4 Mohr's stress circle transformation model

4.1 Plane stress transformation equations analysed

You may have found using the transformation equations a little tedious, especially in SAQ 27. Fortunately there is an easy way to analyse plane stress at a point. Firstly, though, we have to perform a little more algebraic manipulation.

We can rearrange the transformation equations as follows:

$$\sigma_\theta - \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy}) = \tfrac{1}{2}(\sigma_{xx} - \sigma_{yy}) \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_\theta = \tfrac{1}{2}(\sigma_{xx} - \sigma_{yy}) \sin 2\theta - \tau_{xy} \cos 2\theta$$

We can square both sides and add the equations. This allows us to get one equation featuring the two variables σ_θ and τ_θ, using the trigonometric relation $\cos^2 2\theta + \sin^2 2\theta = 1$:

$$[\sigma_\theta - \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy})]^2 + \tau_\theta^2 = \tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \tau_{xy}^2$$

Now compare this equation with that of a circle for two variables x and y, whose centre is at a, b and whose radius is r (Figure 44):

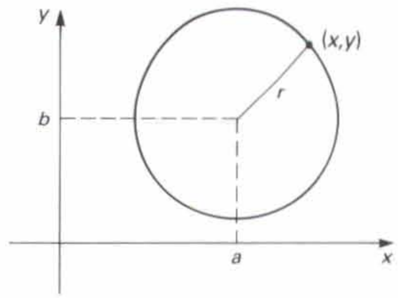

Figure 44

$$[x - a]^2 + [y - b]^2 = r^2$$

Mohr's circle

You can see that the equation relating our two stress variables (σ_θ, τ_θ) is that of a circle, known as *Mohr's circle*, where the equivalent parameters are:

$$a = \tfrac{1}{2}(\sigma_{xx} + \sigma_{yy})$$

$$b = 0 \qquad \text{(effectively we have } [\tau_\theta - 0]^2\text{)}$$

and $\quad r = \sqrt{\tfrac{1}{4}(\sigma_{xx} - \sigma_{yy})^2 + \tau_{xy}^2}$

So if we plotted σ_θ against τ_θ for a range of values of θ we should expect to see a circular graph. Moreover our graph of stress plots will always have its centre on the horizontal axis ($\tau_\theta = 0$), because the equivalent b parameter value is 0. This gives us a quick graphical way of determining stresses in other directions (i.e. on other elements at the same point).

SAQ 28

Plot the graph of σ_θ against τ_θ for the results obtained in SAQ 27. Use the axes as shown in Figure 45.

If we know the stress components on one element face, therefore, we can determine the stress components on any plane (or element face) at the same point inclined at an angle to the original element.

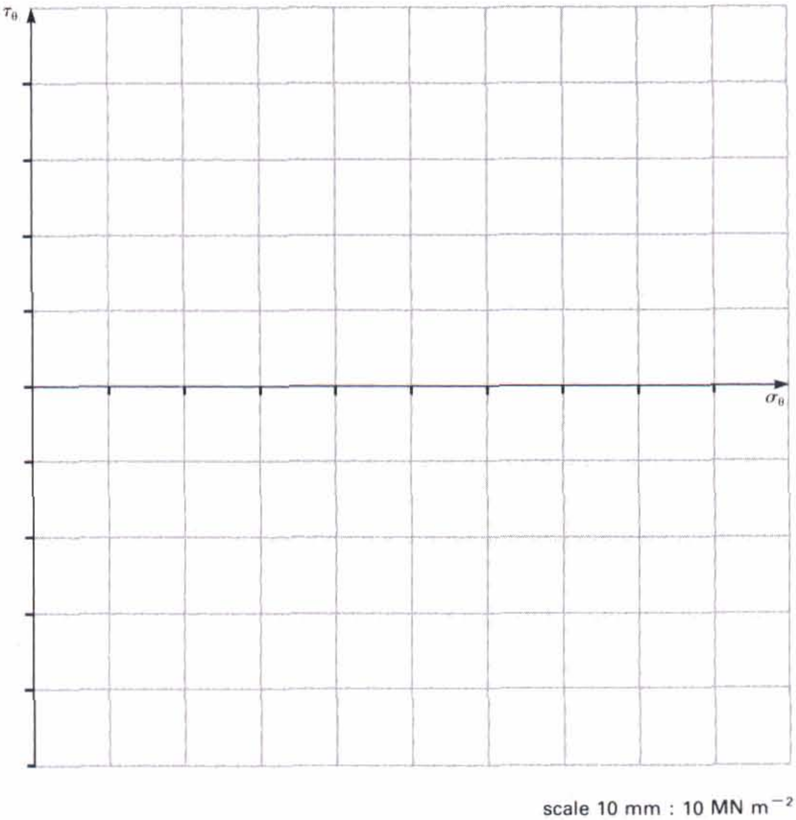

Figure 45

4.2 Drawing Mohr's stress circle

Figure 46 shows a stress element with a known pair of stresses on it. Stresses on the other faces have been omitted for clarity. Also shown is another stress element at the same point, inclined at angle θ to the first element. If we have sufficient data to draw the circle graph the stresses on the other element face can easily be shown as another point on the circle, Figure 47. Note that although plane A′B′ is located at angle θ anti-clockwise from plane AB, the corresponding stress plot is located at 2θ anticlockwise from the AB stress plot. (The 2θ relationship was explained in Section 3.2.) This complication of doubling the angle on the circle graph always applies with Mohr's circle. The stress plots of planes θ apart on elements appear as 2θ apart on the circle (in the same direction, clockwise or anticlockwise).

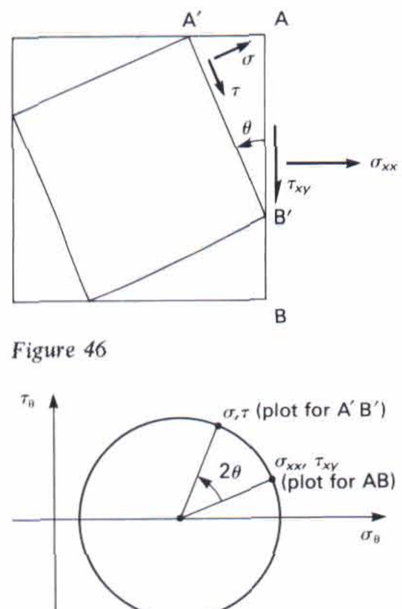

Figure 46

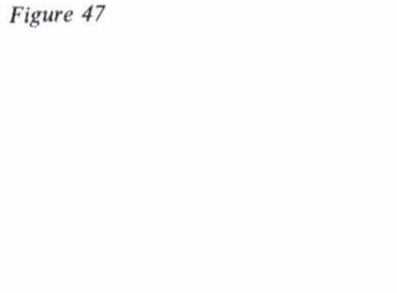

Figure 47

Notice that if we have planes on elements at 90° to one another, the corresponding stress plots will be at 180° to one another on the circle. This gives us a very quick way of drawing the circle graph, because such stress plots will be at the ends of a diameter, and we already know that the centre of the circle will always be on the $\tau_\theta = 0$ axis (Section 4.1).

The circle is drawn, not by plotting many $(\sigma_\theta, \tau_\theta)$ points, but by a geometrical construction after finding the centre and radius by plotting the end-points of a diameter. Figure 48 shows an element in a state of plane

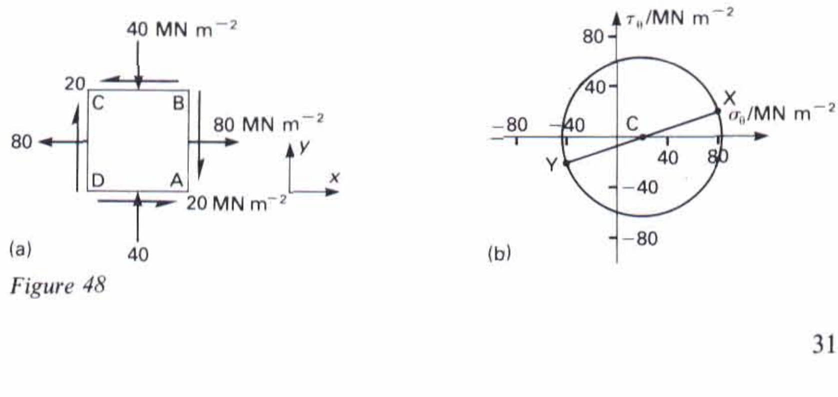

Figure 48

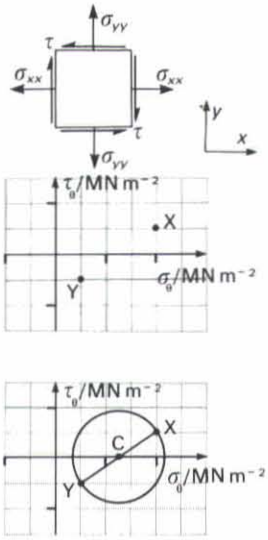

Figure 49

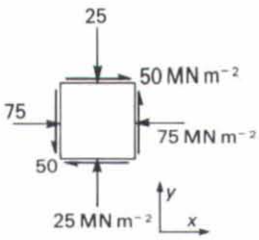

Figure 50

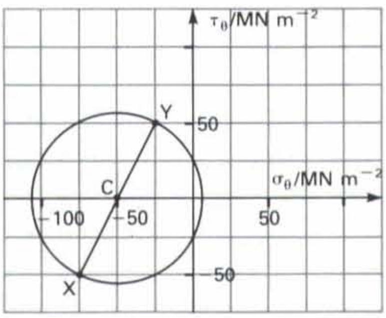

Figure 51

stress and the corresponding Mohr's circle. The opposite faces AB and CD of the element have stress components (80, 20) $MN\,m^{-2}$ and these are plotted as the point X. The faces BC and DA have stress components $(-40, -20)$ $MN\,m^{-2}$ and are plotted as the point Y. Plots X and Y are the end-points of a diameter of Mohr's circle, so the centre C is located as the mid-point of this diameter. The circle is then drawn with the centre at C and radius CX.

Mohr's circle procedure (Figure 49)

1. Draw axes, σ_θ horizontal and τ_θ vertical, with the same scale on both axes.
2. Plot X (σ_{xx}, τ) and Y $(\sigma_{yy}, -\tau)$.
3. Join XY and mark the intersection with the σ_θ axis. This is the centre C. Check by calculation that C is at $\{\frac{1}{2}(\sigma_{xx}+\sigma_{yy}), 0\}$.
4. Draw Mohr's circle, centre at C, through X and Y.

Note that the scale should be chosen to give a circle of reasonable size and the scale must be the same on both axes. This choice of scale will not be difficult after some practice in drawing Mohr's circles. (I have omitted the suffices for τ_{yx} and τ_{xy} to avoid cluttering up the element sketch.)

Example

Draw Mohr's circle for the state of stress shown in Figure 50.

Solution

Using the procedure,

1. Scale: 10 mm : 50 $MN\,m^{-2}$ on both axes (Figure 51).
2. Plot X $(-75, -50)$ $MN\,m^{-2}$ and Y $(-25, 50)$ $MN\,m^{-2}$.
3. Join XY; mark in C. Check C is the point $(-50, 0)$ $MN\,m^{-2}$.
4. Draw Mohr's circle.

SAQ 29

Draw Mohr's circles for each of the eight states of stress shown in Figure 52. All stresses are in units of $MN\,m^{-2}$.

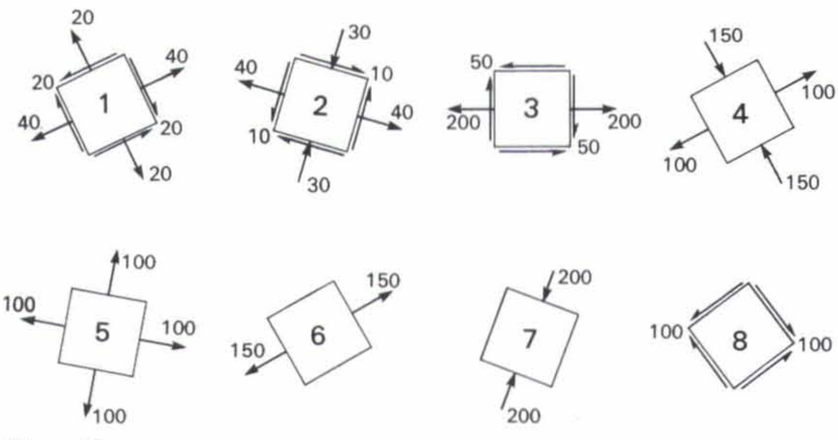

Figure 52

Given a Mohr's circle of stress, with the end-points X and Y of a diameter marked, you should also be able to work back and draw the corresponding element.

Example

For the Mohr's circle of Figure 53 draw the element aligned with the x and y axes, and indicate the stresses.

Solution

Stress coordinates of X are $(-100, 100)$ MN m^{-2}.

Stress on faces AB and CD is $\sigma_{xx} = -100$ MN m^{-2} and $\tau = 100$ MN m^{-2}.

Stress coordinates of Y are $(50, -100)$ MN m^{-2}.

Stress on faces BC and DA is $\sigma_{yy} = 50$ MN m^{-2} and $\tau = -100$ MN m^{-2}.

Figure 54 shows this state of plane stress.

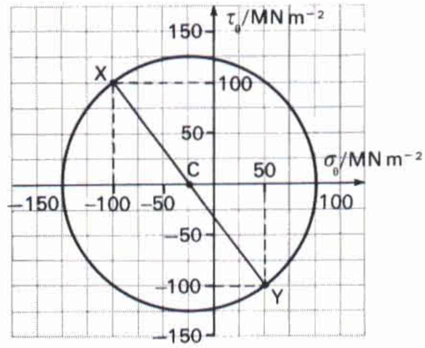

Figure 53

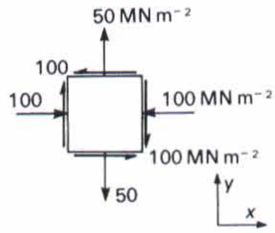

Figure 54

SAQ 30

Draw the states of stress on the element, aligned with the x and y axes, from each of the Mohr's circles in Figure 55.

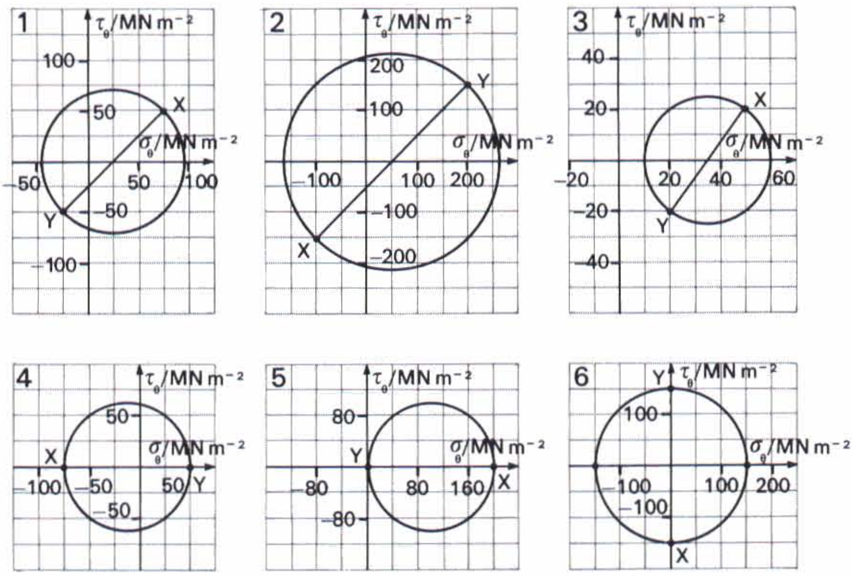

Figure 55

We have seen in Section 4.1 how the equation representing the stress field at a point is that of a circle giving a graph of all the normal and shear stress coordinates, and have practised drawing a variety of such circle graphs. With reference to the equation, you should be able to quickly compute sufficient data to draw the circle without having to measure anything to scale.

SAQ 31

In Figure 48, what are the coordinates of the centre of the circle, and what is the radius?

SAQ 32

From Figure 48, what are the values of maximum shear stress, maximum normal stress and minimum normal stress at the point?