2 Material behaviour under stress

2.1 Stress–strain curves

Material properties and testing is a vast subject area, so I shall restrict the discussion to the key points of interest to the design engineer. The characteristic most commonly used by engineers is the stress–strain curve, obtained experimentally by carefully loading a material specimen in a single uni-axial direction until it breaks. Figure 21 shows resulting stress–strain curves for a mild steel and for PVC (polyvinyl chloride) samples undergoing uni-axial tensile strength tests. Note that for steel there is a stress at point A below which the stress–strain curve is linear, so that stress is proportional to strain in this region. For the PVC however, there is no such linear region, and the relationship is not a constant, varying with the stress and strain (as well as time and temperature).

Concentrating on the steel characteristic beyond A, the limit of proportionality, stress is no longer proportional to strain and the stress–strain curve continues to B, the elastic limit, which is the maximum stress that can be sustained without producing a permanent ‘plastic’ deformation or permanent set, when the load is removed. With steel more curious things happen with further increase in load. At C the stress reaches an upper yield point, followed by a sudden release in stress to its lower yield point at D. From D to F the strain increases even though the stress stays at the lower yield stress. After F strain-hardening occurs and higher stresses can be carried at the expense of large plastic strains until G when the ultimate stress is reached. At this point the specimen visibly thins at one point (or necks), more marked plastic strain occurs until eventually the specimen fractures at H. The ability to hold strength in a plastic mode is called ductility. Ductile materials include aluminium alloys, copper, most mild and medium strength steels and many thermoplastics.

Figure 22 on the other hand shows (to a different scale) a stress–strain characteristic for a material which is not ductile. There is no obvious yielding and relatively little plastic strain before fracture occurs. This is characteristic of brittleness, and examples of brittle materials are cast iron, high-strength steels, concrete, wood, ceramics and glass.
2.2 The linear elastic model

In general for design purposes (rightly or wrongly!) the limit of proportionality, the elastic limit and yield points are usually grouped together and quoted as a yield stress below which the material is assumed to be elastic and linear. Linearity and elasticity are basic assumptions often made in stress analysis, which are fine for most metals, but not for all structural materials (e.g. PVC as you have seen). The usual objective in the design of structures is to make sure that all stresses stay within the linear elastic region, well below the yield stresses, if the structure is to safely carry its loads over any period of time.

Using a linear elastic model we can assume that all stresses and strains are linearly dependent on the applied loads; thus for example, however complicated an internal pattern of stresses and strains may be, if the loads are increased by 10% then so will be the stresses and strains. For our simple uni-axial test case, in the linear elastic region the stress is proportional to strain, so we can say that for uni-axial conditions

\[ \text{stress} = \text{a constant} \times \text{strain} \]

This constant of proportionality is called the modulus of elasticity (or Young's modulus), \( E \), and:

\[ \sigma = E \varepsilon \]

It is also an assumption often made in the linear elastic model that this relationship applies equally well in both tensile and compressive test cases, with \( E \) having the same value in each, but again whilst this may be valid for most metals it isn’t necessarily true of other materials.

**SAQ 19**

What are the SI units of Young's modulus, \( E \)?

The modulus \( E \) is often referred to as the ‘stiffness’ of the material but it may be helpful to think of it as the stress required to double the length of a sample under test — assuming the material to be infinitely strong and linear. (In practice of course it would yield long before this stress was reached.) Interestingly the stiffness of a material depends upon the atomic bonds within it, so the stiffness of most materials is not greatly affected by small amounts of alloying or by heat treatments. However, these do have a considerable effect on the yield and failure stresses. Figure 23 shows the comparative stress–strain curves for a mild (low carbon) steel and a higher-grade steel. Note that the initial slope of the curve, that is the \( E \) value, is much the same, but the yield stresses are vastly different. ‘Steel’ is a generic term covering a vast range of different alloys.

This is also true of other materials, for example aluminium and its alloys. The addition of alloying elements, such as copper to give dural, can considerably increase the yield and failure stresses, but leaves the modulus essentially the same. Without detailed information on the exact alloy and its processing history (including heat treatments and working, such as rolling into sheet or forging) then the yield stress cannot be stated with any accuracy. Manufacturers' and suppliers' technical literature usually gives such design data with reference to appropriate British and international standards.

Apart from ensuring strength we may need to predict the shape changes of a structure under load. The simplest example of all is to estimate the extension of a uniform rod under tension. You should be able to see how to do that. The applied force divided by the cross-sectional area gives the stress \( (\sigma = F/A) \). The stress–strain curve for the appropriate material will give the corresponding strain (or show that it will have yielded!). Provided
the material is in its linear elastic region the actual elongation can then be calculated from the strain \( e = \varepsilon L \). Note that we can use the linear approximation \( \sigma = E\varepsilon \) directly. You may prefer to work with the equation in this form, or else to substitute the definitions of \( \sigma \) and \( \varepsilon \), giving \( F/A = E\varepsilon/L \). (If your material’s yield stress has been exceeded then you’ll have to redesign the rod to ensure that the stress stays within the linear elastic region.)

These equations can also be applied to compression, in which case \( F \) and \( e \) (and \( \sigma \) and \( \varepsilon \)) are negative. Real materials do not necessarily behave the same in compression as in tension despite our modelling assumption, of which I shall have more to say in Unit 11. For examples, the fracture stress of wood is rather lower in compression; on the other hand concrete can withstand large compressive stresses but virtually zero tensile stresses — hence the common use of reinforced concrete, with steel bars to take the tensile stresses. Regarding stiffness properties the Young’s modulus of most materials can be considered the same in tension and compression, although in the case of wood it is usually lower for compression.

For this course I shall assume that my structural materials have properties that do not vary from one point to another, and that the properties are the same in all directions from a point. That is to say that each of my materials is homogeneous and isotropic.

### 2.3 Typical material properties and their use in design

There are no perfect materials with ideal properties all round. Performance in one aspect may be at the expense of another. For example, steel can be modified by alloy constituents, heat treatment, ageing, working (method of manufacture) etc. to give much higher yield stresses, but often at the expense of ductility (it gets more brittle). It is most important to specify and consult standards (e.g. BSI) when selecting materials, and even then you must make sure the material is actually available in the form you want from suppliers. Table 1 gives some representative values for comparative reference only, but please use the appropriate standards if you are designing something for real!

**Table 1** Typical mechanical properties

<table>
<thead>
<tr>
<th>Material</th>
<th>Young’s modulus ( E ) GN m(^{-2} )</th>
<th>Yield stress ( \sigma_Y ) MN m(^{-2} )</th>
<th>Fracture stress ( \sigma_F ) MN m(^{-2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel (mild)</td>
<td>210</td>
<td>240</td>
<td>480</td>
</tr>
<tr>
<td>Steel (high yield)</td>
<td>210</td>
<td>450</td>
<td>600</td>
</tr>
<tr>
<td>Dural</td>
<td>70</td>
<td>300</td>
<td>400</td>
</tr>
<tr>
<td>Brass</td>
<td>105</td>
<td>200</td>
<td>400</td>
</tr>
<tr>
<td>Rubber</td>
<td>0.01</td>
<td>—</td>
<td>20</td>
</tr>
<tr>
<td>Concrete</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(compression)</td>
<td>14</td>
<td>—</td>
<td>40</td>
</tr>
<tr>
<td>(tension)</td>
<td>—</td>
<td>—</td>
<td>5</td>
</tr>
<tr>
<td>Wood</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(compression)</td>
<td>8</td>
<td>—</td>
<td>60</td>
</tr>
<tr>
<td>(tension)</td>
<td>12</td>
<td>—</td>
<td>80</td>
</tr>
</tbody>
</table>
**Example**

An aircraft elevator control rod, made of a material with a yield stress of 350 MN m\(^{-2}\), is 10 mm in diameter and 4 m long. It is tested to a tension of 25 kN. Estimate the stress, the strain and the extension.

**Solution**

Cross-sectional area, \(A = 78.54 \times 10^{-6} \text{ m}^2\).

Stress, \(\sigma = \frac{F}{A} = \frac{25 \times 10^3 \text{ N}}{78.54 \times 10^{-6} \text{ m}^2} = 318.3 \text{ MN m}^{-2}\).

This is safely below the yield stress, so I can use \(\sigma = E\epsilon\).

From the tables, \(E = 70 \text{ GN m}^{-2}\).

Strain \(\epsilon = \frac{\sigma}{E} = \frac{318.3 \times 10^6}{70 \times 10^9} = 4.55 \times 10^{-3}\).

Elongation \(\epsilon = \epsilon L = 4.55 \times 10^{-3} \times 4 \text{ m} = 18 \text{ mm}\).

**SAQ 20**

A concrete column has 0.1 m\(^2\) cross-section and is 10 m in length. If it were to carry a load producing one half of the compressive failure stress, estimate the change of length. Use data from Table 1.

**SAQ 21**

A mine hoist cable of 300 mm\(^2\) cross-section high-yield steel, length 2 km carries a 3000 kg load. Estimate the extension, due to the load only. (In practice the cable's own weight would also be significant in a case like this.) Use data from Table 1.

The solutions of the previous two SAQs simply required manipulation of the basic equations. Figure 24 shows some more difficult examples. In Figure 24(a) there are two different cross-sections. In Figure 24(b) the cross-section is uniform but the material changes. In Figure 24(c) there is a concrete block with reinforcing steel bars in it. In this latter case the cross-section does not vary along the bar but each cross-section has a mixture of materials. In such cases as these three examples the equation \(\frac{F}{A} = \frac{E\epsilon}{L}\) cannot be applied directly to the whole member. To determine the relationship between force and extension requires some extra information. It is necessary to break the member down into components to which the formula can be applied.

![Figure 24](image-url)
Note that in cases (a) and (b) the components carry the same force, so have different extensions. In case (c) the extension is the same for each component material so they must carry different forces. To complete the solution you need to use the following procedure:

**Step 1:** Choose the variable (force or extension) which is equal on the separated components.

**Step 2:** Use \( F/A = Ee/L \) to obtain the other variable (extension or force) for each component.

**Step 3:** Add these derived variables in the appropriate way to find the force or extension for the whole member.

**Step 4:** Substitute numerical values.

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**Example**

Figure 25 shows part of a reinforced concrete column of 0.21 m\(^2\) total cross-section and length 5 m containing ten steel rods of cross-section 1000 mm\(^2\) each. Using \( E_s = 200\) GN m\(^{-2}\) and \( E_c = 15\) GN m\(^{-2}\), for a compressive load of 1 MN estimate (a) the change of length of the column, (b) the stress in each material, (c) the magnitude of the force exerted by each material.

**Solution**

(a) **Step 1:** The common variable is the extension \( e \).

**Step 2:** For the steel \( F_s = A_s E_s e/L \)

For the concrete \( F_c = A_c E_c e/L \)

**Step 3:** \( F_s + F_c = F, \quad (A_s E_s e/L) + (A_c E_c e/L) = F \)

\( (A_s E_s + A_c E_c) e/L = F \)

\( e = FL/(A_s E_s + A_c E_c) \)

**Step 4:** \( F = -10^6\) N (compression, given),

\( L = 5\) m, \( A_s = 0.01\) m\(^2\), \( A_c = 0.20\) m\(^2\)

\( e = \frac{-10^6 \times 5}{(0.01 \times 2 \times 10^{11}) + (0.20 \times 1.5 \times 10^{10})} = -10^{-3}\) m = -1 mm

(b) In this case for both materials the strain is

\( \varepsilon = e/L = -10^{-3}\) m/5 m = \(-2 \times 10^{-4}\)

So the steel stress is \( \sigma_s = E_s \varepsilon = -40\) MN m\(^{-2}\)

Concrete stress is \( \sigma_c = E_c \varepsilon = -3\) MN m\(^{-2}\)

(c) Steel \( F_s = \sigma_s A_s = -40\) MN m\(^{-2}\) \times 0.01 m\(^2\) = -0.4 MN

Concrete \( F_c = \sigma_c A_c = -3\) MN m\(^{-2}\) \times 0.20 m\(^2\) = -0.6 MN

As a check you can see that the total force is 1 MN (compressive).

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**SAQ 22**

Figure 26 shows a 20 kg load supported by two vertical wires, which meet at the load without slack when unstressed. Both wires are mild steel and of diameter 1 mm. Estimate (a) the extension of the wires, (b) the force magnitudes exerted by each.
SAQ 23

4 m of 0.6 mm diameter wire is connected in line to 12 m of 0.4 mm diameter wire. (The wire is cold drawn steel with a yield stress over 1600 MN m\(^{-2}\).) A tensile load of 200 N is to be applied. Estimate the total extension.

Just as there is a modulus \(E\) associated with the normal stress–strain curve, there is another modulus associated with the shear stress–strain curve. The \(\tau-\gamma\) curve for a material has a similar shape to its \(\sigma-\varepsilon\) curve, with an approximately linear region. Hence within this region we can say that

\[
\tau = G\gamma
\]

where \(G\) is called the shear modulus or modulus of rigidity. The value of \(G\) is typically 0.4\(E\). As we shall see later in this Unit, it happens that information on shear stresses can be derived from knowledge of applied normal stresses without recourse to shear stress and shear strain curves. In practice shear stress and shear strain curves are not needed, so we hardly ever refer to them, but the shear modulus \(G\) is important in the analysis of torsional loads, covered in Unit 11.

2.4 Longitudinal and transverse strains; Poisson’s ratio

In discussing the relationship between material stress and strain, I was referring to the strain in the same direction as the applied stress, i.e. uniaxial stress and strain. However, when an object is stretched, not only does it become longer, but also narrower, as the volume of material doesn’t really change (Figure 27). This is particularly noticeable with a rubber band because of the large strains that are possible, and I suggest that you try it. With other materials, it may not be so noticeable. It happens that with most materials this transverse strain is some ratio of the longitudinal strain. A positive longitudinal strain, i.e. an extension, gives a negative transverse strain, i.e. a narrowing. The ratio of the numerical values of transverse strain, \(\varepsilon_t\), to longitudinal strain, \(\varepsilon\), is called Poisson’s ratio, represented by \(\nu\), the Greek letter nu, so:

\[
\nu = -\varepsilon_t/\varepsilon \quad \text{or} \quad \varepsilon_t = -\nu\varepsilon
\]

Values of \(\nu\) lie between 0 and 0.5, with steel about 0.3 and rubber nearly 0.5. Because of this transverse strain the cross-sectional area of a component reduces under tension and this raises a minor difficulty. For example, a circular metal bar with \(\varepsilon = 0.001\) and \(\nu = 0.4\) will have \(\varepsilon_t = -0.0004\). The diameter will then be 0.9996 of its unstressed value and the reduced cross-sectional area means that the stress is actually higher than we thought before. For a material like rubber, achieving large strains, this can be very important. However, you can see from the figures quoted that for normal engineering materials the change in area in elastic stress is less than 1%, so engineers are normally happy to calculate the stress based on the initial area. This is what you should do in this course unless clearly indicated otherwise. In fact you should not worry about transverse strain unless the question explicitly involves it.
2.5 Safety factors

When designing, an engineer needs to know at what limiting stress a particular material can be safely used. Is this the yield stress? Unfortunately it is not quite that simple. The main problems are:

1. **Stress calculation** — the stress is calculated and is not known with exactitude.

2. **Stress raisers** — some shapes and internal flaws result in local stresses much higher than the calculated average; these are called stress raisers.

3. **Crack propagation** — cracks can propagate at stresses much lower than the yield stress, especially for varying or dynamic loads.

It is worth looking at each of these factors in a little more detail.

**Stress calculation**

The loads on a structure may not be known precisely. A good example of this is the effect of air movement. The forces exerted on high buildings and bridges by high winds can be very important, but the extremes of weather over the life of the building cannot be predicted accurately. Also small-scale gustiness can be important. Of course the latter also applies to aircraft even though the nominal flying speeds are well established by the detail design stage.

In a complicated structure it may be difficult to determine how the loads are shared by the members, and also there can be forces arising due to thermal expansion. The forces in the members can only be estimated; the theoretical calculations are an idealized approximation.

**Stress raisers**

A simple example of a stress raiser is a sudden change of cross-section of a simple tension member (Figure 28). The larger cross-sectional area is 200 mm², giving an average stress of \(20 \times 10^5 \text{N}/200 \times 10^{-6} \text{m}^2 = 100 \text{MN m}^{-2}\). The smaller cross-section has half the area, so twice the average stress giving 200 MN m\(^{-2}\). However, if such a member is actually tested it is found to fail at unexpectedly low loads. The reason is that at the narrow section the stress is not distributed uniformly over the section, but is concentrated at the corners. In the case of a cut, the sharp edge can cause a large increase in the local stress (Figure 29). Such stress raisers can be useful in everyday life, for example a piece of paper continues to tear at the end of the existing tear. In engineering a stress raiser is usually a liability and to be avoided. Hence it is most desirable to transmit forces through smoothly varying cross-sections, avoiding notches and sharp internal corners.

Even a round hole in a member will act as a stress raiser, for example a hole for a bolt or rivet, so where possible these should be placed in regions of low average stress. In the case of a tensile test specimen the sharp serrations on the jaws needed to grip the test piece cause locally increased stresses. Therefore the gripped part is made wider, or a larger diameter, and carefully tapered into the test section (Figure 30). On an ordinary bar intended for tension loading such shaping may not be economic and the whole bar must be of the larger size. Figure 31 shows the stress patterns at various sections of a typical bolted tension member. At a distance away from the bolt which is equal to the width of the member the stress has distributed itself across the width in a fairly uniform way. This redistribution of stress is called *St Venant's principle*. It means that outside the immediate region near the application of the load, the average stress is a good approximation to the stress across the cross-section.
Crack propagation

A crack or even a surface scratch can be a worrying stress raiser. In fact the theoretical analysis of the stress raiser due to a sharp-edged scratch shows that the raised stress can be relaxed as the crack advances. However, if the crack is big enough then it can shoot through the component leaving it in two pieces. The critical size of the crack at which this happens depends upon the average material stress and the shape of the material stress-strain curve. This theory is known as the Griffith crack theory. A large crack will 'go' at a smaller average stress, so a large structure such as a bridge or ship, which needs to be safe against a large crack, needs to be designed for a correspondingly low stress.

One advantage of the suspension type of bridge is that the cables are made of many thin strands of high-yield (strong) steel. These thin strands physically do not contain a large crack so they can safely operate at higher stress, and even if a smaller crack was present it could not grow into the neighbouring strand. Therefore the cables require less material and give a lower final cost.

There are certain ways in which a crack can propagate even though it is smaller than the Griffith size for a particular material and stress. One example is stress corrosion where the stressed metal is more prone to react chemically and is thereby weakened. Another is fatigue, which means that the stress is not steady but fluctuating. These fluctuations can slowly advance a crack until Griffith size is reached, at which point total failure occurs.

So, because of uncertainties in the loads on the member, in the material capabilities, from the presence of stress raisers and so on, it is not practical to design components to withstand exactly their service loads. To take account of such realities and to allow calculation to provide safe and economic components and structures, safety factors are used. The values used for safety factors depend upon the application. For example it may be economically attractive (and no less safe) to have relatively small safety factors coupled with very careful design and monitoring of materials (for example in the aerospace industries), whereas in other areas it is more attractive to have larger safety factors and worry less about the precise condition of the materials (e.g. in civil engineering).

Safety factors are implemented in various ways. One way is to compare the expected failure load (EFL) to the expected load (EL) giving a load safety factor

\[ \text{LSF} = \frac{\text{EFL}}{\text{EL}} \]

or to compare the material's failure stress, FS, (e.g. the yield stress) to the predicted stress (PS) for the expected load, giving a stress safety factor

\[ \text{SSF} = \frac{\text{FS}}{\text{PS}} \]

If the expected failure is by material stress, then the SSF will equal the LSF. However, if the expected failure is other than a stress failure (e.g. by buckling, explained in Unit 11) then the LSF and SSF will have different values for a given structure.

Nowadays it is becoming common to apply a slightly more complex method called the resistance factor – load factor method (RF–LF method). The possible load (PL) is the expected load (EL) times the load factor (LF):

\[ \text{PL} = \text{EL} \times \text{LF} \]
The expected failure load is the possible load times a resistance factor, RF:

\[
EFL = PL \times RF = (EL \times LF) \times RF
\]

so the total safety factor is

\[
\frac{EFL}{EL} = RF \times LF
\]

just the product of the load and resistance factors. Hence the total safety factor is the result of a load factor to account for uncertainties in the load, and a resistance factor to account for uncertainties in the structure, e.g. the quality of materials, joints, straightness of members and so on, just as in the older system different resistance factors may be applied for different types of failure. For example, there may be an expected load of 100 kN, a load factor of 1.4, a resistance factor of 4.0 against material yield, and a resistance factor of 1.6 against buckling. This gives a possible load of 140 kN, an EFL of 560 kN for yield, and an EFL of 224 kN for buckling. The LSF against material yield (i.e. the SSF) is thus 5.6, and the actual LSF, because of buckling, is 2.24. If buckling were expected at a higher load than yielding, then the LSF would be equal to the SSF, not higher, because yield would be the actual expected failure mode.

The RF–LF method is more rational, in that the need for the safety factors and their values can be more closely related to the actual service conditions, giving a better performance/cost relationship than more crude blanket safety factors. The values of the factors applied in practice depend upon the particular application, and the likelihood of the loading condition actually occurring, i.e. a likely load will be allowed a higher factor. The glossary includes brief explanations of many of the terms that you might meet in this and other texts.

The conclusion to be drawn from all this is that there is a stress level at which a material is safe for a given application, but that safe working stress depends upon the precise material, its heat treatment and working, its environment (vibration and corrosion), the actual size of the members of the structure, how the loads are applied and transmitted through joints and fasteners, etc. This safe working stress might be looked upon as a line of demarcation between the areas of Engineering Mechanics and of Materials Science. If you want to know more about the reasons for the value of a safe working stress than you will have to study 'materials'. As far as this course is concerned you now know enough to appreciate why the safe stress is not just the yield stress and why it can vary so much for, say, 'steel'. What I want to concentrate on in this Unit is how we determine whether the safe working stress will be exceeded or not in a particular design. To do this you must be able to look at the combination of shear and normal stresses at a point and interpret them in the simplest way. The next section begins to tell you how to do that.