one link fixed, the number of degrees of freedom is given by $F = 3n - 2j - 3$. This method applies only to plane-linkage mechanisms.

Answers to revision exercise

2, 3, 5 are all true.

1. Certain mechanisms have more than one input motion. The number of inputs required is equal to the number of degrees of freedom of the mechanism, which may be more than one.

4. An even number of links $n$ is required in the ‘formula’ to give $F = 1$. However, if the link dimensions are specially chosen, some ‘apparent’ structures can have constrained motion.