4 Struts

A member in compression is usually called a strut or column, or sometimes a stanchion. For moderate forces, the member gets a little shorter because it is not perfectly rigid \( (F/A = Ee/L) \). However, when a critical load is reached the member suddenly behaves differently — it undergoes a new type of failure (Figure 35). This is called buckling. It is the most common type of stability failure. A thin member in compression may support something like only one thousandth of its compressive yield stress capability. Indeed for thin wire and string it may be hard to measure any compressive strength at all. A member which will be loaded in compression, therefore, requires careful attention to its design.

The useful compressive strength of a strut is dependent on its length. If samples of a given material and diameter but different lengths are tested, a curve like Figure 36 is found. For very short samples the strength is fairly constant (A), but for long ones it is much less and decreases rapidly (B). The type of failure is also different. In region A the material is irreparably crushed and the strength corresponds quite well to the material compressive failure stress times the cross-sectional area, \( F = \sigma A \). In region B, the initial buckling occurs elastically, so the member — if not allowed to collapse beyond initial failure — can recover its shape when the forces are removed. The phenomenon of buckling is very important structurally, but can be observed on household objects too, such as a ruler or a strip of cardboard. You may have noticed it when pushing on a thin knife blade or using a woodsaw.

For compressive members in practical structures, buckling is a serious problem, and there is often a clearly visible distinction between the members in tension, which can be thin, and those in compression which need to be much thicker, even for the same value of force. One example is yacht rigging (Figure 37b) where the mast is much thicker than the tension wire supports. Another is the crane (Figure 37a), where the control wires and hoist cable are in tension, but the jib is in compression and so needs to be much bulkier.
Engineers need to be able to estimate the buckling load for a compression member. We can gain an initial insight into this by analysing the arrangement of Figure 38(a), two light rigid pin-jointed links with a sprung hinge in the centre. For each radian of deflection at the hinge, this torsion spring gives a torque of \( K \) (N m), i.e. it has a torsional stiffness of \( K \) (N m rad\(^{-1}\)). I shall neglect the weight of the links, so it does not matter which way up it is. Figure 38(b) shows it in a ‘buckled’ position. Figure 38(c) is the free-body diagram for one half. For a small angle \( \theta \) the vertical deflection of the centre is approximately \( L \theta \). For rotational equilibrium of the member, I shall take moments about the centre hinge to avoid the effects of other forces there:

\[
M - P \times L \theta = 0
\]

The total angular deflection of the centre hinge is \( 2\theta \) so the moment exerted by the spring is \( M = K \times 2\theta \)

\[
\therefore \quad 2K\theta - PL\theta = 0
\]

\[
\therefore \quad (2K - PL)\theta = 0
\]

\[
\therefore \quad \theta = 0, \text{ or else } (2K - PL) = 0, \text{ i.e. } P = 2K/L
\]

What is the physical significance of these mathematical solutions? Either \( \theta = 0 \) or \( P = 2K/L \). Therefore, either the whole member is straight, or \( P = 2K/L \). In other words, for an end force smaller in magnitude than \( 2K/L \), the member will stay straight. If the end force reaches the value \( 2K/L \) (called the critical load), then \( \theta \) can be anything — the strut can collapse (Figure 39). This simple model exhibits typical buckling behaviour. For struts of longer \( L \) the critical buckling load \( (P = 2K/L) \) is consequently smaller.

Now we can begin to make sense of Figure 36. If the member is short (small \( L \)), the buckling load is greater than the material limitations, so buckling does not occur because the material itself fails first. As \( L \) becomes greater the critical buckling load will fall below the material failure capability so buckling becomes the predominant mode of failure, and as \( L \) increases the maximum possible applied load declines (Figure 40).

The model that I have used to introduce buckling is a very simple one. All the flexibility was concentrated at the centre (a ‘lumped parameter’ model). This model gives a qualitative understanding of buckling, but is not good enough to make useful quantitative predictions about real struts with the flexibility distributed all the way along the member. Figure 41 shows a more realistic uniform strut in a buckled position. Each section of the strut has a bending moment in it. This bending moment is the same as you looked at in detail when studying beams but this time is caused by the offset effect of the line of action of \( \bar{P} \). At a point of deflection \( y \) the moment is \( M = Py \). The largest deflection is near to the centre (for a symmetrical strut, exactly so). To prevent buckling this is, therefore, the most important place to increase the thickness. The crane in Figure 37 has a jib thick in the centre where required, thinning at the ends to save material.

Mathematical analysis of the strut model of Figure 41 predicts a critical load of

\[
P = \frac{\pi^2 EI}{L^2}
\]

The physical argument underlying the analysis is similar to that for the hinged strut that we looked at. This is called the Euler formula after Leonard Euler (pronounced oiler), the Swiss mathematician (1707–1783). The product \( EI \) is a measure of the member’s stiffness in bending. \( E \) is the material’s Young’s modulus and \( I \) is the second moment of area of the cross-section. This term is sometimes called flexural rigidity.
Example
A metal bar \((E = 200 \text{ GN m}^{-2}, \sigma_y = 200 \text{ MN m}^{-2})\) has length 5 m and diameter 20 mm. Estimate (a) the material yield load and (b) the buckling load, and compare them.

Solution
(a) The yield load estimate is \(P = \sigma_y A = 63 \text{ kN}\).
(b) The buckling load estimate from the Euler formula is
\[
P = \frac{\pi^2 E I}{L^2} = \frac{\pi^2 E \frac{1}{4} \pi R^4}{L^2} \quad (I = \frac{1}{4} \pi R^4 \text{ from Table 2}).
\]
\[
= 620 \text{ N}
\]
The buckling load is less than one hundredth of the material stress limit load.

SAQ 26
(a) If the diameter of the metal bar is increased to 63.5 mm, what is the buckling load?
(b) How many times the amount of material of the previous bar is needed?

SAQ 27
The control linkage of an aircraft elevator includes an aluminium alloy rod \((E = 70 \text{ GN m}^{-2})\) of length 2.4 m and solid circular cross-section 20 mm diameter, required to transmit a compression force of 400 N. (a) What is the buckling load? (b) What is the load safety factor? (c) Estimate the diameter of solid rod that would be on the verge of buckling.

This weakness of members in compression due to buckling is an expensive state of affairs, requiring the use of a lot of extra material. What can we do to improve the resistance to buckling without using so much material? From the Euler formula \(P = \frac{\pi^2 E I}{L^2}\) you can see that we must either increase \(E\), increase \(I\) or decrease \(L\). Increasing \(E\) is rarely a practical solution. The materials are usually predetermined by other factors. It might be possible to change from say aluminium to steel, but the aluminium was probably being used for lightness, so the denser steel will not be welcome. Reducing \(L\), the overall length, may not be practical as it would probably defeat the whole purpose of the member.

Increasing the second moment of area, \(I\), looks an interesting possibility. To do this means moving the material further way from the bending axis to make it more effective in resisting bending. For a rectangular section (Figure 42), \(I = ab^3/12\). By doubling \(b\) and halving \(a\) we can use the same amount of material but increase \(I\) by a factor of four. However, when \(b\) becomes greater than \(a\), then buckling can occur by bending about the other axis. The resistance to buckling depends on the least \(I\) value for the given section. However, there are other ways. The material can be spread out by making a hollow section, for example a circular tube.

SAQ 28
The solid 20 mm diameter 5 m long rod of the last example is to be replaced by a tube with 50 mm average diameter and 2 mm wall thickness. (a) By what proportions are the cross-sectional area and mass changed? (b) Estimate the new buckling load. (c) By what factor has this changed?
The hollow tube is a considerable improvement in buckling resistance, although often still far from the material stress limit. The modern lamp-post shows that the hollow tube can be a good way to achieve better $I$ values and bending resistance. Other sections which one sees commonly are the rolled-steel joist I-section, angle, channel and tee (Figure 43). These are cheaper to produce than hollow tube, but still retain the advantage of better $I$ values than round or square solid sections.

Another way to increase $I$ is to split the member up into several parallel separated pieces. This way each piece is less prone to local damage than thin tube. The jib of a crane is usually made this way.

**SAQ 29**

(a) Calculate the $I$ values of the three cross-sections of Figure 44. They have the same area so would use the same amount of material.

(b) Express the two better ones as ratios of the smallest value.

*(Hint: the $I$ value of a shape can be found by subtracting the value for a ‘missing’ piece from that for a simple shape.)*

The contribution to bending resistance of a member of a strut not actually located on the bending axis can be expressed mathematically in another useful way. Figure 45 shows the member whose own neutral axis is $G$ but it is being bent about the axis $XX$. The small area $\delta A$ is $y$ from the neutral axis. Measured from the bending axis $XX$, the second moment of area of $\delta A$ is, by definition,

$$\delta I_X = \delta A (h + y)^2$$

$$= \delta A h^2 + \delta A y^2 + 2 \delta A h y$$

The total $I$ contributed by that member is found by integration over the area

$$I_X = \int \delta I_X = \int h^2 \, dA + \int y^2 \, dA + \int 2h y \, dA$$

Now

$$\int h^2 \, dA = h^2 \int dA = h^2 A,$$

$$\int y^2 \, dA = I_G \quad \text{by definition.}$$

$$\int 2h y \, dA = 0 \quad \text{because $y$ goes equally positive and negative (G is the neutral axis.)}$$

So $I_X = I_G + Ah^2$.

In this case, for the complete strut $I = 4I_X$.

This result corresponds to the Parallel Axis Theorem for second moments of mass, which you met in Block 4 (Section 6.5).

With a good separation of the members nearly all the second moment of area comes from the $Ah^2$ term. Of course this is just another way of saying that $I_X$ is much larger than $I_G$. Crane jibs, power pylons, the Eiffel tower and many other structures bear witness to the importance of this in improving bending, and hence buckling, resistance.

Returning to the crane jib, if we actually built a jib with this four-cornered construction, but without any interior support, what would happen? Each of the four corner members would have to carry one-quarter of the total axial load. But the buckling resistance of each corner on its own is only provided by $I_G$, not $I_X$, so they will buckle easily. Our calculated value of $I_X$ does not on its own assure a high load capability for the whole strut. We must make the four corners act together by linking them with bracing. This is why steel structures such as power pylons are full of triangulation bracing, and bracing on the bracing and so on. It gets complicated, but on
a large structure it is cheaper because the job can be done with so much less material than a single solid piece. Such a structure can also be assembled on site in remote areas.

There are a few other points that you should be aware of. Our model of Figure 42 was ideally straight when unloaded, and the load was applied exactly axially. A real strut will always be bent to some degree or another even before the load is applied. The result of this is that the deflection–load curve shows an increasing value all the way, and the effective collapse load is reduced. Similarly the loads are never applied exactly axially, and this has the same effect. In general the best performance of struts is achieved by ensuring that the loads are, as accurately as possible, applied axially to a straight strut.

I have only discussed one mode of buckling, but others are possible. Details of these can be found in reference books but for this course you do not need to know more than their existence. Figure 46 shows some examples.

---

**SAQ 30**

Figure 47 shows an I-section of aluminium alloy \(E = 70\) GN m\(^{-2}\).

(a) Estimate the second moment of area about each of the axes shown.

(b) Under what load do you estimate a 2.5 m long section would buckle?

---

**Summary of Section 4**

Compression members (struts or columns) may fail by strength, stiffness or stability. A moderate load on a strut will cause length change \(e\) (estimated from \(F/A = Ee/L\), stiffness). A larger load will cause buckling on a long member (estimated from the Euler formula, \(F = \pi^2 EI/L^2\), for stability). For very short members buckling does not occur but the material itself will fail (estimated from \(F = \sigma A\), for strength).

Compact section members are very prone to buckling and so in practice rolled I, L, T, U sections or tube are used, and for larger members built-up structures, to improve the second moment of area.